# Chapter 3

# Wednesday, July 13, 2011

## 3.1 Sets of Real Numbers - Rational and Irrationals

This section aims to give a more detailed description of sets of numbers larger than just the integers $\mathbb{Z}$. These larger sets, the rationals, reals, and complex numbers, are intuitively simple yet have several striking properties.

### 3.1.1 The Rationals.

The rational numbers are the set of fractions that we know and love from elementary school. One would like to define the rationals as

$$\mathbb{Q} = \left\{ \frac{p}{q} \,\middle|\, p, q \in \mathbb{Z}, q \neq 0 \right\}.$$

Of course, this definition of $\mathbb{Q}$ gives us too many numbers, as we want for $\frac{1}{2}$ and $\frac{2}{4}$ to be seen as the same number. Thus, two elements of our set $\frac{p_1}{q_1}$ and $\frac{p_2}{q_2}$ are declared to be the same if there is an integer $k \in \mathbb{Z}$ such that

$$\frac{p_1}{q_1} = \frac{k \cdot p_2}{k \cdot q_2}.$$

Equivalently, we can say that our two elements are the same if

$$p_1 q_2 = p_2 q_1.$$

If we agree to have $p$ and $q$ relatively prime (i.e., the $\gcd(p, q) = 1$), then we have a unique representation of each rational number:

$$\mathbb{Q} = \left\{ \frac{p}{q} \,\middle|\, p, q \in \mathbb{Z}, q > 0, \gcd(p, q) = 1, \right\}.$$

### 3.1.2 The Reals.

The reals, though visually easy to describe as a complete line, are a bit more subtle to define. We usually think of a real number as a (possibly infinite) string of decimal places. We will only define these informally. The reals, denoted $\mathbb{R}$, are integers plus a (possibly infinite) string of decimal places, with the proviso that infinite strings of ...99999 will be rounded up to the next decimal place.

### 3.1.3 Not all Reals are Rational.

Clearly, every rational number is a real number, since we can just use long division to obtain a decimal expansion for our rational number. However, there are real numbers that are not rational; such numbers are called *irrational.* Oddly enough, there are many more irrational numbers than there are rationals. One of the easiest ways to construct irrational numbers is to take square roots of non-perfect squares. For example, we will show below that $\sqrt{2}$ is irrational. Further, it is true, though we will not show, that $\sqrt{a}$ is irrational for any $a$ which is not a perfect square (i.e., there is no $b \in \mathbb{Z}$ such that $b^2 = a$).

**Theorem.** $\sqrt{2}$ is irrational.

What follows is a proof by contradiction. These proofs are frequently employed in Mathematics. This proof assumes that the opposite of what we want to prove is true; then, using this, we logically arrive at a statement which cannot be true. Thus, we are forced to conclude that our initial assumption is false and thus our statement is proved true.

**Proof.** To obtain a contradiction, assume that $\sqrt{2}$ is rational. Then, we may write

$$\sqrt{2} = \frac{p}{q}$$

with $p, q \in \mathbb{Z}$, $q \neq 0$, and $p$ and $q$ have no common divisors. Squaring both sides, we obtain that

$$2 = \frac{p^2}{q^2}$$

and thus $p^2 = 2q^2$. Since $2q^2$ is even (having a factor of 2 in it), $p^2$ must also be even. Since $p^2 = p \cdot p$, then $p$ must be even (if $p$ was odd, then $p \cdot p = p^2$ would also be odd). Thus, $p = 2r$ for some $r \in \mathbb{Z}$. Substituting back in, we have that

$$4r^2 = (2r)^2 = p^2 = 2q^2.$$

Dividing by 2, we obtain that $q^2 = 2r^2$. As before, this tells us that $q^2$ is even and thus $q$ is even. But, now both $p$ and $q$ are even and thus both have a common divisor of 2. This, of course, contradicts our choice of $p$ and $q$ as having no common divisors. Thus, by contradiction, our initial assumption that $\sqrt{2}$ is rational is false and we must conclude instead that $\sqrt{2}$ is irrational, as desired.

## 3.2 Complex Numbers.

### 3.2.1 Motivating the Complex Numbers.

Complex numbers, despite their name, are in many ways easier to work with than real numbers. They arose from the need of many mathematicians to meaningfully define $\sqrt{-1}$; such a number cannot be contained in $\mathbb{R}$, so instead we must expand our notion of a number to the complex plane.

**Proposition.** There are no real solutions to $x^2 + 1 = 0$.

**Proof.** This will again be a proof by contradiction. Assume $x \in \mathbb{R}$ was indeed a real solution to $x^2 + 1 = 0$; then,

$$x^2 = -1.$$

Now, any real number is either zero, positive, or negative. Clearly, $x = 0$ is not a possibility since then $0 = -1$. Further, if $x$ is either positive or negative, $x^2 > 0$, but $-1 < 0$ and thus

$$-1 < 0 < x^2,$$

contradicting the fact that $x^2 = -1$. Thus, we are forced to conclude that any solution to $x^2 + 1 = 0$ cannot be a real number.

### 3.2.2 Constructing the Complex Numbers.

In order to rectify the fact that $x^2 + 1 = 0$ has no real solutions, we construct a larger space, called complex numbers and denoted by $\mathbb{C}$, that contain a solution to this equation. As a set, the complex numbers are given by

$$\mathbb{C} = \{x + iy \,|\, x, y \in \mathbb{R}\}.$$

Intuitively, $\mathbb{C}$ contains two copies of $\mathbb{R}$: the real part of a complex number $z = x + iy$ is given by $x$ (the real number not associated to $i$); this is usually written $\text{Re}(z) = a$. The imaginary part of a complex number $z = x + iy$ is given by $y$ (the real number attached to $i$); it is usually written $\text{Im}(z) = b$.

