Freshman Summer Research Institute
Homework Solutions - Monday, July 18, 2011

Question 1. Find a general formula for i^k for any $k \in \mathbb{Z}$. Be sure to include the case when k is negative. Use your formula to compute $i^{1827361}$.

Solution 1. The general formula is that i^k is equal to $1, i, -1$, or $-i$, depending on if the remainder of k is $0, 1, 2$, or 3 when we divide by 4, respectively. Another way of saying this is by using *modular arithmetic*:

$$i^k = \begin{cases} 1 & k \equiv 0 \mod 4 \\ i & k \equiv 1 \mod 4 \\ -1 & k \equiv 2 \mod 4 \\ -i & k \equiv 3 \mod 4 \end{cases}$$

Using this, we can see that $1827361 = 456840 \cdot 4 + 1$ and, since the remainder is 1, we have that

$$i^{1827361} = i.$$

Question 2. Write the following complex numbers in the form $a + bi$.

(a) $(1+i)^3$

(b) $\left[\dfrac{2+i}{6i-(1-2i)}\right]^2$

(c) $\dfrac{3}{i} + \dfrac{i}{3} + \dfrac{8i-1}{i}$

Solution 2a. Using the binomial theorem, we have that

$$(1+i)^3 = 1^3 + 3 \cdot 1^2 i^1 + 3 \cdot 1^1 i^2 + i^3 = 1 + 3i - 3 - i = -2 + 2i.$$

Solution 2b. First simplifying the inside and then squaring, we obtain

$$\left[\frac{2+i}{6i-(1-2i)}\right]^2 = \left[\frac{2+i}{-1+8i}\right]^2 = \left[\frac{2+i}{-1+8i} \cdot \frac{-1-8i}{-1-8i}\right]^2 =$$

$$\left[\frac{6-17i}{65}\right]^2 = \frac{253-204i}{4225}$$

Solution 2c. Obtaining a common denominator and clearing the i's out of the denominators, we have

$$\frac{3}{i} + \frac{i}{3} + \frac{8i-1}{i} = \frac{9 + i^2 + 3(8i-1)}{3i} = \frac{5+27i}{3i} = \frac{5}{3i} + 9 = 9 - \frac{5}{3}i.$$

Question 3. In the below problems, you may use the following facts:

$$\sin(-\theta) = -\sin\theta \quad \text{and} \quad \cos(-\theta) = \cos\theta.$$

(a) Use Euler's equation to prove that

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}.$$

(b) Use Euler's equation to prove that

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

]**Solution 3a.** Starting with the right-hand side of the equation and using the Euler equation twice, we get

$$\frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{\cos\theta + i\sin\theta + \cos(-\theta) + i\sin(-\theta)}{2} = \frac{\cos\theta + i\sin\theta + \cos(\theta) - i\sin(\theta)}{2} =$$

$$\frac{2\cos\theta}{2} = \cos\theta.$$

Solution 3b. Again, starting with the right-hand side of the equation and using the Euler equation twice, we get

$$\frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{\cos\theta + i\sin\theta - (\cos(-\theta) + i\sin(-\theta))}{2i} =$$

$$\frac{\cos\theta + i\sin\theta - \cos\theta + i\sin\theta}{2i} = \frac{2i\sin\theta}{2i} = \sin\theta.$$

Question 4.

(a) Show that the set of irrationals is not *closed under multiplication*. That is, show that there exist irrational numbers a and b such that $a \cdot b$ is rational.

(b) Furthermore, show that the set of irrationals is not *closed under addition* by finding irrational numbers a and b such that $a + b$ is rational.

(c) Show that the set of irrationals is *closed under multiplicative and additive inverses* by showing that if x is irrational, then $\frac{1}{x}$ and $-x$ are also irrational.

(d) Show that between every two distinct rational numbers $x, y \in \mathbb{Q}$, there exists another rational number. That is, if $x, y \in \mathbb{Q}$ are rationals, then there exists some rational $z \in \mathbb{Q}$ such that $x < z < y$.

