String Stretching, Frequency Modulation, and Banjo Clang

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(Dated: April 22, 2014)

The banjo's floating bridge, string break angle, and flexible drumhead all contribute to substantial audio range frequency modulation. From the world of electronic music synthesis, it is known that *modulating* higher frequency sounds with lower acoustic frequencies leads to metallic and bell-like tone. The mechanics of the banjo does just that quite naturally, modulating fundamentals and harmonics with the motion of the bridge. In technical terms, with a floating bridge, string stretching is first order in bridge motion, in contrast to it being second order in the vibration amplitude of the strings.

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I. WHAT IS A BANJO?

A banjo is a drum with strings mounted on a neck. With minor caveats, that's what makes it a banjo. So that's what must be responsible for its characteristic sound. A reasonable question then is: what is it in the mechanics of sound production by drum and strings that distinguishes banjo sound from other stringed instruments? While trying to understand the mechanics of tailpiece and break angle, I came to understand some aspects of the workings of drumhead and floating bridge which may well be a partial answer to that question.

A. Needle Points of Sound

Pete Seeger liked to say, "This banjo can do things that a guitar can't do. You get these needlepoints of sound..." (The "needlepoints" metaphor appears in many interviews in his later life.)

In 1865, Mark Twain wrote, "...When you want genuine music — music that will come right home to you like a bad quarter, suffuse your system like strychnine whisky, go right through you like Brandreth's pills, ramify your whole constitution like the measles, and break out on your hide like the pin-feather pimples on a picked goose, — when you want all this, just smash your piano, and invoke the glory-beaming banjo!"

I'm not in their league or even in the Minors, but "clang" is a word I use to describe the ideal banjo sound, at least for bluegrass or Scruggs-style playing.

Besides clang in general, there's a particular clang that fascinates me. It appears only occasionally in the picking of bluegrass greats, and I've tried hard to figure out just what it is. I concluded that it comes from the open D 1st string when it's used as a drone (i.e., with the melody and chording appearing on the inner strings). So it appears at relatively random times within the ever-changing rhythm of three-finger rolls, sometimes ringing out with a conspicuously clear clang. To me, it sounds like a steel hammer striking a cold steel chisel. As best I can tell, it needs a strong pick, a bit further from the bridge than normal playing. My clang hero is J.D. Crowe.

B. Tailpiece and Break Angle Sound

I built/assembled two banjos that should have sounded the same — and they almost did. They were open-backs for frailing, but one had a metallic, slightly more piercing sound in the attack. The only obvious difference between the two instruments was the tailpiece, an item to which I had heretofore not paid much attention. The minimal, 19^{th} Century-style “No-Knot” tailpiece gave a very small break angle of the strings over the bridge (about 6^o) and was the plunkier of the two. The other was a standard Deering Goodtime tailpiece, which reaches farther toward the bridge and gives a break angle of about 15^o. It gave the more metallic attack. (These two tailpieces are in the photo on the title page.)

It is commonly offered advice: Choose a tailpiece or set an adjustable tailpiece to get a sharper break angle if you want more “snap” or “ping.” For example, Roger Siminoff wrote in his book on set-up and more recently in *Banjo NewsLetter*, July 2010, that sharp angles give “good brightness and ‘snap,’ ” *versus* smaller angles, which give “warm, round tone” (https://banjonews.com/2010-07/more“˙string“˙break“˙angles.html). And Barry Hunn wrote for the Deering March 21, 2014 blog that sharper angles make “the sound a bit sharper or snappier,” while smaller angles give “mellow, warm tone” (http://blog.deeringbanjos.com/6-string-banjo-set-ups). One is also warned that tightening down too hard on the tailpiece will ultimately have the effect of limiting the range of bridge motion and perceptibly decreasing the overall sound volume.

Hear the No-Knot tailpiece (click-here-if-blue-box-is-visible) or http://www.its.caltech.edu/~politzer/FM/no-knot-stinsons2.mp3 . And, on the same banjo (same set-up, microphone, room arrangement, etc.) hear the Goodtime tailpiece (click-here-if-box-is-visible) or go to http://www.its.caltech.edu/~politzer/FM/goodtime-stinsons2.mp3. On the fifteen year old Goodtime banjo being played there, the difference is more subtle than it was with the $5\frac{5}{8}''$ deep pot shown in the title page photo. Also, good speakers help. But to my ear, the No-Knot gives a thumpier sound, while the Goodtime tailpiece has more sparkle.

