Chapter 14

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14.1 Exact Equations

Consider the first order differential equation given by

$$M(x,y) + N(x,y)y' = 0$$

and suppose that we can find some function $\Psi(x,y)$ such that

$$\frac{\partial \Psi}{\partial x} = M(x,y) \text{ and } \frac{\partial \Psi}{\partial y} = N(x,y),$$

where the above derivatives are *partial derivatives*. First, we note that partial derivatives are almost synonymous with ordinary derivatives, except that we must denote which variable we are taking the derivative with respect to; knowing this, we treat all other variables as constants and then just differentiate with respect to just one variable. Thus, after finding this function $\Psi(x,y)$ (called the ***potential function***), we can re-write our equation as

$$\frac{\partial \Psi}{\partial x} + \frac{\partial \Psi}{\partial y}\frac{dy}{dx} = 0.$$

We note that by the chain rule:

$$\frac{d\Psi}{dx} = \frac{\partial \Psi}{\partial x}\frac{dx}{dx} + \frac{\partial \Psi}{\partial y}\frac{dy}{dx} = \frac{\partial \Psi}{\partial x} + \frac{\partial \Psi}{\partial y}\frac{dy}{dx}.$$

Thus, our differential equation reduces even more to

$$\frac{d}{dx}\left(\Psi(x,y)\right) = 0.$$

When we can find this function $\Psi(x,y)$, we say that our differential equation is ***exact***. These exact equations are then easily integrable once Ψ is identified and our solution is given *implicitly* by the equation

$$\Psi(x,y) = c.$$

Example

Consider the differential equation

$$2x + y^2 + 2xyy' = 0.$$

We wish to find the function $\Psi(x, y)$ such that the following two equations hold:

$$\frac{\partial \Psi}{\partial x} = M(x, y) = 2x + y^2$$

$$\frac{\partial \Psi}{\partial y} = N(x, y) = 2xy.$$

Consider the two variable function

$$\Psi(x, y) = x^2 + xy^2.$$

Clearly, its two partial derivatives agree with M and N. Thus, our analysis above tells us that our solution y satisfies the implicit equation

$$x^2 + xy^2 = c.$$

Of course, we can solve here by algebraically manipulating to obtain the explicit form for our solution:

$$y = \sqrt{\frac{c}{x} - x}.$$

14.2 Second Order Linear Equations

With the introduction of a higher order derivative $\frac{d^2y}{dt^2}$, the difficulty in solving for y greatly augments. These second order differential equations are those which can be written in the form

$$\frac{d^2y}{dt^2} = F\left(t, y, \frac{dy}{dt}\right).$$

14.2.1 Describing Second Order Differential Equations

By far, the most important subclass of Second Order Differential Equations are those which are *linear*. These are those differential equations of order 2 that can be written as

$$y'' + p(t)y' + q(t)y = g(t).$$

Thus, this equation is linear because it is linear in the 0-th, 1-st, and 2-nd derivatives y, y', and y''. Any second order differential equation that cannot be written in this way is called *nonlinear*. Note that if we are presented with a second order differential equation where y'' has some function as a coefficient, we may simply divide through by this function to obtain one of the form above.

The second most important distinction is that between *homogenous* and *non-homogenous* equations. A homogenous equation is one where the t-dependent term $g(t)$ is equal to zero; thus a second order linear homogenous differential equation is given by

$$y'' + p(t)y' + q(t)y = 0$$

14.2.2 Second Order Linear Homogenous Differential Equations with Constant Terms

In general, second order linear homogeneous ODEs are difficult to solve. So, we restrict ourselves to the case when the coefficient functions are all constant. So, let us consider the second order homogenous differential equation given by

$$ay'' + by' + cy = 0.$$

Since the solution y appears with all its derivatives in such a simple form, it is reasonable that y would have the term e^{rt} and some of its variants. For now, let us assume that $y = e^{rt}$ and let us plug this into our differential equation:

$$ar^2e^{rt} + bre^{rt} + ce^{rt} = 0.$$

Of course, each of these terms has a common e^{rt}, so we can thus factor to obtain

$$(ar^2 + br + c)e^{rt} = 0.$$

Note that since $e^{rt} > 0$, the left hand side is zero if and only if the quadratic polynomial $ar^2 + br + c = 0$ (called the ***characteristic polynomial***). Thus, we can find the appropriate r using the quadratic equation.

Now that we have our characteristic polynomial and have a way to find the two roots r, we need to make sense of what they mean. First, we note that the homogenous nature of this equation makes the solution space a *vector space*. Clearly, if y_1 and y_2 are two solutions to this differential equation, then so is $y_1 + y_2$. Further, if y is a solution, then αy is a solution for any $\alpha \in \mathbb{R}$. Thus, the solution space of these second order linear homogenous equations is a vector space. Thus, we must find a basis for this vector space, and this is exactly what the two roots of our characteristic polynomial will give us.

Of course, we must also make sense of what happens when our roots are not real. If they are not real, then we must make sense of what a complex root will correspond to; we will come to this soon.

