Chapter 15

Wednesday, August 3, 2011

15.1 Parametric Equations

Usually, when describing the a curve in $\mathbb{R}^2$, it is given by some function $f(x)$ and is sketched on the xy-plane by plotting all pairs of points $(x, f(x))$. However, since these curves are described by functions, their shapes are very constrained. One key feature of f being a function is that it must pass what is frequently called the *vertical line test*, which simply means that above every input value x there is at most one value $f(x)$ associated to it. Graphically, this means that for any vertical line we draw, we must hit at most one point on the curve. This limited way to describe curves means that we cannot, for example, use a function to describe the circle of radius 1 about the origin since it will certainly fail the vertical line test. However, if we use parametric equations instead of functions to descibe our curves, we see that we may succintly descibe several other kinds of curves (including the circle).

15.1.1 Representing Curves as Parametric Equations

A ***parametic equation*** for a curve on the xy-plane is a way to describe how the two coordinates x and y change as a function of a time parameter t. In other words, at any point parametric equation in the xy-plane is given by a 2-tuple of functions in the variable t:

$$(x(t), y(t)).$$

Of course, similar constructions are available in $\mathbb{R}^n$ for any n; in general, these parametric equations will trace out a one-dimensional curve in $\mathbb{R}^n$; we will be focusing on parametric equations in $\mathbb{R}^2$, however.

With this in mind, let us return to our example of the circle. Assume that we want to begin at the point $(1, 0)$ at time $t = 0$ and proceed counterclockwise around the circle of radius 1. Since our circle is radius 1, for every time t, it must be true that our point $(x(t), y(t))$ is distance 1 from the origin and thus (by the Pythagoream Theorem)

$$x(t)^2 + y(t)^2 = 1.$$

Thus, we must find functions $x(t)$ and $y(t)$ that satisfy this equation for all t. Our trigonometric experience tells us that $\sin t$ and $\cos t$ are good candidates. To choose which one will be x and y, we recall that we indicated that at time $t = 0$, we should have that $(x(0), y(0)) = (1, 0)$. Thus, our parametric equation becomes

$$(x(t), y(t)) = (\cos t, \sin t).$$

Checking several values, we see that this curve traces out a circle in a counter-clockwise manner at t increases. In fact, at $t = 2\pi$, we see that we return to $(1, 0)$ and continue the process over. Thus, we may simply constrain ourselves to using $0 \leq t \leq 2\pi$.

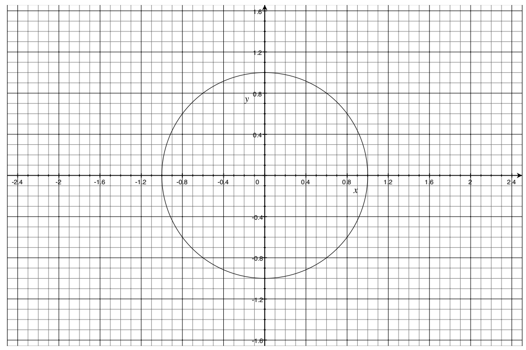


Figure 15.1: The circle as a parametric curve

In a similar spirit, we may construct numerous beautifully complicated curves in the xy-plane by expressing them as parametric equations. One other example is the curve is sketched below and is given by the parametric equation

$$(x(t), y(t)) = (3 \sin 5t, 3 \cos 3t).$$

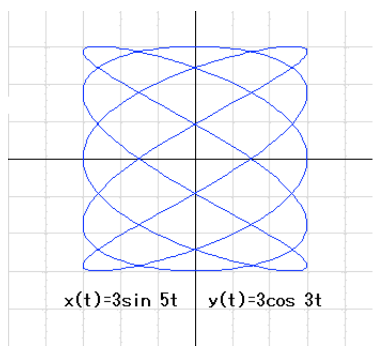


Figure 15.2: A parametric curve

15.1.2 Functions as Parametric Equations

One important feature of parametric equations is that they are a more general way of drawing curves than simply using a function $f(x)$; in other words, we may represent any curve that comes from a function $f(x)$ by turning it into a parametric equation. To do this, we notice that these functional curves come from plotting points $(x, f(x))$. Thus, to turn this into a parametric equation, we simply let $x(t) = t$ and then $y(t) = f(t)$. Thus, any curve coming from a function may be represented by

$$(x(t), y(t)) = (t, f(t)).$$

15.2 Systems of Differential Equations

Since parametric equations allow us to generalize curve drawing from simply functions to more exotic shapes, we wish to also broaden our understanding of differential equations to provide a method for describing parametric equations via differential equations. Since parametric equations require understanding two different functions $x(t)$ and $y(t)$, we need a ***system of differential equations*** to describe their behavior using derivatives. We will be restricting ourselves to systems of first-order differential equations. Thus, our systems will take on the from

$$\frac{dx}{dt} = f(t, x, y)$$
$$\frac{dy}{dt} = g(t, x, y)$$

As usual, both f and g are simply expressions that use only the parameter t and the two coordinate variables x and y.

