Chapter 2

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2.1 Mathematical Objectives and Foundations

2.1.1 Counting Elements

The area of Mathematics focusing of counting elements in a given set is known as *Combinatorics*.

Suppose we are given a set S of n elements (we say S has *cardinality* n) and we want to form a collection of k elements (with $k \leq n$). A natural question is "How many possible collections are there?" First, we need to decide if the *order* of our collection matters. For example, are we to regard the collection $\{a, b, c\}$ to be the same as $\{b, c, a\}$?

If we want order to matter, then we are interested in counting *permutations*. In this case, we are trying to fill k slots with a possibility of n elements. So, for our first choice, we have n possibilities; for our second choice, we have $n-1$ choices (since we can't use the one we chose for the first slot again). In general, if we are trying to fill the j-th slot, we will have $n-j$ choices left. Thus, the total number of permutations of length k in a set with n elements is given by

$$_nP_k = n(n-1)(n-2)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!},$$

where $n! = n(n-1)(n-2)\cdots(3)(2)(1)$.

Now, if we ask that order does not matter (e.g., we wish to consider $\{a, b, c\}$ to be the same as $\{b, c, a\}$), then we are looking for the number of *combinations* of k elements. In this case, if we were to use the number of permutations, we would be overcounting. To correct for this, we divide by the total number of possible ways of ordering k elements, which is given by $k!$. Thus, the total number of combinations is given by

$$\binom{n}{k} = \frac{_nP_k}{k!} = \frac{n!}{k!(n-k)!}.$$

Recall that we consider sets (and therefore subsets) to be unordered objects. So, if we wish to calculate the number of subsets with k elements from a set S with n elements, it makes sense to use *combinations*. Thus, the number of subsets of size k in a set S of size n is given by

$$\#\text{ Subsets of size } k = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Of course, we see with the above example of the set $T = \{a, b, c\}$ (with $n = 3$) that there is

$$\binom{3}{0} = \frac{3!}{0!3!} = 1$$

subset with 0 elements (the empty set). For $k = 1$, there are

$$\binom{3}{1} = \frac{3!}{1!2!} = 3$$

subsets of size 1 (the singletons $\{a\}, \{b\}$, and $\{c\}$). Next, there are

$$\binom{3}{2} = \frac{3!}{2!1!} = 3$$

subsets of size 2 (given by $\{a, b\}, \{a, c\}$, and $\{b, c\}$). Lastly, there is

$$\binom{3}{3} = \frac{3!}{3!0!} = 1$$

subset of order 3 (given by the entire set $\{a, b, c\}$).

2.2 Induction.

Induction is a very useful method of proof for statements. Like a line of falling dominoes, induction proves a statement about some particular n by knowing it is true for $n - 1$.

2.2.1 Motivating Induction.

In our above combinatorics computations, we wished to understand all of the subsets of a given set S. The collection of subsets of a set S is called the **power set** of S and is denoted by $\mathcal{P}(S)$. Thus, the elements of $\mathcal{P}(S)$ are subsets of the original set S. In what follows, we will find a relationship between the size of the original set S and the size of its power set $\mathcal{P}(S)$; that is, we will see how many subsets a set of size n has.

Let's say we have some statement $A(n)$ about every natural number n with $n \geq 0$. We'll keep the following example in mind. Let's say that $P(n)$ is the statement that "the cardinality of the power set $\mathcal{P}(S)$ of a set S of n elements is 2^n". Our goal is to show that the statement $A(n)$ is true for all integers $n \geq 0$ (as this statement doesn't make sense for $n < 0$ or non-integers).

The method of proof we use will be called induction and it is broken up into a few easy steps.

a) **Identify statement** $A(n)$ We have already done this in our example. This statement is what we want to prove and it should depend on the index n, where n is a natural number.

b) **Base Case.** Here, we prove that our statement A is true for our beginning case $n = 0$ (note that we can start at any n_0, but this n_0 is usually 0 or 1). In our example, $A(0)$ is the statement that "the cardinality of the power set $\mathcal{P}(\varnothing)$ of the set with zero elements (i.e., the empty set $\varnothing$) is $2^0 = 1$".

c) **The Inductive Step.** In this step, we *assume* that our statement $A(k)$ is true for some $k \geq 0$ and use this to *prove* $A(k+1)$ is true. In our example, we *assume* for some $k \geq 0$ that "the cardinality of $\mathcal{P}(S)$ of a set S with k elements is 2^k" and use this to *prove* that "the cardinality of $\mathcal{P}(T)$ of a set T with $k + 1$ elements is 2^{k+1}."

The reason why induction works as a proof is essentially the domino analogy. We imagine a line of dominoes, each representing a different $A(n)$; knocking over a domino means proving that that corresponding $A(n)$ is true. Our base case $A(0)$ means that we knock down the starting domino. Our inductive step means that since $A(0)$ is true, then $A(1)$ is true; that is, knocking down the first domino causes the second domino to fall. Using the inductive step again, since $A(1)$ is true, $A(2)$ is true as well; that is, knocking down the $A(1)$ domino will then in turn knock over the $A(2)$ domino. Continuing in this fashion, we see that eventually every domino $A(n+1)$ is is knocked over since its predecessor $A(n)$ is knocked over. Thus, we are eventually able to prove the statement $A(n)$ is true for every $n \geq 0$ (or $n \geq n_0$ for any other choice of starting point).

2.2.2 Power Set Cardinality Proof.

Here, we will give the detailed example of the power set cardinality proof. Note the structure of the proof and mimic this structure in your own proofs.

Theorem. Let S be a set of n elements. Its power set $\mathcal{P}(S)$ has cardinality 2^n.

Proof. Let $A(n)$ be the statement that the cardinality of $\mathcal{P}(S)$ of a set S with n elements is 2^n for $n \geq 0$. We will prove that $A(n)$ is true for all $n \geq 0$.

For our base case, we wish to prove $A(0)$, that the cardinality of $\mathcal{P}(\varnothing)$ is $2^0 = 1$ (as the set S with 0 elements is the empty set $\varnothing$). As there are no elements in $\varnothing$, the only subset is the entire set itself, the empty set $\varnothing$. So, $\mathcal{P}(\varnothing) = \{\varnothing\}$, and thus the cardinality of $P(\varnothing)$ is 1.

Now, we take the inductive step. So, we assume that $A(k)$ is true for some $k \geq 0$. Our goal is to show that this implies that $A(k+1)$ is true. Thus, we assume that for any set S with k elements, its power set $\mathcal{P}(S)$ has cardinality 2^k; using this, we will show that if T is a set with $k+1$ elements, then the cardinality of its power set $\mathcal{P}(T)$ is 2^{k+1}. Let a be any element in T. Since T has $k+1$ elements, we see that $T = \{a\} \cup S$, where $a \notin S$ and S is a set of cardinality k. Clearly, every subset B of T either does contain a or does not contain a. If it does not contain a, then B is a subset of S and we know that there are 2^k such subsets by the inductive assumption. If B does contain a, then it is of the form $a \cup B'$ where B' is a subset of S; thus, there are also 2^k of these subsets as there are 2^k possibilities for B'. In total, there are $2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$ subsets of T. Thus, the cardinality of the power set $\mathcal{P}(T)$ is 2^{k+1}. Thus, we have proven the inductive step.

By induction, our statement $A(n)$ is true for all $n \geq 0$. So, the cardinality of the power set $\mathcal{P}(S)$ of an n-element set S is 2^n.