FSRI Mathematics 2011
A Note on Convergence

In class and in the workshop, we saw that the convergence of a sequence a_n was phrased in a "for all - there exists" statement. Logically, these statements can be a bit tricky, so this note will elucidate this very important concept.

Recall that a sequence a_n converges to A (written $a_n \to A$) if for all $\varepsilon > 0$, there exists an N such that for all $n > N$, the the following inequality holds:

$$|a_n - A| < \varepsilon.$$

In proofs, we are frequently in one of two scenarios when it comes to using convergence:

(1) **If we want to show that** $a_n \to A$, then we are given a $\varepsilon > 0$ (not of our choosing), and our job is to choose an N and show that for all $n > N$, the inequality $|a_n - A| < \varepsilon$ holds.

(2) **If we know that** $a_n \to A$, then for any $\varepsilon > 0$ *of our choosing*, we are given back an N and we know that for all $n > N$, the inequality $|a_n - A| < \varepsilon$ holds.

In many cases, we must prove that a certain sequence converges knowing that some other sequence converges. Thus, we must keep straight what we can choose and what we cannot. In the below two proofs, we will utilize the above two observations.

Theorem. Show that if $a_n \to A$, then $c \cdot a_n \to c \cdot A$.

Discussion. Here, we know that $a_n \to A$ and we wish to show that $ca_n \to cA$.
`We want to show:` $ca_n \to cA$. Thus, we will start our proof with "Let $\varepsilon > 0$" (which is not of our choosing). Our job is to find an N such that for all $n > N$, the inequality $|ca_n - cA| < \varepsilon$.

`We know:` $a_n \to A$. Thus, for any $\varepsilon' > 0$ of our choosing, we will be automatically given back an N' so that whenever $n > N'$, then $|a_n - A| < \varepsilon'$.

`What we'll do:` Since we are free to choose any $\varepsilon' > 0$, let's choose $\varepsilon' = \frac{\varepsilon}{|c|}$. We will be given back an N' so that for all $n > N'$, the inequality $|a_n - A| < \varepsilon' = \frac{\varepsilon}{|c|}$ holds. So, the N that we choose will be this N' (thus we will let $N = N'$). We will show that for all $n > N = N'$, then inequality $|ca_n - cA| < \varepsilon$.

Note: We will skip the situation where $c = 0$, which is the case when we have the constant 0 sequence converging to 0].

Proof. Given $\varepsilon > 0$, we will find an N such that for all $n > N$, $|ca_n - cA| < \varepsilon$. Since $a_n \to A$, then for $\varepsilon' = \frac{\varepsilon}{|c|} > 0$, there exist an N' such that for all $n > N'$, the inequality $|a_n - A| < \varepsilon' = \frac{\varepsilon}{|c|}$ is true. Let $N = N'$ Thus, for all $n > N = N'$, we have that

$$|a_n - A| < \frac{\varepsilon}{|c|}.$$

Multiplying by $|c| > 0$ and distributing inside the absolute value gives $|c(a_n - A)| < \varepsilon$ and thus $|ca_n - cA| < \varepsilon$, as desired. Thus, $c \cdot a_n \to c \cdot A$, as desired.

Theorem. If $a_n \to A$ and $b_n \to B$, then $a_n + b_n \to A + B$.

Discussion. Here, we know that $a_n \to A$ and that $b_n \to B$. We wish to show that $a_n + b_n \to A + B$.

`We want to show:` $a_n + b_n \to A + B$. Thus, we will start our proof with "Let $\varepsilon > 0$" (which is not of our choosing). Our job is to find an N such that for all $n > N$, the inequality $|(a_n + b_n) - (A + B)| < \varepsilon$ holds.

`We know:` $a_n \to A$. Thus, for any $\varepsilon_a > 0$ of our choosing, there exists an N_a such that for all $n > N_a$, the inequality $|a_n - A| < \varepsilon_a$ holds.

`We know:` $b_n \to B$. Thus, for any $\varepsilon_b > 0$ of our choosing, there exists an N_b such that for all $n > N_b$, the inequality $|b_n - B| < \varepsilon_b$ holds.

`What we'll do:` Since we are free to choose ε_a and ε_b, let's choose $\varepsilon_a = \varepsilon_b = \dfrac{\varepsilon}{2}$. Thus, we are given back an N_a and N_b so that

- for all $n > N_a$, $|a_n - A| < \varepsilon_a = \dfrac{\varepsilon}{2}$.
- for all $n > N_b$, $|b_n - B| < \varepsilon_b = \dfrac{\varepsilon}{2}$.

Since we want to use both statements, we want simultaneously for $n > N_a$ and for $n > N_b$. Thus, we will choose $N = \max\{N_a, N_b\}$. Then, we will use the above, in conjunction with the triangle inequality, to conclude that for all $n > N = \max\{N_a, N_b\}$, then the inequality $|(a_n + b_n) - (A + B)| < \varepsilon$ holds.

Proof. Let $\varepsilon > 0$. Since $a_n \to A$, then for $\varepsilon_a = \frac{\varepsilon}{2} > 0$, there exists an N_a so that for all $n > N_a$, the inequality $|a_n - A| < \frac{\varepsilon}{2}$ holds. Similarly, since $b_n \to B$, then for $\varepsilon_b = \frac{\varepsilon}{2} > 0$, there exists an N_b so that for all $n > N_b$, the inequality $|b_n - B| < \frac{\varepsilon}{2}$ holds. So, we choose $N = \max\{N_a, N_b\}$. So, for $n > N = \max\{N_a, N_b\}$ then both $n > N_a$ and $n > N_b$. Thus, we know that $|a_n - A| < \frac{\varepsilon}{2}$ and that $|b_n - B| < \frac{\varepsilon}{2}$. Thus, for $n > N$, we can consider $|(a_n + b_n) - (A + B) = |(a_n - A) + (b_n - B)|$ along with the triangle inequality and get that

$$|(a_n + b_n) - (A + B) = |(a_n - A) + (b_n - B)| \leq |a_n - A| + |b_n - B| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, $|(a_n + b_n) - (A + B)| < \varepsilon$, as desired. Thus, $a_n + b_n \to A + B$.