Freshman Summer Research Institute
Homework Solutions - Wednesday, July 13, 2011

Question 1. Prove the following statements about sets and subsets.

(a) If $A \subset C$ and $B \subset C$, show that $A \cup B \subset C$.

(b) $A \cup A = A$ and $A \cap A = A$.

(c) $A \cup \varnothing = A$ and $A \cap \varnothing = \varnothing$.

(d) If $A \subset B$ and $B \subset C$, then $A \subset C$.

Solution 1a. We will show that $A \cup B \subset C$ by showing that every element in $A \cup B$ is also an element of C. Let $x \in A \cup B$. Then, x is an element of A or B. Either way, since $A \subset C$ and $B \subset C$, $x \in C$. Thus, $A \cup B \subset C$.

Solution 1b. To show $A \cup A = A$, we will show that $A \cup A \subset A$ and $A \subset A \cup A$. First, to show that $A \cup A \subset A$, let $x \in A \cup A$. Then, by definition x is an element of A or A is an element of A. Either way, x is an element of A and thus $A \cup A \subset A$. Conversely, to show that $A \subset A \cup A$, if $x \in A$, then it also an element of A or A and thus an element of $A \cup A$. Thus, we may conclude that $A \cup A = A$.

A similar proof show that $A \cap A = A$ just be replacing every "or" with "and".

Solution 1c. To show that $A \cup \varnothing = A$, we will show that $A \cup \varnothing \subset A$ and $A \subset A \cup \varnothing$. First, to show that $A \cup \varnothing \subset A$, if $x \in A \cup \varnothing$, then x is an element of A or the empty set. Clearly, x cannot be in the empty set since $\varnothing$ contains no elements. Thus, it must be true that $x \in A$. So, $A \cup \varnothing \subset A$. Conversely, to show that $A \subset A \cup \varnothing$, we note that it is always true that a set is a subset of the "larger" union $A \cup \varnothing$. Thus, we conclude that $A \cup \varnothing = A$.

To show that $A \cap \varnothing = \varnothing$, we will show that $A \cap \varnothing \subset \varnothing$ and $\varnothing \subset A \cap \varnothing$. First, to show that $A \cap \varnothing \subset \varnothing$, let $x \in A \cap \varnothing$. Then, x is in A and $\varnothing$. Since $x \in \varnothing$, x does not exist (since $\varnothing$ contains no elements). So, since no such x exists, $A \cap \varnothing$ contains no elements and is thus the empty set $\varnothing$. For the other inclusion $\varnothing \subset A \cap \varnothing$, $\varnothing$ is a subset of every set trivially. Thus, we conclude that $A \cap \varnothing = \varnothing$.

Solution 1d. To show that $A \subset C$, we must show that if $x \in A$, then $x \in C$. To this end, let $x \in A$. Since $A \subset B$, we this implies that $x \in B$. Furthermore, since $B \subset C$, we conclude that also $x \in C$. Thus, we have shown that any element $x \in A$ is also an element of C. Thus, $A \subset C$.

Question 2. Note that

$$1 = 1$$

$$1 - 4 = -(1 + 2)$$

$$1 - 4 + 9 = 1 + 2 + 3$$

$$1 - 4 + 9 - 16 = -(1 + 2 + 3 + 4)$$

Guess the general law (using n's) suggested by the above and prove it using induction.

Solution 2. The general statement is that

$$\sum_{k=1}^{n} (-1)^{k+1} k^2 = (-1)^{n+1} \sum_{k=1}^{n} k,$$

which we will show is true for all $n \geq 1$.

First, we check the base case of when $n = 1$. In this case, we have

$$\sum_{k=1}^{1}(-1)^{k+1}k^2 = (-1)^{1+1}\sum_{k=1}^{1}k,$$

which is equivalent to the statement

$$1^2 = 1,$$

a true statement.

To fulfill our inductive step, we assume that the $A(n)$ statement

$$\sum_{k=1}^{n}(-1)^{k+1}k^2 = (-1)^{n+1}\sum_{k=1}^{n}k$$

is true and use it t prove that $A(n+1)$ is true as well. To ease computations, we will break this up into the cases when n is even and when n is odd.