(e) Use (d) to deduce that between 0 and 1, there are infinitely many rational numbers.

Solution 4a. Consider $a = \sqrt{2}$ and $b = \sqrt{2}$, which are irrational numbers (proven in class). Their product is $\sqrt{2} \cdot \sqrt{2} = 2$, which is a rational number (in fact, an integer). Thus, the product of two irrationals is not necessarily an irrational number.

Solution 4b. Similarly, we can use $a = \sqrt{2}$ and $b = -\sqrt{2}$, both of which are irrational. Then, their sum is $\sqrt{2} + -\sqrt{2} = 0$, which is rational.

Solution 4c. We may assume that x is an irrational number; of course, we may also assume that $x \neq 0$ since 0 is a rational number. We will prove by way of contradiction that $1/x$ is also irrational. So, we assume that $1/x$ is rational. Thus, $\dfrac{1}{x} = \dfrac{p}{q}$, where $p, q \in \mathbb{Z}$ and both $p, q \neq 0$ (since $x \neq 0$). Then, by taking reciprocals of both sides, we see that $x = \dfrac{q}{p}$, which is a rational number. Of course, this contradicts the fact that x is irrational. This contradiction forces us to conclude that $1/x$ must also be irrational.

Similarly, to show that $-x$ is irrational, we assume it is rational. Then, $-x = \dfrac{p}{q}$ with $p, q \in \mathbb{Z}$ and $q \neq 0$. Negating both sides of our equation, we see that $x = \dfrac{-p}{q}$, which implies that x is rational, contradicting its irrationality. Thus, we are forced to conclude that $-x$ is irrational.

Solution 4d. Let $x = \frac{p}{q}$ and $y = \frac{r}{s}$. Since the two numbers x and y are distinct, we know that $ps \neq qr$. To find a point in the middle, we take the "average" of the two numbers

$$\frac{ps + qr}{2qs}.$$

Clearly, this is another rational number since $q, s \neq 0$. Furthermore, this point is distinct from both $\frac{p}{q}$ and $\frac{r}{s}$ since $ps \neq qr$ and this point is strictly in between our two initial points.

Solution 4e. (d) implies that between 0 and 1 (in fact, any two rational numbers), there is a middle rational point. Using this middle point and one of the initial point, we can build another point between the first middle point and one of the initial points. Continuing in this manner, can build a sequence of rational points that never repeat but that are all between 0 and 1 (or any two rationals). Thus, between any two rationals there are infinitely many other rational numbers.

Question 5.

(a) Recall that complex conjugation negates the imaginary part of a complex number:

$$\overline{a + bi} = a - bi.$$

Use Euler's equation to prove that

$$\overline{re^{i\theta}} = re^{-i\theta}.$$

Geometrically, why does this make sense?

(b) Use (a) to prove that $z \cdot \overline{z}$ will be a non-negative real number. [This simple statement, by the way, tells us that Quantum Mechanics produces real-valued physical measurements].

Question 5a. Using Euler's equation, we have that

$$\overline{re^{i\theta}} = \overline{r\cos\theta + ir\sin\theta} = r\cos\theta - ir\sin\theta = r\cos(-\theta) + ir\sin(-\theta) = re^{-i\theta}.$$

Geometrically, this makes sense because conjugation reflects our complex number about the positive real axis. in terms of polar notation, though, this reflection may be obtained by rotating clockwise instead of counterclockwise (or vice versa) by an angle of θ.

Question 5b. To use (a), we first write $z = re^{i\theta}$ and then compute

$$z \cdot \overline{z} = re^{i\theta} \cdot \overline{re^{i\theta}} = re^{i\theta} \cdot re^{-i\theta} = r^2 e^{i(\theta - \theta)} = r^2,$$

which is a positive real number.

Question 6.

(a) Prove that $z \in \mathbb{C}$ is *real* if and only if $\overline{z} = z$.

