

Chapter 16

Thursday, August 4, 2011

16.1 Springs in Motion: Hooke's Law and the Second-Order ODE

We have seen already that differential equations are powerful tools for understanding mechanics and electro-magnetism. In particular, we have used differential equations in previous workshop questions to understand how Kerchoff's Law can be used to find the current of a circuit and how to use Newton's Laws of Motion to compute terminal velocity. Today, we will be focusing on using second-order differential equations to determine the motion of a spring via *Hooke's Law*. Solving such systems will require a review of how to solve general second-order, linear, homogeneous systems with constant coefficients as well as develop a technique called *the method of undetermined coefficients* to solve the non-homogeneous equations.

16.1.1 Understanding Vibrating Springs

The physical system we will be considering is that of an object of mass m hanging on a spring, which is allowed to stretch and contract within certain reasonable limits. A similar situation occurs when instead of a hanging mass on a spring, we have a mass attached horizontally to a spring that is allowed to push and pull on the mass. We will be focusing, however, on the former situation of the hanging mass.

The goal is to obtain a function $x(t)$ such that at every time t , $x(t)$ will read how far the mass has moved (or displaced) from its resting location. Since this distance is relative, we must stress that as the mass lays hanging with no motion, it is at its equilibrium displacement and thus $x = 0$. We will be using the convention that if the mass moves x inches below this equilibrium, then x will be positive. Thus, contrary to our usual coordinate system, downward will be considered the positive direction.

When we pull the mass below its equilibrium (resting) point, we will feel a tension or force pulling upwards trying to restore the mass to its initial equilibrium. In contrast, if we push the mass above the resting point, then the spring will try to push it back to rest. Of course, if the mass is at rest, then there is no net force acting on it and it remains at rest. Using these observations, Hooke argued that this **restoring force** (the one that tries to push or pull the mass back to its resting point) is proportional to how far the mass is from its equilibrium. Thus, Hooke's law, written algebraically, gives us that

$$\text{restoring force} = -kx, \quad k > 0.$$

This equation is certainly convincing since, the further we pull our mass from rest, the stronger we feel the restoring force. We further notice that since $k > 0$, $-k < 0$ and thus the restoring force always acts in the opposite direction of our pull. Of course, this simply means that the restoring force is acting to move the mass back towards its equilibrium position. Furthermore, when the mass is at its resting location of $x = 0$, we see that there is no restoring force, and the mass will thus not move.

Newton's Laws of Motion indicate that force is equal to the product of the mass of an object and its acceleration. Of course, acceleration is simply the derivative of velocity and, in turn, velocity is the derivative of displacement or distance. Thus, since $x(t)$ is our displacement, combining Hooke's Law and Newton's Laws of Motion gives us the differential equation

$$m \frac{d^2x}{dt^2} = -kx.$$

Of course, this equation assumes that there are no other forces acting on our system. With this assumption, we may solve our differential equations using second-order methods from earlier this week.

Thus, given the differential equations $mx'' + kx = 0$, we find its corresponding characteristic equation $mx^2 + k = 0$ and thus have

$$r = \pm i \sqrt{\frac{k}{m}}.$$

Notice that since both mass m and the spring constant k are positive, these two roots are purely imaginary. The number $\sqrt{k/m} = \omega$ is called the **frequency** and has an important physical interpretation, as we will see below. So, using our previous analysis of second-order ODEs, the general form for our solution will be

$$x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t).$$

From common experience, though, we realize that the motion of a mass on a spring is relatively simple and looks like a sine or cosine wave. In the above format, though, we see that it is expressed as a linear combination of $\sin(\omega t)$ and $\cos(\omega t)$. However, after some algebraic manipulations, we see that we can indeed transform the above into the more familiar function $A \cos(\omega t + \delta)$. We do this by setting our cosine expression equal to the original general solution and use the cosine angle-sum formula:

$$A \cos(\omega t) \cos \delta - A \sin(\omega t) \sin \delta = A \cos(\omega t + \delta) = c_1 \cos(\omega t) + c_2 \sin(\omega t).$$

Using this expression, we can see that we can write A and δ in terms of the general form by having

$$A = \sqrt{c_1^2 + c_2^2} \quad \text{and} \quad \cos \delta = \frac{c_1}{A} \quad \& \quad \sin \delta = -\frac{c_2}{A}.$$

The above equations are sufficient to solve for A and δ . The quantity A is known as the **amplitude** and it measures the maximum displacement of the spring. δ is known as the **phase angle** and relates with the initial placement of the mass. A spring on a mass that undergoes such a displacement is said to be in **simple harmonic motion**.