We begin with the case when n is even. In this case, our inductive assumption may be written

$$1 - 4 + 9 - \cdots - n^2 = -(1 + 2 + 3 + \cdots + n).$$

Adding $(n+1)^2$ to both sides of the equation yields

$$1 - 4 + 9 - \cdots - n^2 + (n+1)^2 = -(1 + 2 + 3 + \cdots + n) + (n+1)^2.$$

Rewriting the right-hand side using our power sum formula, we get

$$1 - 4 + 9 - \cdots - n^2 + (n+1)^2 = -\frac{n(n+1)}{2} + (n+1)^2.$$

Continuing our manipulations of the right-hand side, we obtain

$$1 - 4 + 9 - \cdots - n^2 + (n+1)^2 = (n+1)\left(n + 1 - \frac{n}{2}\right)$$

$$1 - 4 + 9 - \cdots - n^2 + (n+1)^2 = (n+1)\left(n + 1 - \frac{n}{2}\right)$$

$$1 - 4 + 9 - \cdots - n^2 + (n+1)^2 = (n+1)\left(\frac{2n + 2 - n}{2}\right)$$

$$1 - 4 + 9 - \cdots - n^2 + (n+1)^2 = (n+1)\left(\frac{((n+1)+1}{2}\right).$$

We notice that the right-hand side is equal to the power sum formula for $\sum_{k=1}^{n+1} k$. Thus, we have deduced our $A(n+1)$ statement

$$1 - 4 + 9 - \cdots - n^2 + (n+1)^2 = \sum_{k=1}^{n} k.$$

The case where n is odd is almost identical, but with opposite signs, so it is omitted here. Thus, we conclude by induction that

$$\sum_{k=1}^{n}(-1)^{k+1}k^2 = (-1)^{n+1}\sum_{k=1}^{n}k,$$

is true for all $n \geq 1$.

Question 3. Note that

$$1 + \frac{1}{2} = 2 - \frac{1}{2}$$

$$1 + \frac{1}{2} + \frac{1}{4} = 2 - \frac{1}{4}$$

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = 2 - \frac{1}{8}$$

Guess the general law (using n's) suggested by the above and prove it using induction.

Solution 3. The general pattern is

$$\sum_{k=0}^{n} \frac{1}{2^k} = 2 - \frac{1}{2^n}$$

for $n \geq 0$. We will prove this statement using induction.

As our base case, we see that when $n = 0$, we have the statement

$$\sum_{k=0}^{0} \frac{1}{2^k} = 2 - \frac{1}{2^0}$$

is equivalent to

$$\frac{1}{1} = 2 - \frac{1}{1},$$

which is obviously true.

Next, we perform our inductive step. That is, our inductive assumption states that we assume that $A(n)$ is true and prove that $A(n+1)$ is also true. So, we assume that

$$\sum_{k=0}^{n} \frac{1}{2^k} = 2 - \frac{1}{2^n}$$

for some n and we want to show that

$$\sum_{k=0}^{n+1} \frac{1}{2^k} = 2 - \frac{1}{2^{n+1}}.$$

To prove the statement $A(n+1)$, we will begin with the left-hand side of the statement and reach the right-hand side while using our inductive assumption. So,

$$\sum_{k=0}^{n+1} \frac{1}{2^k} = \sum_{k=0}^{n} \frac{1}{2^k} + \frac{1}{2^{n+1}} = \left(2 - \frac{1}{2^n}\right) + \frac{1}{2^{n+1}} = 2 - \frac{2}{2^{n+1}} + \frac{1}{2^{n+1}} = 2 - \frac{1}{2^{n+1}}.$$

Thus, we have used our assumption that $A(n)$ is true to show that $A(n+1)$ is also true. Thus, by induction we have proven that the statement

$$\sum_{k=0}^{n} \frac{1}{2^k} = 2 - \frac{1}{2^n}$$

hold for every $n \geq 0$.

Question 4.

(a) Prove the following statement using induction:

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2 .$$

(b) Use your previously proven Sum of Powers formulae for $\sum_{k=1}^{n} k$ and $\sum_{k=1}^{n} k^3$ to verify that the above statement is true.