(b) Prove that for any $z, w \in \mathbb{C}$,

$$\overline{z+w} = \overline{z} + \overline{w} \text{ and}$$
$$\overline{z \cdot w} = \overline{z} \cdot \overline{w}.$$

(c) Prove that for any $z \in \mathbb{C}$, $z \cdot \overline{z}$ and $z + \overline{z}$ are real.

(d) Prove that if z is a root of the *real* polynomial

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$

then $\overline{z}$ is also a root. Thus, we may assume that the coefficients a_i are all real numbers while z and $\overline{z}$ may be complex numbers.

(e) Prove that any polynomial with coefficients in $\mathbb{R}$ can be written as a product of linear and quadratic polynomials in $\mathbb{R}$.

(f) Prove that any odd-degree polynomial with coefficients in $\mathbb{R}$ has at least one real root.

Solution 6a. If $z = a + bi$ is real, then $b = 0$ and thus $\overline{z} = \overline{a + 0i} = a - 0i = z$. Conversely, if $z = \overline{z}$, then $a + bi = a - bi$. This, of course, implies that $b = -b$, which is only true if $b = 0$. Thus, $z = a + 0i$, which is a real number.

Solution 6b. Let $z = a + bi$ and $w = c + di$, then

$$\overline{z+w} = \overline{(a+bi)+(c+di)} = \overline{(a+c)+(b+d)i} = (a+c)-(b+d)i = (a-bi)+(c-di) = \overline{z}+\overline{w}.$$

Furthermore,

$$\overline{z \cdot w} = \overline{(a+bi)\cdot(c+di)} = \overline{(ac-bd)+(bc+ad)i} = (ac-bd)-(bc+ad)i =$$
$$(a-bi)\cdot(c-di) = \overline{z}\cdot\overline{w}$$

Solution 6c. For any $z = a + bi$,

$$z \cdot \overline{z} = (a+bi)\cdot(a-bi) = a^2 + b^2,$$

which is a real number. Furthermore,

$$z + \overline{z} = (a+bi) + \overline{a+bi} = (a+bi) + (a-bi) = 2a,$$

which is also real.

Solution 6d. First, we note that $\overline{z^n} = \overline{z} \cdots \overline{z} = \overline{z}^n$ by (b). Now, since if z is a root of the real polynomial, then we have that

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0.$$

Taking the conjugate of both sides, we have

$$\overline{a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0} = \overline{0}.$$

Using the above rules to break up conjugates and the fact that 0 is real, we get

$$\overline{a_n}\,\overline{z}^n + \overline{a_{n-1}}\,\overline{z}^{n-1} + \cdots + \overline{a_1}\,\overline{z} + \overline{a_0} = \overline{0} = 0.$$

Since our polynomial is real, the coefficients a_k are all real and thus, by (a), $\overline{a_k} = a_k$. Using this, we obtain

$$a_n \overline{z}^n + a_{n-1} \overline{z}^{n-1} + \cdots + a_1 \overline{z} + a_0 = 0.$$

Of course, the above means that $\overline{z}$ is a root of the original polynomial p.

Solution 6e. (d) tells us that if α is a root to a real polynomial, then $\overline{\alpha}$ is also a root. If α is *real*, then $\alpha = \overline{\alpha}$ and, according to the Fundamental Theorem of Algebra, we have a factor of $(z - \alpha)$. If α is a non-real root, then α is a different complex number but is also a root of the polynomial. Thus, $(z - \alpha)(z - \overline{\alpha}) = z^2 + (\alpha + \overline{\alpha})z + \alpha \cdot \overline{\alpha}$ is also a real factor (since $\alpha + \overline{\alpha}$ and $\alpha \cdot \overline{\alpha}$ are real numbers). Since there are the only two possibilities, we may write our polynomial as a product of these linear and quadratic factors.

Solution 6f. Notice that by (d), roots to real polynomials come in complex conjugate pairs. Since the degree of the polynomial is odd, then, by the Fundamental Theorem of Algebra, there must be an odd number of roots. Since they come in conjugate pairs, we must have that one of these "pairs" is actually the same number and thus one of the roots α has $\alpha = \overline{\alpha}$, and thus must be real, as desired.