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# Homework Assignment #2

( Due: **Friday, 18 Oct 2019 by midnight** / Please e-mail your work directly to rmkatti@caltech.edu )

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1. As a measure of power level in electrical circuits, the dBm is very useful for practical measurements. As you know, decibels are logarithmic ratios, and for power can be expressed as
# dB's = 10 $\log_{10}(P / P_{ref})$

   For dBm, this ratio is specified as the power relative to a reference level of one milliwatt: $P_{ref}$ = 1mW. Hence, a power level, $P$, of 1 mW corresponds to 0 dBm, 1 μW corresponds to -30dBm, and 1 nW corresponds to -60dBm, *etc*.

   Given the following resistors at room temperature (*T*=300K), calculate the thermal noise power (in dBm); and the sq-rt noise voltage noise spectral density of these resistors, and the noise voltage with the given bandwidth values (BW) shown.

   a) *R*=50Ω, BW=1 Hz;

   b) *R*=1kΩ, BW=1 kHz;

   c) *R*=2MΩ, BW=10 kHz.

   d) Please comment on the *relative* thermal noise powers you have calculated for these different resistors.

   e) Review the slide entitled "*Nyquist's Derivation*" from Lecture 3. Should there be a factor of 4 in your formula for noise power? Why or why not?

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2. **Scenario:** I perform frequency response measurements on the best amplifier in my laboratory and determine that its bandwidth is flat from 1Hz-1MHz. To evaluate the amplifier's voltage noise RTI, I then short circuit its input port, and measure the voltage noise at the amplifier's output. I observe that its r.m.s. magnitude is 100μV. The manufacturer assures me in the owner's manual that the amplifier's noise is <u>white</u> over its aforementioned bandwidth, that its power gain is 60dB, and that its input impedance is 100MΩ in parallel with 3pF.

   a. What is the voltage noise spectral density RTI for this amplifier?

   b. I now remove the short and replace it with a 10kΩ resistor that has been cooled to 40mK=0.04K (again placing it across the amplifier's input terminals). Upon doing this I see the r.m.s. voltage noise at the amplifier's output go up by a factor of two. You've heard in the lectures the instructor assert that we can model this effect as arising, in part, from a white *current noise* spectral density that emanates outward from the amplifier's input... a kind of backaction from the amplifier.

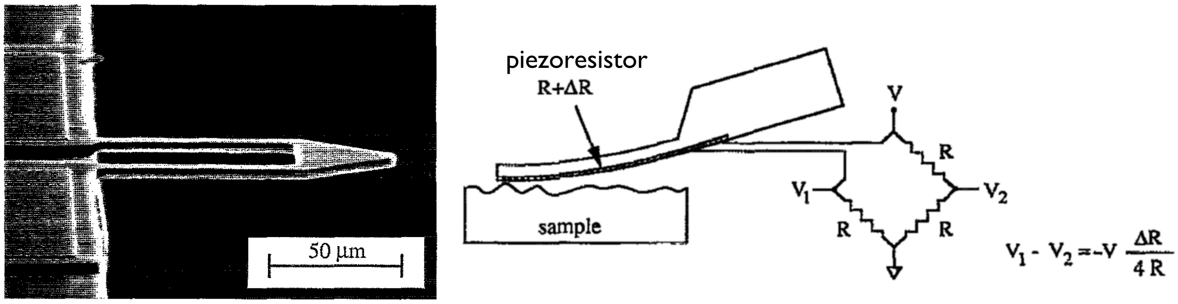
   c. Does what I said make any sense? Why? Draw a little circuit diagram to explain your arguments with representative noise generators hooked to the input and output terminals.

   d. What is the magnitude of the sq-rt current noise spectral density emanating from the noise source at the input terminals that is responsible for this effect?

   e. If this is really happening the current noise source will heat up the 10kΩ resistor! How much power does this noise source spew <u>into</u> this resistor?

   f. Does this amount of backaction power seem significant to you? Why or why not?

