Freshman Summer Research Institute
Homework due Wednesday, July 13, 2011

Note: This HW is due at the beginning of class on Wednesday, July 21, 2010. You are encouraged to work in groups, but write up your own solutions and be sure that you understand all of what you write.

Question 1. Prove the following statements about sets and subsets.

(a) If $A \subset C$ and $B \subset C$, show that $A \cup B \subset C$.

(b) $A \cup A = A$ and $A \cap A = A$.

(c) $A \cup \varnothing = A$ and $A \cap \varnothing = \varnothing$.

(d) If $A \subset B$ and $B \subset C$, then $A \subset C$.

Question 2. Note that

$$1 = 1$$
$$1 - 4 = -(1 + 2)$$
$$1 - 4 + 9 = 1 + 2 + 3$$
$$1 - 4 + 9 - 16 = -(1 + 2 + 3 + 4)$$

Guess the general law (using n's) suggested by the above and prove it using induction.

Question 3. Note that

$$1 + \frac{1}{2} = 2 - \frac{1}{2}$$
$$1 + \frac{1}{2} + \frac{1}{4} = 2 - \frac{1}{4}$$
$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = 2 - \frac{1}{8}$$

Guess the general law (using n's) suggested by the above and prove it using induction.

Question 4.

(a) Prove the following statement using induction:

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2 .$$

(b) Use your previously proven Sum of Powers formulae for $\sum_{k=1}^{n} k$ and $\sum_{k=1}^{n} k^3$ to verify that the above statement is true.

Question 5. The following will demonstrate that *bijection* is an *equivalence relation* on the class of sets.

- Reflexivity: Show that there exists a bijection $f : S \to S$.
- Symmetry: If there exists a bijection $f : S \to T$, then there exists a bijection $g : T \to S$.
- Transivity: If there exist bijections $f : S \to T$ and $g : T \to R$, then there exists a bijection $h : S \to R$.