I learned the tune from Mary Cox (http://www.maryzcox.com), who explains she learned it from dulcimer players, who, in turn, know it from an old fiddle recording, whence it’s present name, “John Stinson’s #2.”

C. Tailpiece Mechanics Challenge

Many people comment that an increase in break angle increases the down-pressure of the strings onto the bridge, and that's certainly true. However, the physics of how that might effect the sound was obscure. I could not make sense of the simple statement that the higher pressure increases the efficiency of high frequency transfer from the strings to the bridge. It was also unclear that banjo tone adjustment was just a matter of turning up the treble. Also, since one retunes the strings after adjusting the break angle, the audible "snap" didn't seem to be be the physical snap of the plucked string because it is always tuned to the same tension.

However, the sharper break angle does increase the amount that the strings have to stretch *while the bridge moves up and down.* This string stretching modulates string tension. And this produces pitch and harmonic frequency modulation. I realized that this variation in string tension might not be negligible. That is in contrast to the tiny variation in tension inherent in the transverse vibration along any plucked string,

The tension variation from bridge motion is a non-linear effect. One consequence is that care must be used when discussing frequencies and Fourier components. So keep in mind that it is actually always the total motion of the bridge that modulates the string tensions. That total bridge motion is not a smooth sinusoidal function of time. And, no matter what harmonic of a particular string we are considering, it will be modulated by the total bridge motion, which generally has a strong component of the fundamental frequencies of the plucked strings, as well as significant amounts of higher ones.

D. Acoustic Frequency Modulation

While trying to figure out what happens to the sound when you modulate tension *while* a string is vibrating, I stumbled on the early literature on acoustic frequency modulation by pioneers of electronic music synthesizers.

In 1973, John Chowning published a remarkable insight (*The Synthesis of Complex Audio Spectra by Means of Frequency Modulation*, J. Audio Eng. Soc. 21, 7, 1973, digital version here or here:
https://ccrma.stanford.edu/sites/default/files/user/jc/fmsynthesispaperfinal_1.pdf). In FM

radio, the carrier radio frequency is modulated with the audio frequencies of the desired signal. Carrier and audio frequencies are many orders of magnitude apart. Chowning found that modulating audio frequencies with lower *audio* frequencies yields sounds that are far more "realistic" than the flat, early electronic sounds of pure sine waves. The abstract math is the same as for FM radio signals: the modulation produces "side bands" around the high frequency. The bands span a width in frequency that is of order of the modulating frequency. (A nice, classic review for the sound case was given by Bill Schottstaedt: *An Introduction to FM*, https://ccrma.stanford.edu/software/snd/snd/fm.html.)

Chowning is a jack of all trades but primarily a musician and composer, with an interest in electronic music and psychoacoustics. The acoustic FM idea became the basis of a patent, which was then licensed to Yamaha. It proved to be one of Stanford University's all-time biggest money makers. (Chowning was on the Stanford music faculty and co-founder in 1975 of Stanford's CCRMA [Center for Computer Research in Music and Acoustics]). The Stanford/Yamaha R&D collaboration produced the first generation of electronic instruments that made sounds that were recognizable (at least vaguely) as particular, standard instruments.

In his paper, Chowning mentions that for struck "bell-like" sounds, the amplitude of the modulation relative to the central frequency should be proportional to the overall decaying amplitude. (That's precisely what the real banjo modulation amplitude does, coming from stretching and string tension modulation due to bridge motion.) Schottstaed writes, "As the index sweeps upward, energy is swept gradually outward into higher order side bands; this is the originally exciting, now extremely annoying 'FM sweep'. The important thing to get from these Bessel functions is that the higher the index, the more dispersed the spectral energy — normally a brighter sound." "Index" is the relative frequency range of the modulation. The central frequency ν_c is modulated at frequency ν_m, i.e., we make the

