Chapter 5

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5.1 The Idea of a Limit

The limit is the building block of differential and integral calculus. Specifically, it makes precise the idea of *approaching a number* or *approaching infinity*. The major application of the limit is the idea of a derivative, the instantaneous rate of change of a function.

5.2 Limits of Sequences

5.2.1 Definitions and Examples of Sequences

A ***sequence*** is essentially an ordered, countably infinite list of (possibly repeating) real numbers. For every natural number n, the sequence $\{a_n\}$ gives us back a real number a_n. A sequence is usually written as follows: $\{a_n\} = a_1, a_2, a_3, \ldots$. The notation $\{a_n\}$ usually refers to the ordered set of all the elements in the set, while a_n refers to the n-th element of the sequence $\{a_n\}$. Of course, a sequence of complex numbers also makes sense and many of the statements and proofs for *complex sequences* follow almost immediately from those of *real sequences*.

One simple example of a sequence is one that has as its n-th entry, the number n:

$$\{a_n\} = 1, 2, 3, \ldots.$$

Of course, our sequences may be as simple as we like (like the constant sequences $c_n = a, a, a, \ldots$) and can become very complicated. An example of the latter are ***recursive sequences***, where the n-th entry can be derived from the previous $n-1$ entries. For example, consider the following relatively easy recursive sequence:

$$x_1 = 1$$

$$x_n = n \cdot x_{n-1}.$$

Running the recursion a few steps, we see that our sequences has as its first terms: $\{x_n\} = 1, 2, 6, 24, 120, 640, \ldots$. A proof by induction can easily give that our sequence can actually be explicitly written as a factorial: $x_n = n!$

There are a variety of standard examples of sequences. Listed below are some:

$$a_n = (-1)^n = -1, 1, -1, 1, -1, 1, \ldots$$

$$b_n = \frac{1}{n} = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$$

$$c_n = \sin\left(\frac{n\pi}{2}\right) = 0, 1, 0, -1, 0, 1, 0, -1, \ldots$$

Note that to make sense of a sequence, we must tell you what the *starting element* is. Usually, this is the term a_0 or a_1, but sometimes our statement does not make sense for $n = 0$ or $n = 1$.

5.2.2 Describing Limits - Monotonicity and Boundedness

There are many adjectives that mathematicians use to qualitatively describe sequences. A sequence is ***monotone increasing*** if $a_n \leq a_{n+1}$ for all n. Similarly, a sequence is ***monotone decreasing*** if $a_n \geq a_{n+1}$ for all n. Thus, as is implied in the name, a sequence is increasing if its value goes up as n goes up; conversely, a sequence is decreasing if its value goes down as n goes up. We say that a sequence is ***monotone*** if it is either monotone increasing or monotone decreasing.

A sequence is ***bounded below*** if there exists some constant $C_1 \in \mathbb{R}$ such that $a_n \geq C_1$ for every n. Similarly, a sequence is ***bounded above*** if there exists a constant $C_2 \in \mathbb{R}$ such that $a_n \leq C_2$ for all n. Thus, a sequence is bounded above if every entry in the sequence is below some fixed number; similarly, a sequence is bounded below if every entry is above some fixed number. We say a sequence is ***bounded*** if it is both bounded above and bounded below.

5.2.3 Convergence of Limits

Intuitively, we see that many sequences a_n begin to get really close to some fixed number L as n gets big. This is the idea motivating the definition of the convergence of a sequence.

We say that a sequence a_n ***converges to*** L if for every $\varepsilon > 0$, there exists some N such that for every $n > N$, $|a_n - L| < \varepsilon$.

This definition seems convoluted at first until we begin to break it apart. To prove convergence of a sequence a_n, we are given a small number ε as a challenge. Given this ε, we must provide some N such that from then on (i.e., for $n > N$) the distance between a_n and L is as small as they wanted (i.e., $|a_n - L| < \varepsilon$). Note that the N that we provide depends on the ε that is requested. Intuitively, a smaller ε requires a larger N because we must go further along in the convergent sequence to get within the smaller radius.

This definition is best illustrated by an example. We will prove that the sequence $a_n = 1/n$ will converge to 0. Intuitively, this is clear since as n gets very large, $1/n$ gets very close to zero; in fact, it becomes as close to zero as we like. Note that in this proof, we first have a *discussion*, in which we will decide, given $\varepsilon > 0$, what value of N we should choose. Needless to say, N will depend on ε.

Proposition. The sequence $a_n = 1/n$ converges to 0.

