Freshman Summer Research Institute
Homework - Monday, July 18, 2011

Note: This HW is due at the beginning of class on Monday, July 26, 2010. You are encouraged to work in groups, but write up your own solutions and be sure that you understand all of what you write.

Question 1. Find a general formula for i^k for any $k \in \mathbb{Z}$. Be sure to include the case when k is negative. Use your formula to compute $i^{1827361}$.

Question 2. Write the following complex numbers in the form $a + bi$.

(a) $(1+i)^3$

(b) $\left[\dfrac{2+i}{6i-(1-2i)}\right]^2$

(c) $\dfrac{3}{i}+\dfrac{i}{3}+\dfrac{8i-1}{i}$

Question 3. In the below problems, you may use the following facts:

$$\sin(-\theta) = -\sin\theta \quad \text{and} \quad \cos(-\theta) = \cos\theta.$$

(a) Use Euler's equation to prove that
$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}.$$

(b) Use Euler's equation to prove that
$$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}.$$

Question 4.

(a) Show that the set of irrationals is not *closed under multiplication*. That is, show that there exist irrational numbers a and b such that $a \cdot b$ is rational.

(b) Furthermore, show that the set of irrationals is not *closed under addition* by finding irrational numbers a and b such that $a + b$ is rational.

(c) Show that the set of irrationals is *closed under multiplicative and additive inverses* by showing that if $x \neq 0$ is irrational, then $\frac{1}{x}$ and $-x$ are also irrational.

(d) Show that between every two distinct rational numbers $x, y \in \mathbb{Q}$, there exists another rational number. That is, if $x, y \in \mathbb{Q}$ are rationals, then there exists some rational $z \in \mathbb{Q}$ such that $x < z < y$.

(e) Use (d) to deduce that between 0 and 1, there are infinitely many rational numbers.

Question 5.

(a) Recall that complex conjugation negates the imaginary part of a complex number:

$$\overline{a+bi} = a - bi.$$

Use Euler's equation to prove that

$$\overline{re^{i\theta}} = re^{-i\theta}.$$

Geometrically, why does this make sense?

(b) Use (a) to prove that $z \cdot \overline{z}$ will be a non-negative real number. [This simple statement, by the way, tells us that Quantum Mechanics produces real-valued physical measurements].

Question 6.

(a) Prove that $z \in \mathbb{C}$ is *real* if and only if $\overline{z} = z$.

(b) Prove that for any $z, w \in \mathbb{C}$,

$$\overline{z+w} = \overline{z} + \overline{w} \text{ and}$$

$$\overline{z \cdot w} = \overline{z} \cdot \overline{w}.$$

(c) Prove that for any $z \in \mathbb{C}$, $z \cdot \overline{z}$ and $z + \overline{z}$ are real.

(d) Prove that if z is a root of the *real* polynomial

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$

then $\overline{z}$ is also a root. Thus, we may assume that the coefficients a_i are all real numbers while z and $\overline{z}$ may be complex numbers.

(e) Prove that any polynomial with coefficients in $\mathbb{R}$ can be written as a product of linear and quadratic polynomials in $\mathbb{R}$.

(f) Prove that any odd-degree polynomial with coefficients in $\mathbb{R}$ has at least one real root.