3. **Optimization of force sensing using a piezoresistive nanocantilever.**




**Piezoresistive cantilever for atomic force microscopy.** *(left)* Scanning electron micrograph of a piezoresistive cantilever. *(right)* schematic diagram of the piezoresistive detection scheme for AFM.

The figures above are from early atomic force microscope (AFM) research in Professor Cal Quate's group at Stanford, where AFM was first invented [Specifically, the figures are from: M. Tortonese *et al.* Applied Physics Letters **62**, 834 (1993)]. In AFM with a *piezoresistive cantilever*, which was developed later (about the time of this work), the microscopic variations in surface topology are sensed as a sample is scanned beneath a sharp tip at the cantilever's distal end. The forces of interaction between the sharp tip and the sample surface induce minute cantilever deflections. If the cantilever is thin enough the cantilever becomes very sensitive; forces associated with atomic interactions between individual atoms can then be measured. Our analysis proceeds as follows: We first calculate the relationship between applied force, $F_x$, which is the independent variable, and the resulting deflection at the cantilever's distal end (it's "tip"), $\delta x = x - x_0$. Here, $x$ is the deflected position of the cantilever tip, and $x_0$ is its initial position before the force is applied. We then relate strain at the bottom surface of the cantilever (dark shading in the figure above, at the right) to its deflection at the tip. For this method of displacement transduction to work, the piezoresistor must be localized to just one (here, the bottom) surface. (Strain in both surfaces, being of opposite sign, will cancel.) Hence, we need to evaluate the deflection-induced surface strain to determine the change in the piezoresistor's resistance resulting from deflection. So…

For a simple "diving board"-like cantilever the relationship between displacement (at the tip) and applied force (also at the tip) is

$$\delta x = \frac{l^3}{3EI} F_x,$$

where the cantilever's moment of inertia, *I*, can be expressed as

$$I = \frac{wt^3}{12}.$$

Here, $\delta x$ is the deflection, $l$ is the length, $t$ is the thickness, $w$ is the width and $E$ is Young's modulus. Since $F_x = -k\,\delta x$, we can calculate the stiffness (spring constant) as

$$k = \frac{1}{4} E\, w \left( \frac{t}{l} \right)^3$$

For a small deflection, $\delta x \ll l$, the cantilever acquires an angle of deflection of approximately

$$\delta\theta \simeq \frac{\delta x}{l},$$

and therefore, a radius of curvature $r_c$ that is approximately given as

$$r_c = \frac{l}{\delta\theta} \simeq \frac{l^2}{\delta x}.$$

The average length change of the cantilever is the difference in arc length between the deflected cantilever's upper and lower surfaces,

$$\Delta l = l_{upper} - l_{lower} = (r_c + t)\delta\theta - r_c\delta\theta = t\delta\theta = \frac{t\delta x}{l}.$$

Since, for small deflections the strain is equal and opposite at the top and bottom faces, the surface strain at one face is half the above value,

$$\varepsilon = \frac{\Delta l}{l} \simeq \frac{t}{2l^2}\delta x,$$

which we can rewrite in terms of the applied force,

$$\varepsilon \simeq \frac{t}{2kl^2}F_x \simeq \frac{2l}{Ewt^2}F_x.$$

The "gauge factor", $\gamma$, relates induced fractional resistance change to strain,

$$\frac{\Delta R}{R_0} = \gamma\varepsilon = \gamma\frac{2l}{Ewt^2}F_x.$$

Here, $R_0$ is the unstrained piezoresistor resistance. Although early AFM focused on direct contact interactions between the cantilever tip and surface, later work began to focus on detection of miniscule indirect forces that exist between the tip and surface when they are in close proximity – arising from van der Waal's, electrostatic, and Casimir interactions. For such operation, resonant (a.c.) detection is preferable because of the signal boost given for large *Q*'s. (This will be described in class.)