Example. A spring with a mass of 2 kg has a natural length (resting length) of .5 m and a force of 25.6 N (N stands for Newtons and is the unit for force) is required to maintain it stretched at a length of 0.7 m. If the spring is stretched

to a length of 0.7 m and then released with initial velocity 0, find the position of the mass $x(t)$ at time t .

Solution. We may use Hooke's law to find the spring constant k . Since the mass is at rest naturally at .5m and must be pulled downward (in the positive direction) an extra $x = 0.2$ m, then Hooke's law says that

$$25.6 = k(0.2)$$

and thus $k = 128$ N/m. Thus, we may establish our differential equation to be

$$2x'' + 128x = 0.$$

The further information about the initial state of the spring gives us the initial conditions that $x(0) = .2$ (since it started at length 0.7 m) and that $x'(0) = 0$ (since there was no initial velocity). We may also calculate the frequency $\omega = \sqrt{k/m} = \sqrt{128/2} = 8$. Using the above discussion, we see that the general form for the solution is given by

$$x(t) = c_1 \cos 8t + c_2 \sin 8t.$$

We may then plug our initial values into $x(t)$ and $x'(t)$ to obtain our coefficients $c_1 = .2$ and $c_2 = 0$. Thus, the location of our mass at time t is given explicitly by

$$x(t) = .2 \cos(8t).$$

Notice that, since $c_2 = 0$, we did not have to transform this into a single cosine term since it came to us in this form.

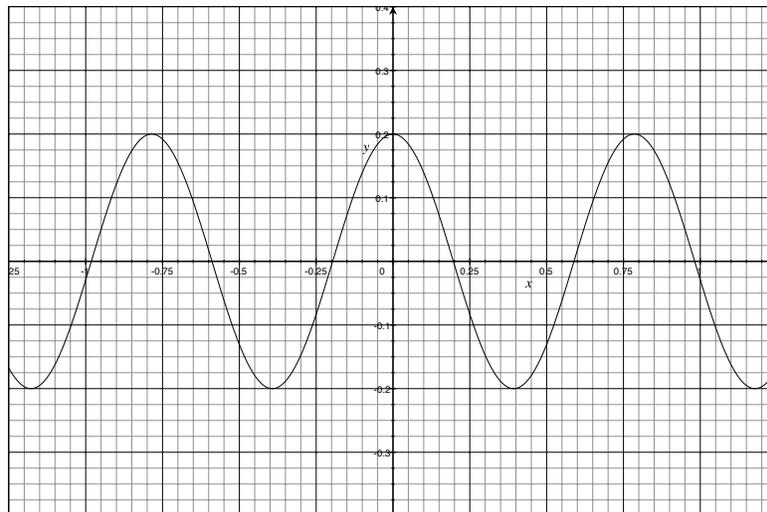


Figure 16.1: The graph of the displacement $x(t)$ for simple harmonic motion

16.1.2 Dampened Vibrations

A more realistic situation that we have encountered in the past when dealing with springs is that they do eventually come to rest after oscillating with decreasing amplitude. Of course, the above situation of simple harmonic motion indicates that the spring will never come to rest and will continue to oscillate with constant amplitude A . The above situation, however, neglected to take

into consideration any kind of damping or frictional force that will eventually slow the spring down. As with our computation of terminal velocity in a previous workshop, we will assume that this damping force is proportional to the *velocity* of the spring itself. Thus, we have

$$\text{damping force} = -c \frac{dx}{dt},$$

where c is once again assumed to be a positive constant. As with the restoring force, we see that the damping force acts in the opposite direction of displacement and thus helps the mass return to its equilibrium.