15.2.1 Linear Systems of Differential Equations

Of particular importance are those systems of equations that are *linear*. We call a system ***linear*** if each both $f(t, x, y)$ and $g(t, x, y)$ are linear functions. Thus, we may write any such system of differential equations as

$$x'(t) = a_{11}(t)x + a_{12}(t)y + g_1(t)$$
$$y'(t) = a_{21}(t)x + a_{22}(t)y + g_2(t)$$

Another important distinction to consider is when our system is *homogeneous*. As above, a system is ***homogeneous*** if both equations have $g_i(t) = 0$. In this case, our system reduces to

$$x'(t) = a_{11}(t)x + a_{12}(t)y$$
$$y'(t) = a_{21}(t)x + a_{22}(t)y$$

Even simpler is when our system is further constrained by asking that the coefficient functions $a_{ij}(t)$ be constant functions. In this case, we may simply write our system as

$$x'(t) = ax + by$$
$$y'(t) = cx + dy$$

When we are dealing with a system of first-order, linear, homogeneous differential equations, the particularly simply form begs us to utilize our knowledge of linear algebra to re-write the above system in the following matrix equation:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

To solve such a system, we must *simultaneously* solve for the functions $x(t)$ and $y(t)$; in other words, when we plug x and y into our system, both equations must be satisfied. Since our differential equations have a particularly simple form, we (as with second-order, linear, homogeneous equations with constant coefficients) assume that x and y have the form $x(t) = x_0 e^{\lambda t}$ and $y(t) = y_0 e^{\lambda t}$. Thus, our solution vector will be of the form

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_0 e^{\lambda t} \\ y_0 e^{\lambda t} \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} e^{\lambda t}.$$

If we take derivatives of both x and y, we are left with

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x_0\lambda e^{\lambda t} \\ y_0\lambda e^{\lambda t} \end{bmatrix} = \lambda \begin{bmatrix} x_0 e^{\lambda t} \\ y_0 e^{\lambda t} \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}.$$

However, from the above formulation of our system, we also have that the derivatives are equal to our coefficient matrix times the vector $[x, y]$. Thus, we are left with the matrix equation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}.$$

This is precisely the eigenvalue equation for the coefficient matrix with eigenvector $[x, y]$ and eigenvalue λ.

Since finding a solution to our system has reduced to finding eigenvalues and eigenspace for the coefficient matrix A, we must review the possibilities for our eigenvalues. Recall that to compute eigenvalues of a square matrix A, we must find the roots to the characteristic polynomial equation

$$\det(A - \lambda I) = 0.$$

Since the characteristic polynomial $\det(A - \lambda I)$ is a real quadratic polynomial, there are exactly three cases we must consider: distinct, real eigenvalues; a repeated, real eigenvalue; and two complex conjugate eigenvalues. For simplicity sake, we will only be investigating those systems with two distinct, real eigenvalues.

15.2.2 Distinct, Real Eigenvalues

The simplest case is when we have two distinct eigenvalues $\lambda_1 \neq \lambda_2$. Of course, each eigenvalue has its corresponding eigenvectors (forming their corresponding eigenspaces). Let us assume that we have already found an eigenvector $[x_1, y_1]$ for each eigenvalue λ_1. Given this linear-algebraic information, we understand that one set of solutions to the system of equations is given by

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_1 e^{\lambda t} \\ y_1 e^{\lambda t} \end{bmatrix}.$$

Thus, the parametric equation $(x(t), y(t)) = (x_1 e^{\lambda t}, y_1 e^{\lambda t})$ gives a solution to our system of equations. While it seems that sketching such a parametric equation may be difficult, we will instead use our understanding of linear algebra to sketch this parametric equation.

Since $x(t) = x_1 e^{\lambda_1 t}$ and $y(t) = y_1 e^{\lambda_1 t}$, for any t, $[x(t), y(t)]$ will be a multiple of the eigenvector $[x_1, y_1]$ and thus remain on the line spanned by this eigenvector. Thus, as t changes, $(x(t), y(t))$ will travel along this line; however the direction in which it travels is dictated by the sign of λ_1. Note that if $\lambda > 0$, then as t grows posititvely $(x(t), y(t))$ will move away from the origin in the direction of $[x_1, y_1]$. As t grows negatively, then $(x(t), y(t))$ will start to approach the origin since both the x and y coordinates are tending towards zero. In contrast, if $\lambda < 0$, we have the exact opposite case: as t grows positive, $(x(t), y(t))$ tends towards the origin while as t grows negative, it moves away from the origin along $[x_1, y_1]$. In the unique case that we have an eigenvector of $\lambda_1 = 0$, we see a completely different phenomenon. In this situation, our solution is given by

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

for *all values of* t. Thus, the points in the eigenspace corresponding to λ_1 are *stationary* or ***equilibrium points***.