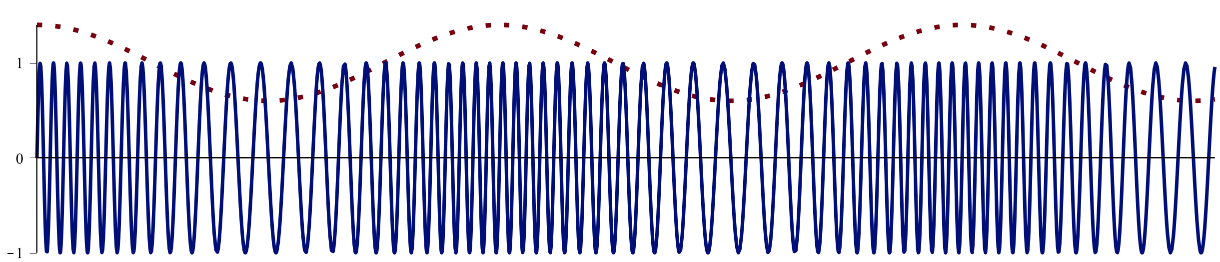

FIG. 1. a frequency modulated signal (solid curve) *versus* time; dots are the modulation

frequency a little bigger and a little smaller at a rate specified by ν_m and by an amount whose maximum is the index. In FIG.1, the index is $0.4\nu_c$ and $\nu_m = 0.04\nu_c$. This is also sometimes called phase modulation. However, I believe that the term "frequency modulation" is conceptually clearer in the present discussion of string stretching.

"Bell-like," "originally exciting, now extremely annoying," and "brighter sound"... I figured I was on the right track for deconstructing banjo sound.

A significant lesson that came out of this era of sound synthesis R&D is a much greater appreciation of the impact of temporal structure on what we hear. Frequency spectrum does not uniquely determine perceived tone. A trivial example is the fact that a simple clap and white noise can have the same spectrum, but they sound nothing like each other. It is also true that our perception of a musical sound lasting just a second or two will depend on the shape in time of its overall amplitude envelope. That's a sensitivity to changes on the scale of tenths or hundredths of a second. More surprising was the realization that perceived tone depends on the relative timing structure of various frequency components on the scale of thousandths of a second. So, hearing on time scales in the range of auditory oscillation periods is not just a matter of frequency spectrum. We are sensitive to the actual time sequence.

Frequency modulation gives side bands in the frequency spectrum. But an important feature of frequency modulation is that the resulting signal as a function of time is as smooth as can be, consistent with that spectrum. Those same frequency components, with the same strengths, can be assembled to give a signal with much greater time-to-time variation. This is just a repeat of the clap-*versus*-white-noise story but all within the scale of a few thousandths of a second. The surprise is that we perceive it. (It is really only surprising if you thought you understood the neuro- and cell biology of hearing.)

E. Clang from Exaggerated Bridge Motion

Here is a sound sample of "clang" that is produced from real banjo tremolo that is sped up digitally into the acoustic frequency range. The banjo has a relatively loose $12\frac{1}{2}''$ skin head and a break angle of only $3\frac{1}{2}^{\circ}$. I pluck the 4^{th} string, tuned down to F(2), i.e., 87 Hz. In the sound sample, there are three plain plucks, followed by six more with a tremolo produced by pushing down repeatedly on the head near the bridge. The tremolo frequency is about 6 Hz. Those are the first nine plucks. That same sound recording is then played again at 3× the original speed. That puts the pitch around C(3) and the modulation around 18 Hz. And finally, the original recording is played at 6× its speed. So the pitch is around C(4). The modulation, at around 36 Hz, is barely into the audio range. Listen: http://www.its.caltech.edu/˜politzer/FM/clang.mp3 .

Moving the bridge up and down modulates the string's vibrational frequencies by stretching the strings. And, when that modulation is sped up into the acoustic range, the wobble disappears and is replaced by a metallic timbre.

F. Synthesized Sound Samples

This section contains some computer-synthesized sounds. Simple samples are easy to produce with modern math software. I mostly used *Mathematica*®. It allows you to construct your own functions of time and then listen to them. The goal is to give a clearer sense of how and why frequency modulations are relevant to timbre. Of course, it doesn't sound like a banjo. A huge number of crucial features are missing. I start with pure sinusoidal tones, which are not the sounds of plucked strings on any real instruments. And the modulation is purely sinusoidal — also not what happens on a banjo. Real plucked string sound contains many strong, high harmonics. These decay at different rates. Also, the real banjo bridge motion has its own complex, non-sinusoidal time structure. But these samples provide an opportunity to describe in words what's going on. That might help as orientation before getting into math and geometry details.