If we used Newton's Law of Motion again, but this time included not just the restoring force but also the damping force, then we obtain the new differential equation

$$m \frac{d^2x}{dt^2} = -kx - c \frac{dx}{dt}.$$

Re-writing we obtain the differential equation

$$mx'' + cx' + kx = 0.$$

It is precisely this cx' term that will make our new equation more complicated but will force it to account for the dampening.

To solve the above system as before, we must first find the roots of the characteristic polynomial $mx^2 + cx + k$. Using the quadratic equation, we see that our roots are

$$r_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}.$$

Thus, we must discuss the three possibilities: two distinct, real roots; one repeated, real root; and two distinct (conjugate) complex roots.

16.1.3 Overdampening - the case of two real, distinct roots

Our spring system is said to be *overdampened* if the roots to the characteristic polynomial are distinct and real. For this to be true, it must be case that $c^2 - 4mk > 0$ so that the square root portion of the quadratic equation yields two different, real answers. Furthermore, since c, m , and k are all positive constants, we have that

$$\sqrt{c^2 - 4mk} < \sqrt{c^2} = c$$

and thus our roots must all be *negative*.

So, we know that in general the form for the displacement will be given by

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

Since our two roots are negative, the exponential terms each tend to 0 and thus $\lim_{t \rightarrow 0} x(t) = 0$ and eventually our system will again be at rest. In these cases, our mass may cross the equilibrium one more time at most, but then quickly return to rest.

16.1.4 Critical dampening - the case of one repeated, real root

Our system is called *critically dampened* if it still acts like the overdamped case, but serves as the bordering situation. This occurs when we begin to

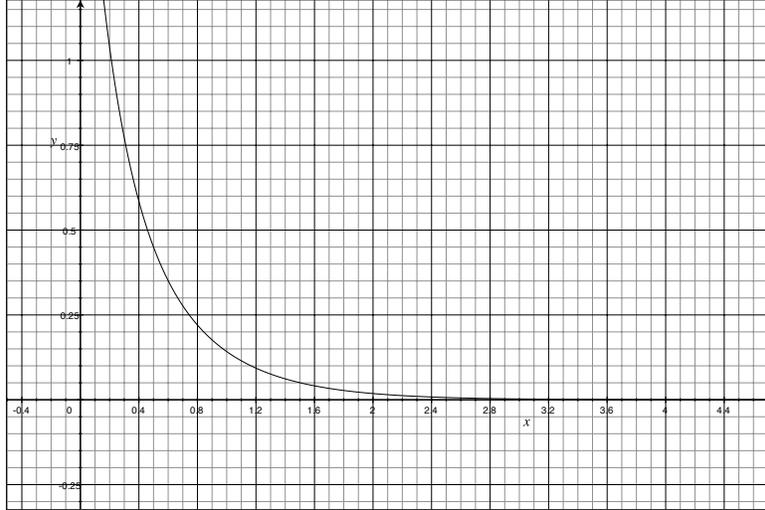


Figure 16.2: An overdamped system

transition from real to complex roots. This intermediate stage manifests in one repeated, real root and occurs when we have the following algebraic equation:

$$c^2 - 4mk = 0.$$

In this case, the square root portion of the quadratic equation is zero we have one real root with multiplicity two. Furthermore, this one root will be precisely

$$r_1 = r_2 = \frac{-c}{2m},$$

which is negative since both c and m are positive.

The general solution for this comes of the form

$$x(t) = c_1 e^{rt} + c_2 t e^{rt}.$$

Clearly, the first term will tend to zero as $t \rightarrow \infty$ since our root r is negative. Also, since an exponential with negative exponent tends to zero faster than a linear t tends towards infinity the second term involving $t e^{rt}$ will also tend towards zero. Thus, in this critical damping case, we still have a quick descent towards equilibrium.