Of course, since the second eigenvalue λ_2 is distinct from λ_1, their corresponding eigenspaces do not overlap (except, of course, for the zero vector $\mathbf{0}$). As with λ_1, we can find a solution to our solution of the form

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} x_2 e^{\lambda_2 t} \\ y_2 e^{\lambda_2 t} \end{bmatrix},$$

where $[x_2, y_2]$ is an eigenvector for the eigenvalue λ_2. Similar to the analysis above, the sign of λ_2 will tell us about the dynamics of the system of equations on the line spanned by $[x_2, y_2]$.

One major advantage of finding these solutions via eigenvectors and eigenvalues is that these form a basis for the solution space. Since our system is homogeneous, the space of solutions will, as usual, form a vector space. Thus, any solution to our system is written as a linear combination of the above two solutions. Thus, in general, a solution to our system of equations is given by

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} e^{\lambda_1 t} + c_2 \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} e^{\lambda_2 t}.$$

Another major advantage of analyzing this system of equation in terms of eigenvalues is that it gives us information about how objects flow around the xy-plane. In particular, if both λ_1 and λ_2 are positive, we have that solutions along the eigenvectors flow away from the origin. In particular, we have that if we start at some point on the plane, it will eventually flow away from the origin; when such a situation occurs, the origin is called a ***source***. Conversely, if both eigenvalues are negative, then eventually every point will flow towards the origin and it is known as a ***sink***. However, when one eigenvalue is postiive and the other is negative, then we have a situation where some solutions begin to tend towards the origin but, as they approach it, suddenly turn away. When this occurs, the origin is known as a ***saddle***. The final case is when one of the eigenvalues is zero. As mentioned above, this gives us a line of solutions that are stationary (called equilibria). Points not on this stationary line will flow either towards or away from this stationary line; they will flow towards this line if the other eigenvalue is negative and away if the eigenvalue is positive.

15.2.3 An example.

To make our above discussion concrete, let us consider the following system of equations:

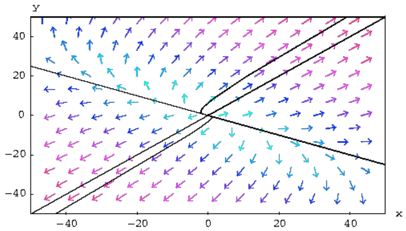
$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}.$$

If we call the above coefficient matrix A, then we must first find the eigenvalues and corresponding eigenspaces of A. The characteristic polynomial for the matrix A is

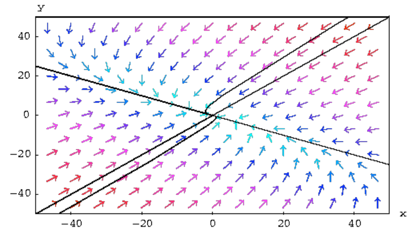
$$\lambda^2 - \lambda - 6 = (\lambda - 3)(\lambda + 2).$$

Thus, we have eigenvalues $\lambda_1 = -2, \lambda_2 = 3$. A quick calculation shows that the corresponding eigenspaces are given by

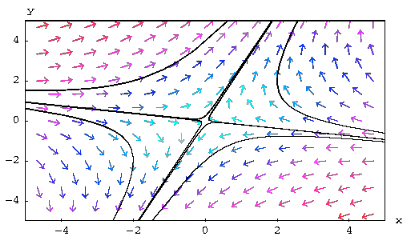
$$E_{-2}(A) = c \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad E_3(A) = c \begin{bmatrix} 4 \\ -1 \end{bmatrix}.$$



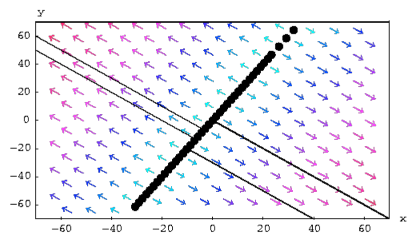
(a) A source: $\lambda_1, \lambda_2 > 0$



(b) A sink: $\lambda_1, \lambda_2 < 0$



(c) A saddle: $\lambda_1 < 0 < \lambda_2$



(d) $\lambda_1 = 0, \lambda_2 > 0$

Figure 15.3: The Dynamics of a System of Equations with Distinct, Real Roots

Thus, the general solution for our equation will be of the form

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 4 \\ -1 \end{bmatrix} e^{3t}.$$

To draw the corresponding ***phase portrait***, must first sketch the eigenspaces we found. Around the eigenspace corresponding the eigenvalue $\lambda_1 = -2$ (which is spanned by $[1, 1]$), our vectors will all be pointing towards the origin (because $-2 < 0$). Conversely, for the eigenspace corrsponding to the eigenvalue $\lambda_2 = 3$ (which is spanned by $[4, -1]$), the vectors near this line will be pointing away from the origin (since $3 > 0$.) Thus, our phase portrait has the form given in the diagram below.