First, here is a pure sine wave D(5); that's 587 Hz, an octave above a common open 1st string tuning: http://www.its.caltech.edu/˜politzer/FM/pureD5.wav

Here is the sine wave D(5) modulated in frequency at 196 Hz [a G(3)] with a maximum fractional excursion in frequency of 0.033:
http://www.its.caltech.edu/˜politzer/FM/FM-D5.wav That means that the "instantaneous" frequency of the signal is smoothly varying between 606 Hz and 567 Hz and back (i.e., about ±3%) at 196 times a second, while the amplitude of the signal is constant. "Instantaneous frequency" could be defined as follows: Take a single cycle of the signal (up, down, and back) and find the closest pure sine wave approximation to that signal. That pure sine wave would have the instantaneous frequency. Perhaps you can recognize in this

simple sample just a bit of the "glory-beaming" sound, especially if you turn up the volume.

There are other ways that D(5) could be modified. I present some here to emphasize how remarkable, in contrast, audio range frequency modulation really is.

1) Simply add a weaker G(3) signal to the D(5). In particular, in this sample the G(3) amplitude is 25% the size of the D(5) amplitude: http://www.its.caltech.edu/~politzer/FM/D5-G3.mp3. (Tiny speakers might require turning up the volume to hear the lower frequencies.)

2) Add 606 Hz and 567 Hz to the pure D(5), all with equal amplitude: http://www.its.caltech.edu/~politzer/FM/D5plus-minus-sss.mp3. Here we get beats at the difference frequencies. If the D(5) is shifted by 90° in phase, i.e., advanced in time by 1.7 thousandths of a second relative to the other two, the frequency spectrum is *identical* to the spectrum of the previous sound, but the sound itself is different: http://www.its.caltech.edu/~politzer/FM/D5plus-minus-scs.mp3.

3) Frequency tremolo: http://www.its.caltech.edu/~politzer/FM/D5tremolo.wav. In terms of math formulas, this is also frequency modulation. However, to hear it as a tremolo, the modulation frequency must be well below the audible range. In this case it's about 10 per second. The frequency modulation is again ±3% of the 587 Hz.

An influential paper by Risset and Matthews in 1969 [Physics Today 22(2), 23 (1969)] emphasized that then new computers could be used to understand sound in ways never possible before. One aspect discussed in that paper was the "envelope" of particular musical sounds. On a scale shorter than the duration of the note but longer than the period of individual acoustic oscillations, how does the volume evolve? Different instruments and other sound sources are quite different. For example, a typical pluck has a sharp onset followed by an exponential decay. So I use the function $t^{0.05}\exp(-8t)$ (with t in seconds) to modify the amplitude.

A pure 587 Hz thus modified sounds as follows: http://www.its.caltech.edu/~politzer/FM/pureD5pluck.wav

Now modulate that 587 Hz signal with 196 Hz but let the index or range in Hz of that modulation also vary at the rate given by the overall amplitude envelope. That makes sense because the physical frequency modulation due to string stretching decreases with decreasing amplitude: http://www.its.caltech.edu/~politzer/FM/FM-D5pluck. (OK. This sounds more like a toy electronic piano than a banjo, but I stick with the contention that it's in the right direction to begin to account for the characteristic sound of the plucked banjo.

Again, you might have to turn up the volume to hear the ring.)

Historically, the focus on frequency modulation was one of first steps in the effort produce traditional musical sounds with electronic components. A lot of trial and error went into assembling parts that together were convincing. With the enormous growth of computer speed and memory in the interim, a totally new kind of synthesizer has come into being. These days, if you want a keyboard to sound like a cello, you can build a machine with a huge bank of recordings of real cello sound clips that the keyboard can access. On the other hand, you could build a *viola organista* following Leonardo Da Vinci's sketches. (Actually, that's been done at least a few times over the centuries; it's well worth tracking down the on-line videos.)

II. THE GENERAL IDEA & A VARIETY OF CONSEQUENCES

A. Break Angle & String Stretch Geometry

"Floating bridge" refers to the bridge's relation to the strings. Specifically, the floating bridge goes up and down relative to the ends of the strings, which are fixed to the rim and the neck. That is to be contrasted with a bridge and saddle, as on a flat-top guitar, where the ends of the strings go up and down with the bridge.