Solution 4a. Our statement $A(n)$ is that

$$\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n}\right)^2$$

for $n \geq 1$ First, we begin with our inductive basis. When $n = 1$, we have the statement

$$\sum_{k=1}^{1} k^3 = \left(\sum_{k=1}^{1} k\right)^2,$$

which is equivalent to the statement

$$1^3 = 1^2,$$

which is trivially true.

Next, we assume that the statement $A(n)$ is true for some n and show that it is true for $A(n+1)$. That is, we wish to use the inductive assumption of

$$\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2$$

to show that

$$\sum_{k=1}^{n+1} k^3 = \left(\sum_{k=1}^{n+1} k\right)^2.$$

To do this, we will begin with the left-hand side and find that it is eventually equal to the right-hand side.

$$\sum_{k=1}^{n+1} k^3 = \sum_{k=1}^{n} k^3 + (n+1)^3 = \sum_{k=1}^{n} k^3 + (n+1)(n+1)^2 = \sum_{k=1}^{n} k^3 + n(n+1)^2 + (n+1)^2 =$$

$$\sum_{k=1}^{n} k^3 + 2(n+1)n\frac{(n+1)}{2} + (n+1)^2 = \sum_{k=1}^{n} k^3 + 2(n+1)\left(\sum_{k=1}^{n} k\right) + (n+1)^2 =$$

$$\left(\sum_{k=1}^{n} k\right)^2 + 2(n+1)\left(\sum_{k=1}^{n} k\right) + (n+1)^2 = \left(\sum_{k=1}^{n} k + (n+1)\right)^2 = \left(\sum_{k=1}^{n+1} k\right)^2.$$

Thus, we have used our inductive assumption that $A(n)$ is true to prove that $A(n+1)$ is also true. Thus, by induction,

$$\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n}\right)^2$$

holds for all $n \geq 1$.

Solution 4b. Recall that in a previous workshop, we proved that

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \quad \text{and} \quad \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}.$$

Thus, it is clear that

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4} = \left(\frac{n(n+1)}{2}\right)^2 = \left(\sum_{k=1}^{n} k\right)^2.$$

Question 5. The following will demonstrate that *bijection* is an *equivalence relation* on the class of sets.

- Reflexivity: Show that there exists a bijection $f : S \to S$.
- Symmetry: If there exists a bijection $f : S \to T$, then there exists a bijection $g : T \to S$.
- Transitivity: If there exist bijections $f : S \to T$ and $g : T \to R$, then there exists a bijection $h : S \to R$.

Solution 5. To show that there exists a bijection $f : S \to S$, consider the function $f(s) = s$, the so-called identity map. Clearly, this map is both injective and surjective (every identity map has these properties), so this map is a bijection.

To prove symmetry, we must show that if $f : S \to T$ is a bijection, there there exists a bijection $g : T \to S$. Since $f : S \to T$ is a bijection, it is both injective and surjective. Since f is surjective, we know that for every $t \in T$, there exists an $s \in S$ such that $f(s) = t$. Since f is also injective, we know that this is the only $s \in S$ such that $f(s) = t$. So, we define $g : T \to S$ by $g(t)$ is the element s such that $f(s) = t$; that is, the image of t under the function g is the pre-image of t under the function f. Since f is surjective, we can use this definition of g for every $t \in T$ (since every t has a pre-image). Furthermore, since f is injective, $g : T \to S$ maps to only one s.

The function $g : T \to S$ is surjective since every $s \in S$ has a pre-image t (which is its image under f). To show that $g : T \to S$ is injective, we will show that whenever $g(t_1) = g(t_2)$, then $t_1 = t_2$. We know that $g(t_1) = s_1$ such that $f(s_1) = t_1$ and similarly that $g(t_2) = s_2$ such that $f(s_2) = t_2$. Thus, since $g(t_1) = g(t_2)$, we know that $s_1 = s_2$. Since f is a function $f(s_1) = f(s_2)$, which is equivalent to $t_1 = t_2$. Thus, g is injective as well.

We have already shown transitivity in a previous workshop, where we showed that the composition map $f \circ g : S \to T$ is bijective if both $f : S \to T$ and $g : T \to R$ are bijective. So, we only need to let $h = f \circ g$ as we are done.

Thus, since all three of these properties hold, we say that bijection is an *equivalence relation* on the class of all sets.