Nanocantilevers can be much more sensitive than the microcantilevers used in early AFM studies. Some time ago, my group determined that metallic piezoresistive (PZM) detection works well as cantilever size is scaled into the nano regime [Li, M., Tang, H.X., & Roukes, M.L., Ultra-sensitive NEMS-based cantilevers for sensing, scanned probe and very high-frequency applications. Nature Nanotechnology **2**, 114-120 (2007).] Let's consider optimization of a hypothetical system to read out minute forces from such a nanocantilever, and ask *"what is the minimum detectable force that is achievable?"*

<u>Now, on to the homework problem!</u>

Assume the nanocantilever has simple diving-board geometry with the following characteristics: $t$ = 50nm, $l$ = 5 μm, $w$ = 200 nm, $\gamma = 2$, and $Q$=10,000 when measured in vacuo at 300K. Also assume it is made from monocrystalline Si, which has $E$ = 2 x 10¹¹ N/m² and $\rho = 2330\ \text{kg/m}^3$.

a) The cantilever's resonant frequency is given as $f_0 = \frac{1}{2\pi}\alpha\sqrt{\frac{E}{\rho}}\frac{t}{l^2}$ where, for the fundamental mode, $\alpha = 0.1615$. What is the cantilever's frequency $f_0$ (in Hz) and force constant, $k$ (in N/m)?

b) At finite temperatures, thermal fluctuations drive stochastic displacement of the device. The fluctuation-dissipation theorem (FDT) establishes the ultimate displacement-noise floor of the cantilever – that arising solely from these so-called thermomechanical fluctuations. Dividing this FDT prediction by the cantilever's (resonant) amplitude response function (as was discussed earlier in class) allows determination of its effective noise RTI in the force domain. On resonance, this yields the sqrt-spectral density of the cantilever force noise,

$$S_F^{\ 1/2}(f_0) = \sqrt{\frac{2k_B T\, k}{\pi\, f_0 Q}} \qquad \left[\mathrm{N}/\sqrt{\mathrm{Hz}}\right].$$

c) This quantity tells us how big an a.c. force on the cantilever is required, on resonance, to overcome its intrinsic thermomechanical fluctuations. What is the value of $S_F^{1/2}(f_0)$ for the cantilever?

As always, a practical measurement system will never precisely attain noise levels given by thermodynamic fluctuations – but, with careful design, we can get close. As described in class, we must also consider transducer noise and readout system noise. Let's first consider the former.

Assume our metallic piezoresistor has a resistance of 100Ω.

d) What is its thermal voltage noise (Johnson noise) spectral density?

e) We must bias the piezoresistor with a d.c. current to transduce the displacement-induced time-varying strain, which results in a time-varying resistance, into a time-varying voltage. The magnitude of this transduced noise voltage – which relates directly to the thermodynamic displacement fluctuations – will be directly proportional to the bias current. At what magnitude of this bias current will, in the voltage domain, the transduced on-resonance displacement fluctuations equal the piezoresistor's Johnson noise? Is the current required to achieve this of reasonable magnitude to apply to the nanoscale piezoresistor? Why or why not?

Now, let's consider the amplifier's noise RTI (*i.e.,* in the force domain).

f) With special matching techniques (to be described in a future lecture) it is possible to match this transducer with a readout amplifier characterized by a total noise (voltage plus current noise) of order ~100 pV/√Hz. Considering this amplifier's noise alone, what is its equivalent contribution in the force domain (i.e. RTI) at the bias current deduced above in part (b)?

Now, let's evaluate the performance of the complete, cascaded measurement system.

g) What is the <u>total</u> force noise RTI – from all aforementioned sources: thermomechanical noise, transducer noise, and amplifier noise – for this measurement system.

h) What is the dominant noise source in this system?

i) How can we improve the system's performance? Can you think of any limitations there might be to attain such improvement?