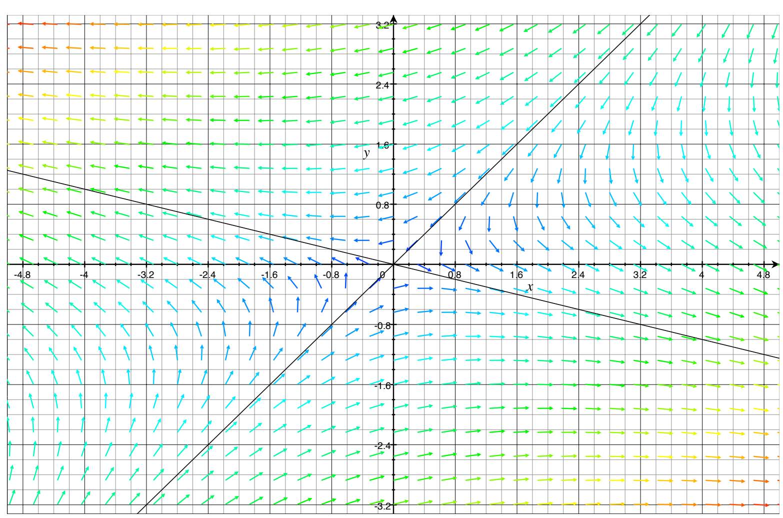


Figure 15.4: The phase portrait for our system of equations.

We may immediately see that, as expected, the origin is a saddle point. The dynamics are such that being on one side or the other of the eigenlines will drastically affect the long-term behavior of a solution. These *unstable equilibria* are of crucial important to chaos theory and dynamical systems. We are not surprised, however, that we do obtain a saddle point since our eigenvalues come with opposite signs.

Of course, the above system gives us infinitely many solutions (corresponding to our choices of c_1 and c_2). To pin down a particular solution, consider the initial values given by $x(0) = 1$ and $y(0) = 0$. Inputting these values into our general form will give us the explicit form for the solution that passes through the point $(1, 0)$. Plugging in, we obtain the system of equations

$$\begin{matrix} c_1 + 4c_2 = 1 \\ c_1 - c_2 = 0 \end{matrix} .$$

Solving, we obtain $c_1 = c_2 = 1/5$. Thus, our particular solution has the form

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + \frac{1}{5} \begin{bmatrix} 4 \\ -1 \end{bmatrix} e^{3t} = \begin{bmatrix} .2e^{-2t} + .8e^{3t} \\ .2e^{-2t} - .2e^{3t} \end{bmatrix} .$$

Plotting this particular solution on our phase portrait, we obtain the following diagram.

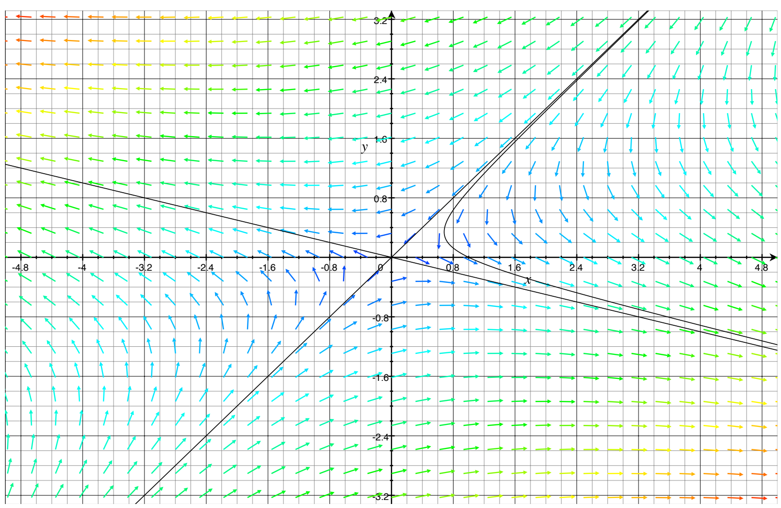


Figure 15.5: A particular solution to our system of differential equations