Discussion. To show $a_n \to 0$, our goal is to, given some $\varepsilon > 0$, find an N such that for $n > N$, $|a_n - L| < \varepsilon$. Since $a_n = 1/n$, we need to solve

$$\left|\frac{1}{n} - 0\right| < \varepsilon.$$

Of course, this is equivalent to

$$\frac{1}{n} < \varepsilon;$$

we were able to drop the absolute value because $n > 0$ and thus $1/n > 0$. Solving, we have that

$$\frac{1}{\varepsilon} < n.$$

Thus, we should let $N = \dfrac{1}{\varepsilon}$.

Proof. Given $\varepsilon > 0$, let $N = \dfrac{1}{\varepsilon}$. Then, for $n > N$, we have that

$$n > N = \frac{1}{\varepsilon},$$

which give us that $\dfrac{1}{n} < \varepsilon$. Since $1/n > 0$, this is equivalent to

$$\left|\frac{1}{n} - 0\right| < \varepsilon,$$

which is exactly what we wanted to prove by our definition of a_n, L, and convergence.

One commonly used theorem about converging limits whose proof uses intimately various axiomatic properties of $\mathbb{R}$ is the following.

Theorem. Every bounded, monotone sequence converges.

5.3 Limits of Functions.

5.3.1 Approaching a number.

When we studied the limit of a sequence a_n, we always let $n \to \infty$. In the context of a function $f(x)$, we are interested in what happens as our variable x gets close to some real number a. Of course, the definition of taking a limit as x goes to some number a of $f(x)$ will be phrased in much of the same logic and language as that of sequences.

5.3.2 Defining the limit of a function.

The definition for the limit of a sequence was phrased as an $\varepsilon - N$ statement. In a similar spirit, the definition for the limit of a function is given as an $\varepsilon - \delta$ statement. As before, we are given $\varepsilon > 0$, and we are asked to produce a $\delta > 0$ so that our function $f(x)$ is close to our limit L.

We say that ***the limit of*** $f(x)$ ***as*** x ***approaches*** a ***is*** L if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $0 < |x - a| < \delta$ implies that $|f(x) - L| < \varepsilon$. We write

$$\lim_{x \to a} f(x) = L.$$

Thus, we are given the challenge of getting $f(x)$ ε-close to L; to do this, we need to give them a small region around a (of radius δ) such that for x in this region, $f(x)$ is ε-close to L. Of course, an example is the best way to illustrate the mechanics of this definition. As before, we will calculate the δ we need (in terms of ε) in a discussion.

Proposition. The constant function $f(x) = b$ has $\lim_{x \to a} f(x) = b$.

Discussion. To prove this proposition, we need to find, given some $\varepsilon > 0$, a $\delta > 0$ such that $|x - a| < \delta$ implies that $|f(x) - b| < \varepsilon$. However, we see that since $f(x) = b$, then we have the following equality $|f(x) - b| = |b - b| = 0$. Thus, we may choose any $\delta > 0$ since any x will give us that

$$|f(x) - b| = |b - b| = 0 < \varepsilon$$

since ε was taken to be strictly greater than 0.

Proof. Given any $\varepsilon > 0$, let $\delta = 1$ (we can actually let δ be *any* positive number. Then, for any x (in particular, for those with $|x - a| < \delta$), we have that $|f(x) - b| = |b - b| = 0 < \varepsilon$. Thus, $\lim_{x \to a} f(x) = b$.

Proposition. Let $f(x) = 2x + 6$. Then

$$\lim_{x \to a} f(x) = 2a + 6$$

for any $a \in \mathbb{R}$

Discussion. Assume we are given $\varepsilon > 0$. Our goal is to find a $\delta > 0$ such that if we are in the region $|x - a| < \delta$, then our function will satisfy $|(2x + 6) - (2a + 6)| < \varepsilon$. Starting with this latter inequality, we will try to solve for δ in terms of ε by finding a term of $|x - a|$. Thus, the inequality

$$|(2x + 6) - (2a + 6)| < \varepsilon$$

simplifies to

$$|2(x - a)| < \varepsilon.$$

Using properties of absolute values, we we can reduce this to

$$2\,|x - a| < \varepsilon.$$

Now, we can find our $|x - a|$ term by dividing by 2 to obtain

$$|x - a| < \frac{\varepsilon}{2}.$$

Thus, a wise choice of δ will be this bounding constant $\frac{\varepsilon}{2}$.

Proof. Given any $\varepsilon > 0$, let

$$\delta = \frac{\varepsilon}{2}.$$

Then, if $|x - a| < \delta$, then

$$|x - a| < \frac{\varepsilon}{2}.$$

In particular, we have that

$$2\,|x - a| = |2(x - a)| < \varepsilon.$$

Adding a subtracting a 6 gives us

$$|(2x + 6) - (2a + 6)| < \varepsilon.$$

Of course, since $f(x) = 2x + 6$ and $L = 2a + 6$, we have that

$$\lim_{x \to a} 2x + 6 = 2a + 6.$$