Chapter 1

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1.1 Mathematical Objectives and Foundations

1.1.1 Purpose of Mathematics

Mathematics is the science of placing order on the world by establishing axioms (or *truths*) and logically deducing meaningful information from formal operations.

1.2 Set Theory

The basic object in Mathematics is the set. The purpose of this section is to acquire the basic tools for describing and manipulating sets, especially sets of real and complex numbers.

1.2.1 Definitions.

At its most basic level, a **set** is an infinite or finite collection of objects. Members of a set are referred to as **elements**. If x is an element of a set S, we write

$$x \in S.$$

The notion of a set is devoid of any ordering or any other mathematical structure; that is, the set is the most basic concept in Mathematics.

Furthermore, if we have a set S, we say that A is a **subset** of S (denoted $A \subset S$) if every element of A is also in S. Intuitively, the subset A is contained in the set S. Following are some basic examples.

Example - The Empty Set.
The **empty set** is the set which has no elements. It is denoted by $\varnothing$ or $\{\}$. The empty set is a subset of *any* set S: $\varnothing \subset S$.

Example - A finite set of numbers.
An instance of such a set is $S = \{1, 4, 56, -89, \sqrt{2}, \pi, e^2\}$. This set S is *finite* because there is only a finite number of elements. One possible subset is $A = \{1, -89, \pi\}$. Note that A is indeed a subset of S since every element of A is also in S.

Example - The Natural Numbers.
Consider the set $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ of all non-negative whole numbers. This set is called the **natural numbers**.

Example - The Integers.

Now, consider a set larger than $\mathbb{N}$ called the **integers**:

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3\}.$$

We can give an equivalent definition of $\mathbb{Z}$ as

$$\mathbb{Z} = \{x \mid |x| \in \mathbb{N}\}.$$

In general, mathematicians frequently describe sets using this notation:

$$S = \{x \,|P\},$$

where P is some statement about x and the vertical line is read as "such that". In our example, P is the statement "$|x| \in \mathbb{N}$".

1.2.2 Operations on Sets

If we are given two sets S and T, we may form their **intersection** $S \cap T$:

$$S \cap T = \{x \big| \, x \in S \text{ and } x \in T\}.$$

Thus, $S \cap T$ is the set of elements that are *both* in S and in T. Thus, the intersection of two sets is a subset of each set:

$$S \cap T \subset S \;\text{ and }\; S \cap T \subset T.$$

Similarly, we can form the **union** of two sets S and T by

$$S \cup T = \{x | \, x \in S \;\text{ or }\; x \in T\}.$$

This time, we note that S and T sit inside of $S \cup T$; thus, in subset notation, we have the following:

$$S \subset S \cup T \;\text{ and }\; T \subset S \cup T.$$

As a last operation, we may take the **complement** of a set. If we are given a subset $A \subset S$, then the complement of A in S is given by

$$\overline{A} = \{x \in S | \, x \notin A\},$$

where $x \notin A$ means x is *not* an element of A. It is important to note that we must take the complement of A *in some larger set* S. In terms of subsets and the above operations, we have

$$\overline{A} \subset S$$
$$A \cup \overline{A} = S$$
$$A \cap \overline{A} = \varnothing.$$

1.2.3 Equality of Sets.

In the previous section, we noted that $A \cup \overline{A} = S$, but we have not formally defined the equality of two sets. We say that two sets S and T are equal if $S \subset T$ and $T \subset S$. For example, it is clear that $A \cup \overline{A} \subset S$ since both A and $\overline{A}$ are subset of S, so their union is also a subset of S. On the other hand, any element $x \in S$ is either in the subset A or it is not in A; thus, $x \in A \cup \overline{A}$. Thus, by our definition, $A \cup \overline{A} = S$.

One of the main theorems of basic set theory is the following distribution law.

Theorem. - Distributive Law Let A, B, and C be sets. Then, $A\cap(B\cup C) = (A\cap B)\cup(A\cap C)$.

Proof. To prove that that $A\cap(B\cup C) = (A\cap B)\cup(A\cap C)$, we will prove the following two set inclusions:

$$A\cap(B\cup C)\subset(A\cap B)\cup(A\cap C) \text{ and } (A\cap B)\cup(A\cap C)\subset A\cap(B\cup C).$$

For the first inclusion, we assume $x\in A\cap(B\cup C)$. Then, by definition, x is in A and x is in B or C. In the first case, $x\in A\cap B$; in the second case, $x\in A\cap C$. Thus, by definition $x\in(A\cap B)\cup(A\cup B)$. Since x is any element of our starting set $A\cap(B\cup C)$, we conclude that

$$A\cap(B\cup C)\subset(A\cap B)\cup(A\cap C).$$

Next, we assume that $x\in(A\cap B)\cup(A\cap C)$. Then, by definition, x is in A and B or x is in A and C. Either way, $x\in A$; further, x will be in B or C. Thus, $x\in A\cap(B\cup C)$. Again, since x was an arbitrary element of the set $(A\cap B)\cup(A\cap C)$, we conclude that

$$(A\cap B)\cup(A\cap C)\subset A\cap(B\cup C).$$

By the definition of equality of sets, we conclude that we have our desired equality of sets:

$$A\cap(B\cup C) = (A\cap B)\cup(A\cap C).$$

1.2.4 Functions on Sets

One way of relating two sets S and T is to describe functions

$$f: S\to T$$

that assigns to every element $s\in S$ an element $f(s)\in T$. The set S is called the **domain** of the function f and T is called the **co-domain** or **target**.

There is no guarantee that every element in $t\in T$ will have some $s\in S$ that maps to it; that is, there is no guarantee that there exists some $s\in S$ such that $f(s)=t$. Another way of saying this is that we have no guarantee that t has a **preimage**. In fact, the subset of T of elements that are mapped to by f is known as the *image* of f. It has the following set-theoretic description:

$$\text{Image}(f) = \{t\in T \mid \exists\, s\in S \text{ s.t. } f(s)=t\}.$$

The image of a function is always a subset of the target set T. In the special case that the image is the entire set T (that is, that every element in T has a preimage), we say that the function is **onto** or **surjective**.

If an element $t\in T$ is in the image of f, there is also no guarantee that there is exactly one preimage. Many times, multiple elements in S are mapped to the same t. When this does not happen (that is, when every $t\in T$ has *at most* one preimage), we say that the function f is **one-to-one** or **injective**. Another way of formulating one-to-one-ness is given as follows: A function $f: S\to T$ is **one-to-one** or **injective** if whenever $f(s_1)=f(s_2)$, then $s_1=s_2$. This practical definition (i.e., the one that you would use in a proof) ensures that if you thought that multiple elements of S were mapping to the same t (i.e. $f(s_1)=f(s_2)$), then in fact these elements were all the same one element (i.e. $s_1=s_2$).

It should be clear that a function can be one-to-one but not onto and vice versa. The special functions that are both one-to-one and onto are called **bijections**. In set theory, these kinds of functions are very powerful; if there exists a bijection between S and T, then the two sets are considered to be the same. The onto-ness ensures that to every $t \in T$ is associated at least one $s \in S$; the one-to-one-ness ensures that there is at most one $s \in S$ associated to every $t \in T$. Putting these two concepts together, we see that a bijection ensures that to ever $t \in T$ there is associated *exactly* one $s \in S$.