"Break angle" is the angle the strings make going over the bridge. It is determined by the bridge height and tailpiece geometry, as roughly illustrated in FIG. 2

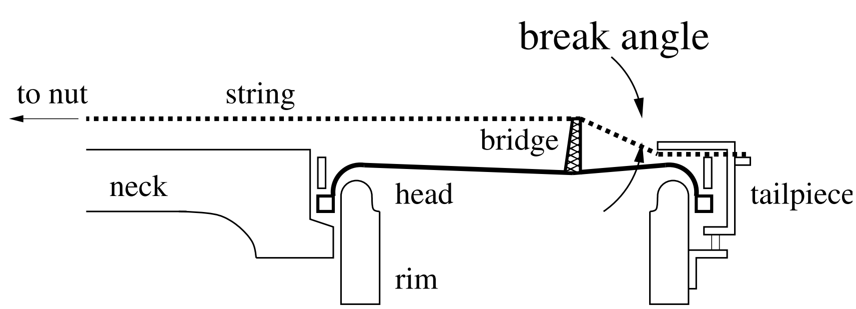

FIG. 2.

String tension is determined by scale length (nut to bridge), string gauge (or density), and chosen pitch of the open string. To a good approximation, the tension is uniform from the tuners to where the string is anchored on the tailpiece. The string tension produces a downward force on the bridge that increases as the break angle increases. In particular, if T is the tension and θ is the angle, the downward force on the tailpiece side is $T \sin\theta$, a function that is zero when θ is zero and increases in proportion to θ as θ increases (at least over the relevant range of angles). However, when the bridge is at rest, this is exactly countered by an equal upward force from the tension in the distorted head.

Is it a big deal if these forces are large? If the banjo is held horizontally, there are also about 1400 lbs of air weighing down on the head as well. This weight doesn't effect the head motion because there is an equal force of the air below pushing up.

So what's different about the string break-angle down force? The head adjusts itself to provide an exactly equal upward force on the bridge when it is at rest. When the string vibrations cause the bridge to move up and down, the break angle changes slightly. In particular, when the bridge moves up relative to the rim, the break angle increases, and so does the downward force of the strings on the bridge. At the same time, when the bridge moves up, the head break angle decreases, and so does the upward force of the head on the bridge. Hence, the forces on the bridge are unbalanced, and the net force is down. Therefore, this configuration supplies a "restoring" force to the bridge up-and-down motion, pushing it back toward equilibrium.

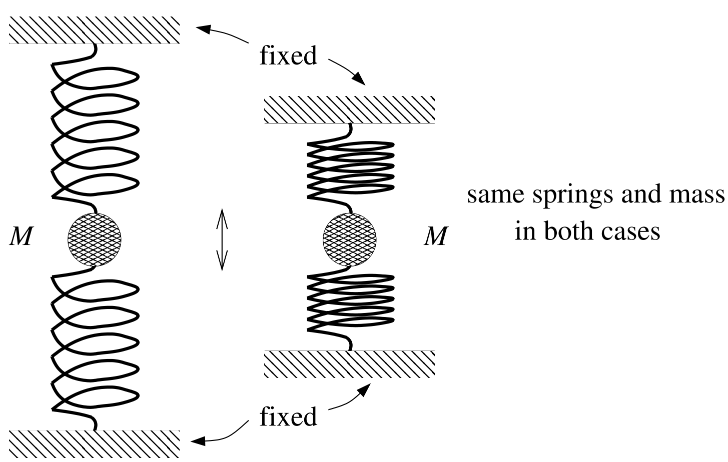

FIG. 3. With *ideal* springs, the restoring force and oscillation frequency of the central mass is independent of the equilibrium compression.

FIG. 3 gives a cartoon description of the bridge, M, held in place by the force of springs from above and below. Shown is the equilibrium or resting position, where the forces of the two springs cancel. When the bridge is displaced, the springs' forces are unequal, and their sum pushes the bridge back towards equilibrium. If the fixed ends of the springs are held closer together, each spring's force is greater than before. However, if they are ideal springs (each providing a force proportional to its stretch or compression), the net force of the two is the same as it was with the fixed ends in the original wider position — not only at equilibrium but also at any position displaced away from equilibrium. Hence, the resulting frequency of free oscillations is also the same as it was with the original wider position.

Every oscillating, macroscopic physical system has a region about its equilibrium position for which the restoring forces are well-approximated by ideal springs. It is always worth asking how large that region is and whether the excursions from equilibrium are small enough that we stay within that region. In almost all acoustic situations, that is the case, and the spring-like behavior gives a faithful representation of the motion. However, the present situation is one in which the various parts conspire to magnify a non-spring-like (*aka* non-linear) behavior. Even if the string stretching along its length is itself well within the linear range, the interaction between the bridge and the tailpiece enhances a non-linear aspect of bridge motion.