14.2.3 Two Real distinct roots

For now, we will consider the case when our characteristic polynomial has *two distinct real roots* $r_1, r_2 \in \mathbb{R}$. In this case, consider the two solutions $y_1 = e^{r_1 t}, y_2 = e^{r_2 t}$. Clearly, these two are actually solutions. As noted before, since our solution space is a vector space, any linear combination of y_1 and y_2 is also a solution to our differential equation. Thus, the solutions to our differential equation take the form

$$y(t) = c_1e^{r_1 t} + c_2e^{r_2 t}.$$

Note that since our roots are distinct, this linear combination is non-trivial in some sense. It turns out that these are all possible solutions to this differential equation. If we wanted to pin down one particular solution, we would have to be able to solve for c_1 and c_2; to be able to do this, we must have two initial conditions about y and y'.

Example. Consider the initial value problem

$$y'' + 5y' + 6y = 0$$

with initial conditions $y(0) = 2$ and $y'(0) = 2$. First, we solve the general differential equation by considering the characteristic polynomial

$$r^2 + 5r + 6 = (r + 3)(r + 2).$$

Thus, the two distinct real solutions to this characteristic polynomial are $r = -2, -3$. Thus, all of our solutions will be of the form $y(t) = c_1e^{-2t} + c_2e^{-3t}$. To find our particular solution to this initial value problem, we use $y(0) = 2$ and $y'(0) = 2$. Using the first value, we have that

$$2 = c_1 + c_2.$$

Taking a derivative of y and using $y'(0) = 2$, we have that

$$2 = -2c_1 - 3c_2.$$

Solving this system of two equations, we see that $c_1 = 8$ and $c_2 = -6$ and thus the solution that satisfies our initial value problem is

$$y(t) = 8e^{-2t} - 6e^{-3t}.$$

14.2.4 Two Real Repeated Roots

In the case when our characteristic polynomial has two real repeated roots, then the general solution found above

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

is redundant because these exponential terms are no longer linearly independent. To fix this problem, we must add a factor of t to one of these terms. Thus, the general solution to a second order linear homogenous equation with repeated root real root r is given as follows:

$$y(t) = c_1 e^{rt} + c_2 t e^{rt}.$$

Example. Consider the initial value problem with differential equation given by

$$y'' - 4y' + 4y = 0$$

and initial values $y(0) = 1$, $y'(0) = 0$. Then the characteristic polynomial of this equation is given by

$$r^2 - 4r + 4 = (r - 2)^2.$$

Of course, there is one double root $r = 2$. Thus, our general solution will be of the form

$$y(t) = c_1 e^{2t} + c_2 t e^{2t}.$$

Using our first initial condition $y(0) = 1$, we immediately see that $c_1 = 1$. If we differentiate our solution y, we have that

$$y'(t) = 2c_1 e^{2t} + c_2 e^{2t} + 2c_2 t e^{2t}$$

and thus that $c_2 = -2$. Thus, our solution to our initial value problem is given by

$$y(t) = e^{2t} - 2te^{2t}.$$

14.2.5 Two Complex Roots

If at least one of our roots is not real, then the other roots must also be non-real; this follows from the fact that any polynomial with coefficients in $\mathbb{R}$ will have solutions coming in pairs given by complex conjugation. Of course, this does not apply when we have one real root because the complex conjugate of a real number is itself. Thus, if our characteristic equation has the complex root $r_1 = \gamma + i\mu$, then the other root must be $r_2 = \overline{r_1} = \gamma - i\mu$.

Now, we must make sense of $e^{(\gamma+i\mu)t}$. First, we use the exponential law to write

$$e^{(\gamma+i\mu)t} = e^{\gamma t} \cdot e^{i\mu t}.$$

This first factor is a real number and thus makes sense in the context of finding solutions to our differential equation; however, we must employ some complex analysis to obtain more information about the second factor. Recall that

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Thus, our solution becomes

$$e^{(\gamma+i\mu)t} = e^{\gamma t}(\cos\mu t + i\sin\mu t).$$

Of course, this is disconcerting because we are looking for real function solutions. Recall that when two functions are solutions of a homogenous equation, so is their sum (and therefore their difference). Thus, the sum of the expressions coming from the two roots (which differ by complex conjugation) is given as follows:

$$e^{(\gamma+i\mu)t} + e^{(\gamma-i\mu)t} = e^{\gamma t}(\cos\mu t + i\sin\mu t) + e^{\gamma t}(\cos\mu t - i\sin\mu t) = 2e^{\gamma t}\cos\mu t.$$

Taking their difference, we obtain a different result:

$$e^{(\gamma+i\mu)t} - e^{(\gamma-i\mu)t} = 2ie^{\gamma t}\sin\mu t.$$

From this, we can deduce that if the characteristic equation of a second order linear homogenous equation has complex roots $\gamma \pm i\mu$, then the solutions will be of the form

$$y(t) = c_1 e^{\gamma t}\cos\mu t + c_2 e^{\gamma t}\sin\mu t.$$

Example. Consider the differential equation

$$y'' + y' + y = 0.$$

This has characteristic equation $r^2 + r + 1$ and has complex roots

$$r = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}.$$

Thus, the general solution to this differential equation is given by

$$y(t) = c_1 e^{-t/2}\cos\left(\sqrt{3}t/2\right) + c_2 e^{-t/2}\sin\left(\sqrt{3}t/2\right).$$

Example. Here is an example where the real part of the complex roots are 0. Consider the differential equation

$$y'' + 9y = 0.$$

The characteristic equation for this is $r^2 + 9$ and has complex roots $r = \pm 3i$. Thus, the general solution to this differential equation is given by

$$y(t) = c_1\cos 3t + c_2\sin 3t.$$