Of course, since there is a notion of addition and multiplication on $\mathbb{R}$, we should be able to give $\mathbb{C}$ an additive and multiplicative structure. These both follow from the general rule that we keep our real and imaginary parts separate and $i^2 = -1$. Thus, given two complex numbers $z = a + bi$ and $w = c + di$, addition is given by

$$z + w = (a + bi) + (c + di) = (a + b) + (c + d)i.$$

Multiplication is given by FOILing:

$$z \cdot w = (a + bi) \cdot (c + di) = ac + adi + bci + bdi^2 =$$

$$ac + adi + bci - bd = (ac - bd) + (ad + bc)i.$$

Of course, once we have multiplication, we also have a notion of division by a non-zero complex number. In particular, we find the reciprocal $\frac{1}{z}$ of $z \in \mathbb{C}$ by rationalizing the denominator:

$$\frac{1}{z} = \frac{1}{a + bi} \cdot \frac{a - bi}{a - bi} = \frac{a - bi}{a^2 + b^2} = \frac{a}{a^2 + b^2} + \frac{-b}{a^2 + b^2}\, i.$$

Of course, the denominator $a^2 + b^2$ is never zero since $z$ was assumed to not be zero (i.e, $z \neq 0 + 0i$).

### 3.2.3 The Geometry of the Complex Plane.

Since the complex numbers $\mathbb{C}$ seem to contain two copies of the real numbers $\mathbb{R}$, it seems likely that we may employ the geometry of the real plane $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ to more visually understand complex numbers.

Given any complex number $z = a + bi$, we can consider the point $(a, b) \in \mathbb{R}^2$; of course, this is the point given by finding $a$ on the $x$-axis and $b$ on the $y$-axis. By the addition rule, it is clear that complex addition

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

corresponds to vector addition in $\mathbb{R}^2$, for those of us familiar with the concept.

Given any complex number $z = a + bi$, reflecting this point across the $x$-axis corresponds to changing the sign of $b$. This operation is *complex conjugation* and is given by

$$\overline{z} = \overline{a + bi} = a - bi.$$

One geometric concept also worth exploring is distance. By the Pythagorean Theorem, the distance from the point $(a, b)$ to the origin $(0, 0)$ is given by

$$\sqrt{a^2 + b^2}.$$

Thus, we can define the *norm* (also called *modulus*) of a complex number $z = a + bi$ by

$$|z| = |a + bi| = \sqrt{a^2 + b^2}.$$

Using complex conjugation, we have a succinct way of computing the norm:

$$\sqrt{z \cdot \overline{z}} = \sqrt{(a + bi)(a - bi)} = \sqrt{a^2 + b^2} = |z|.$$

In general, if we are given two complex numbers $z, w \in \mathbb{C}$ in the complex plane, we may compute the distance between them using the modulus. Specifically, the distance between $z$ and $w$ is given by

$$\left| z - w \right|.$$

### 3.2.4 Euler's Equation.

Those of us familiar with the geometry of $\mathbb{R}^2$ know that $xy$ coordinates are not the only way to describe a point on the plane. The use of *polar coordinates* $r$ and $\theta$ are very useful in many applications. Essentially, $r$ gives the distance of our point from the origin (so $0 \leq r$) and $\theta$ gives the angle the point makes with the positive $x$-axis (so, $0 \leq \theta \leq 2\pi$).

Euler's equation uses these polar coordinates to describe our complex number $z$ in terms of the exponential number $e$. Euler's equation is given by

$$re^{i\theta} = r\cos(\theta) + ir\sin(\theta).$$

Since the complex number $1 + 0i = 1$ is distance $r = 1$ from the origin and makes an angle $\theta = 0$ with the positive $x$-axis, we can write

$$1 + 0i = 1e^{0i} = e^0.$$

Since the complex number $i = 0 + 1i$ is also distance $r = 1$ from the origin and makes an angle of $\theta = \pi/2$, we write

$$i = 1e^{i\pi/2} = e^{\frac{i\pi}{2}}.$$

Next, the number $-1 = -1 + 0i$ is distance $r = 1$ from the origin and makes an angle of $\theta = \pi$ with the $x$-axis, so is given by

$$-1 = 1e^{i\pi} = e^{i\pi}.$$

In a similar fashion,

$$-i = e^{-\frac{i\pi}{2}}.$$

It is interesting to note that Euler's formula for $-1$ gives us that

$$e^{i\pi} + 1 = 0.$$

This is particularly striking because it includes the five most important numbers in Mathematics: $0, 1, e, i$, and $\pi$.

One striking feature of Euler's Formula is that it gives us an easy way to multiply two complex numbers. Given $z = r_1 e^{i\theta_1}$ and $w = r_2 e^{i\theta_2}$, multiplying gives

$$z \cdot w = (r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = (r_1 r_2) e^{i(\theta_1 + \theta_2)}.$$

Thus, the distance of the product of two complex numbers is the product of their distances; further, the angle that the product makes with the $x$-axis is the sum of the angles of the two complex numbers.

### 3.2.5 de Moivre's Theorem

The polar expression for complex numbers has a variety of stunning applications. Since Euler's equation gives us the relation

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

If we raise this expression to the $n$-th power and use the rule that $\left(e^{i\theta}\right)^n = e^{in\theta}$, we obtain de Moirve's Theorem:

$$(\cos\theta + i\sin\theta)^n = \left(e^{i\theta}\right)^n = e^{i(n\theta)} = \cos n\theta + i\sin n\theta.$$