I believe that the widely accepted changes in sound due to break angle adjustment point to string stretching as a key phenomenon. So that is the subject of this section.

As a simple example of the geometry, consider the stretch required when pulling the center of a taut string sideways. Let the endpoints be fixed and separated by one meter. Then displace the center sideways by one centimeter. The string will have to stretch (about) 0.2 mm or 1/5,000 of its total length. (See FIG. 4.) If there is initially a kink at the center

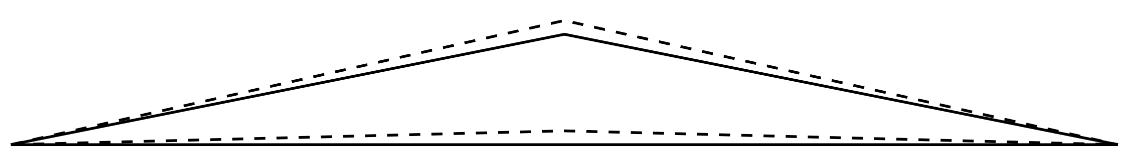

FIG. 4. Additional distortion and stretch (dashed lines), to scale with the numbers in the text

with a displacement of 10 cm from the straight line (which is a break angle of about 11.5°), then the stretch required by an *additional* 1 cm sideways displacement at the center is about 4 mm or about 1/250 of its total length, i.e., 20 times greater than in the previous case. FIG. 5 is closer to an actual banjo. L is the scale length (bridge to nut), l is the bridge to tailpiece distance, and θ_o is the equilibrium (quiet) break angle. The equilibrium length of the string is

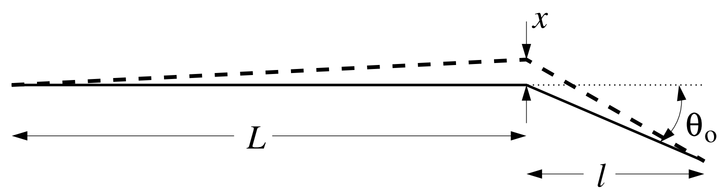

FIG. 5.

$$S_o = L + l \;/\; \cos\,\theta_o\ .$$

If the bridge moves up a distance x, the string must stretch a length ΔS. In practice x is much smaller than l. For example, x could be 0.1 mm and l could be 4 cm. Using that as an approximation, $(x \ll l)$

$$\Delta S \simeq x\ \sin\theta_o$$

Note that $\sin 15^{\rm o} \simeq 0.26$; $\sin 6^{\rm o} \simeq 0.10$; and it gets smaller still for smaller angles: $\sin 1^{\rm o} \simeq 0.02$. The behavior of ΔS as $\theta_{\rm o}$ approaches $0^{\rm o}$ requires a bit more care. In that case, the approximation $x \ll l$ implies that we get a stretch proportional to x^2, i.e., a stretch yet smaller by a factor of x/l.† The stretch that accompanies the normal transverse vibration along the string itself is likewise proportional to the square of the amplitude of that vibration — which is why it is usually ignored in physics descriptions of vibrating strings.

B. String Issues & Details

Changes in string tension start at the bridge and travel along the string at the speed of (longitudinal) sound in that material. For steel, that's about 6100 meters per second. For reference, the speed of sound in air is about 343 m/s. The speed of transverse vibrations along a stretched string of length L tuned to a pitch ν is $2L\nu$. For a banjo string, that might be 270 to 400 meters per second. So it is reasonable in the present discussion to imagine that the tension changes are approximately instantaneous.

All strings on a banjo are subjected to roughly the same stretch by the bridge motion. (That means we're ignoring the simultaneous rocking motion of the bridge; rocking is another story — and actually not inconsequential in creating good banjo sound.) Thicker strings will experience a greater increase in tension for the same stretch. However, the fractional

†The exact formula is $\Delta S = \sqrt{L^2 + x^2} - L + \sqrt{l^2 + (l\ \tan\theta_o + x)^2} - \sqrt{l^2 + (l\ \tan\theta_o)^2}$.

change in pitch is about the same as for thinner strings because it is the ratio of tension to cross sectional area (or diameter-squared) that determines pitch.