16.1.5 Underdampening - the case of two, complex conjugate roots

The final case, called *underdampening* occurs when there is still some residual oscillation, but our system continues to tend towards rest. This situation occurs when we have the algebraic inequality: $c^2 - 4mk < 0$. Since this discriminant will be negative, taking a square root will result in the appearance of an imaginary part and our roots will thus be complex. They are conjugate precisely because the quadratic equation says one can obtain one root from the other by adding or subtracting the imaginary part.

In this complex case, we have the situation where our roots are given by

$$r_{1,2} = \frac{-c}{2m} \pm \omega i,$$

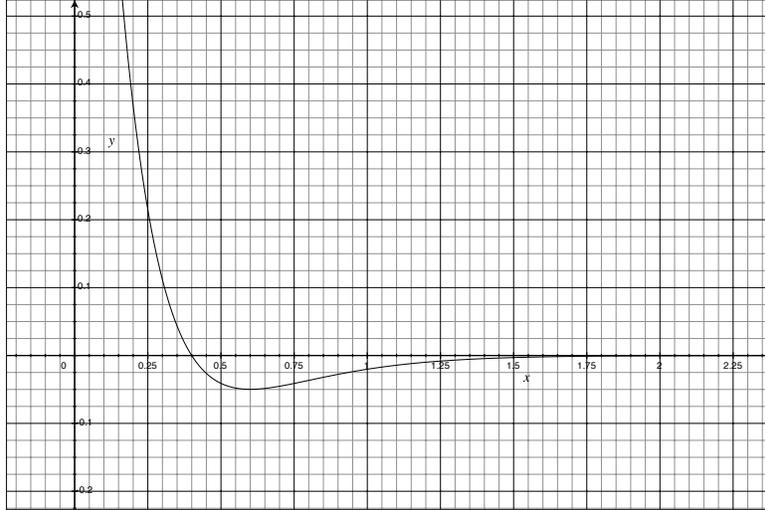


Figure 16.3: An critically dampened system

where

$$\omega = \frac{\sqrt{4mk - c^2}}{2m}.$$

Notice that this is the same ω as in our undamped example because, if $c = 0$ (and thus no damping is taking place), we have that $\omega = \sqrt{k/m}$. Having these roots, we can find the general form to our system to be

$$x(t) = e^{-ct/2m} (c_1 \cos \omega t + c_2 \sin \omega t).$$

The cosine and sine terms indicate that our system will continue to oscillate, while the exponential term will take the entire system back to its equilibrium of $x = 0$.

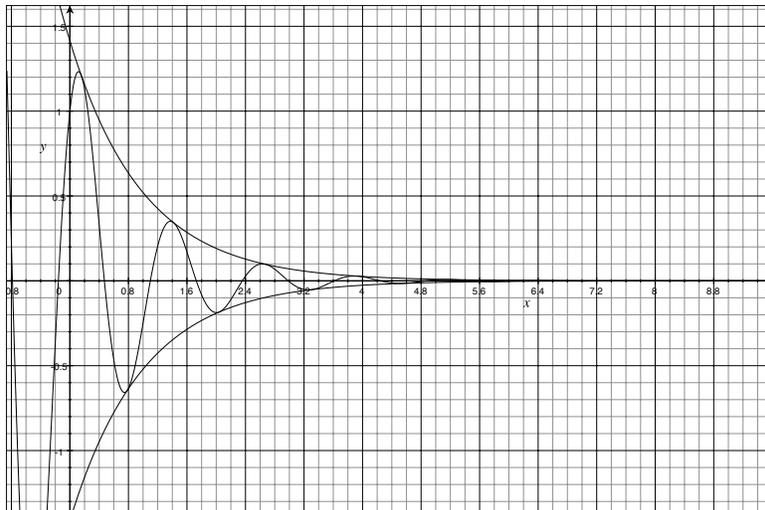


Figure 16.4: An underdamped system bounded by two exponential functions

Note: The motivation for the lecture was obtained from a very well-written section on “Applications of Second-Order Differential Equations” in Stewart’s *Calculus* text.