Gut, nylon, or other synthetic strings are inherently stretchier than steel. Hence, a given amount of bridge and head motion (which means a given loudness) will produce less pitch change in these non-metallic strings than in steel. That's at least part of why they can't produce as much clang, twang, ping, or crack. (Being thicker, they also damp out the higher frequencies faster, relative to metal strings.)

C. Comparison to Other Instruments

The break angle over a floating bridge produces string stretching. That stretching produces tension variation that is much greater than it would be with zero break angle. In technical terms, the stretch is first order in the small string and bridge amplitudes instead of second order. Hence, this variation can produce a frequency modulation as the bridge moves.

As mentioned before, if the bridge is mounted on a saddle, which, in turn, is fixed to the soundboard, the only string stretching is that associated with vibrations along the string. The up-and-down motion of that bridge does not add any further stretching.

There certainly are other acoustic, stringed instruments with floating bridges, where the bridge moves relative to the fixed ends of the string. These include the violin family, mandolins, and arch-top guitars. The difference is that those instruments' soundboards are made of wood (or a facsimile thereof), not skin or mylar. Their bridges do not move nearly as much as on a banjo. We know this even without a high-tech measurement of those movements. The simple fact is that the banjo is **LOUD**, even though the vibrating area of the soundboard is comparable or smaller than on the competition.† Sound volume comes from the motion of the soundboard and/or the motion of air in and out of the soundbox. So we can safely conclude that a typical plucked string motion on a banjo produces far larger bridge motion than it does on other instruments. This is also evident in the shorter sustain of banjo notes: string energy converts more quickly to sound because of the greater bridge

†A favorite demonstration: rub a finger gently on a banjo head; then rub it similarly on anything else — or listen to the accompanying recording-of-just-that: head and adjoining piece of cardboard, both next to the microphone; each rub is about 4 seconds. (Click on the previous box if you're reading this on a Web-enabled device and the box appears or go to http://www.its.caltech.edu/∼politzer/FM/head-rub.mp3.)

and head motion.

The quintessential banjo features should disappear if we replace the skin with wood. Actually, such instruments exist. We call them banjos because they're played as banjos, but the sound is far more dulcimer-like. Here is the sound of a homemade example with a break angle of $6\frac{1}{2}^{\circ}$ and a head made of $1/8''$ thick plywood: (See FIG. 6.)

http://www.its.caltech.edu/~politzer/FM/hexagon-steel-1.mp3

That's with extra light steel strings. In person, it's relatively quiet, with good sustain and little twang or clang. However, I usually have it strung with fishing line:

http://www.its.caltech.edu/~politzer/FM/hexagon-nylon-2.mp3

It was built from a kit from the Hughes Dulcimer Co of Denver CO. (I believe they're no longer in business.)

FIG. 6. The soundboard is $1/8''$ plywood. *"Ceci n'est pas un banjeau"* — ?

III. HEAD TENSION & PING

So far, the main consequences of using a drumhead for a soundboard follow from its flexibility. It moves much more in response to the up-and-down forces of the strings than wood soundboards do. That makes the banjo loud, and it enhances the floating bridge's impact string length and tension. But I suspect that its own dynamics also are important elements in producing banjo timbre.

The drumhead has ways of introducing further, crucial non-linearities into the sound production. In contrast to string tension, which to a good approximation is uniform over the entire string, head tension is a complex matter, and I do not know of any technical description in the world of musical acoustics that does it justice. But banjo players experience it first hand. Consider an extreme example: if there is a small tear in the head, the tension right at the tear, perpendicular to the torn edge is zero. However, the tension along the tear need not be zero, and it can easily be quite large far from the tear. The tension around the rim is not automatically uniform. In the vicinity of the bridge, it resists attempts to get it that way. The technical term for the tension I am describing is "stress," and the math is complicated because the forces within the head can be in various directions, responding to distortions in various directions. The drumhead modeled in college physics courses is actually a reasonable approximation to a soap bubble film in vacuum. The vibrations of a real drumhead are significantly different.

One important aspect in the present context is the general preference to really tighten those banjo heads down. Apocryphal advice often repeated: Tighten it down until it tears and then back off just a quarter turn. An overly tight head (if it hasn't already torn) will diminish the sound output simply because the tension restricts the motion. But something magical happens as one comes close to that range. The head sounds come alive. That is to say, tightening the head doesn't simply raise the frequencies of all its vibrational modes, all in proportion. That's how physics course drumheads work, and their tone would be recognizably constant as the head is tightened — just with higher pitch. My guess is that this very tight domain on real heads enjoys non-linearities which help create ring and ping.

In the discussion of frequency modulation, I did not explicitly comment on the head response, as if to imply that it behaved much like the strings. But the strings are well within their elastic limit, obeying their own Hooke's Law (force strictly proportional to stretch) for longitudinal stretching. In contrast, the head is almost certainly at or often beyond that point.

The way in which the head pushes back against bridge motion also determines very important aspects of banjo sound. I give a brief introduction here and make clear why this is a very important banjo physics issue. However, I leave for future study the geometry of head stress near the bridge and the impact of non-linearities in head stress and strain.

FIG. 7 suggests a simple metaphor for a small part of the string–bridge–head system. The central mass M represents the bridge. Strings push down from the top, represented by the upper spring of "spring constant" k. Their *extra* force F when the bridge moves a

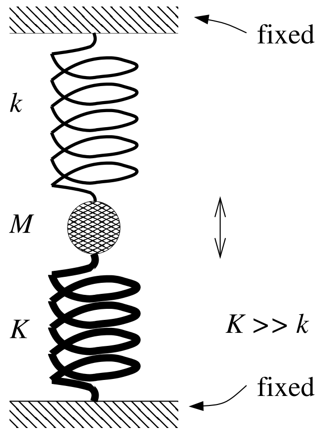

FIG. 7. With very different strength springs, the stiffer one dominates the oscillations.

distance x away from equilibrium is kx. The head pushes up on the bridge from below. And the head is represented by the spring with constant K.

For a *single* spring, if a mass m is attached to one end of the spring and the other is fixed and the mass is displaced from equilibrium and let go, it will oscillate with a frequency ν given by $\nu = 2\pi\sqrt{k/m}$.

In the two spring system of FIG. 6 with the outer spring ends kept fixed, the effective total spring constant is $K + k$. If M is displaced a distance x from its equilibrium position, the springs push back with a total force of $(K + k)x$. And the frequency of free oscillations is $2\pi\sqrt{(K + k)/m}$. If the bottom spring is much stiffer than the top one, then $K \gg k$, and that means that, to a first approximation $K + k \simeq K$.

Just by comparing the deflection or break angle of a banjo head at the foot of the bridge to the break angle of the strings, we can safely conclude that the K of the head is, indeed, much bigger than the k of all the strings combined. At equilibrium, the string and head forces on the bridge are of equal magnitude but opposite direction. They exactly cancel. But, as the bridge moves, the force of the head is a significant determinant of its motion. And, as already mentioned, the relation of that force to the displacement of the bridge away from its equilibrium position is not simple and likely non-linear.

As a teaser of what might be lurking in the dynamics of the head, consider the simplest deviation from ideal spring-like behavior that the head might exhibit. In particular, it is likely that the head has been stretched so tight that typical bridge motions explore its non-linear response. When the bridge goes down a distance x, the extra upward force of the head on the bridge might be greater than the decrease in force that would come with the bridge going up a distance x.

The following audio sample gives a hint of what that can do to the sound. The first note is a pure sinusoidal oscillation, modulated in amplitude to imitate a pluck, i.e. an immediate turn-on followed by an exponential decay. The second sound is the behavior of an oscillator whose restoring force gets stronger on one side of equilibrium and weaker on the other. (E.g., $F = -kx - \beta x^2$.)

http://www.its.caltech.edu/~politzer/FM/sine-stiff-head-pluck.mp3

The second pluck's wave form in time is asymmetric, spending more time above than below equilibrium in each cycle. In terms of frequency spectra, from about 1000 to 6000 Hz the even harmonics are relatively weaker than the odd ones for the sine wave with its pluck amplitude envelope. All integer harmonics are present in the non-linear, anharmonic case.

IV. IN CONCLUSION

A banjo is identified as such by its floating bridge and drumhead soundboard. What I have tried to do in this note is explore the mechanics of how string stretching over the floating bridge can contribute to its characteristic sound. It is also clear that the forces of the head on the bridge are another potential source of crucial non-linearity and deserve further study.

V. ACKNOWLEDGEMENTS

I would like to thank Greg Deering and the Deering Banjo Company® for encouragement and support for the pursuit the physics of banjo sound, thank Frank Rice of Caltech for constructive observations, and thank my wife, Maria, who always has a sympathetic ear, asks good questions, and often suggests good answers.