1 Estimation

When faced with a new situation, a new idea, or a surprising statement, it is valuable to be able to quickly estimate the rough importance or consequences of things. If you always make painstaking, exceedingly accurate estimates, you will waste a lot of your life, since most of your ideas will turn out to be unimportant (as the 18th century Viscount Bolingbroke remarked "Truth lies within a little and certain compass, but error is immense."). So the better strategy is to have lots of ideas, reject the bad ones by the quickest, dirtiest estimates you can, and only follow up with more careful, timeconsuming estimates in the rare cases when the initial rough estimate suggests the idea could be important.

To illustrate the point: a mathematician, an engineer and an order of magnitude physicist are sitting in a car which uses 0.1 liter of gasoline per kilometer driven, and has a 60 liter tank, recently filled with the last of their rubles. They are located at longitude $l_1 = +84°34'15.2''$, latitude $b_2 = 56°36'05.3''$, and trying to get to a party located at longitude $l_2 = -118°08'13.1''$, latitude $b_2 = 34°07'02.3''$. The mathematician says. "We can drive for 600 kilometers on this tankful of gas. Let's see how far away our destination is. I'll assume a spherical earth, convert the degrees, minutes and seconds to radians, and compute the great circle distance d to our destination from spherical trigonometry, $d = R_\oplus \cos^{-1}[\sin(b_1)\sin(b_2) + \cos(b_1)\cos(b_2)\cos(l_2 - l_1)]$." She boots her computer, and prepares to compute the result. The engineer grabs the computer and says -"Hey dummy, roads never follow great-circle routes. We need to get the total distance along real roads. I'll just see if the wireless works here, and we can get driving directions and add the distances from one of those internet mapping services." The order of magnitude physicist, noticing that the longitude difference is nearly 180 degrees, says calmly: "Stop wasting time you two, and help me look for signs to the airport. We're in the middle of siberia, and our party is obviously on the other side of the earth, many thousands of kilometers away. So we don't have enough gas to drive, and even if we did, there hasn't been a land bridge across the Bering Straight for thousands of years."

There are several methods for making quick estimates, and they all require lateral thinking, which, though fun, is an art not a science. The methods can roughly be divided into three main categories: direct multiplication of approximate numbers, sampling, and scaling.

Disclaimer: the next two sections do *not* imply recommended courses of action for unemployed or poorly paid physicists. They are included purely for entertainment value.

1.1 Armed robbery

Before deciding whether to rob a Brinks armored car outside your bank, it may be useful to estimate how much money is in the armored car.

1. The volume of $ 20 bills

   A ream (500 sheets) of paper is about 5 cm thick. Therefore, the volume V of one dollar-denominated note would be about $V = 6 \text{ cm} \times 15 \text{ cm} \times 10^{-2} \text{ cm} = 1 \text{ cm}^3$. The density of paper, like the wood it comes from, is about that of water, so

$$m = \rho V = 1\text{g} \tag{1}$$

2. The volume of the armored car

$$2\text{m} \times 2\text{m} \times 2\text{m} = 10\text{m}^3 = 10^7\text{cm}^3 \tag{2}$$

So if it were full to the ceiling of the \$20's ATMs dispense, it would contain $10^7 \times$ cm$^3 \times$ 1cm^{3}bill$^{-1} \times$ \$20bill^{-1} = \$2 $\times 10^8$. However, 10^7 cm^3 = 10 tons of paper. The armored trucks are not much bigger than a pickup truck, and they have to carry the weight of the armor too. So the trucks can probably not be filled with money —they are mass-limited..

3. We guess the trucks, like heavy pickups, can carry 2 tons = 2000kg = 2×10^6g. Under this assumption, the amount of money they could carry is 2×10^6 one gram bills, or

$$2 \times 10^6\text{g} \times \$20\text{g}^{-1} = \$40 \text{ million in \$ 20's} \tag{3}$$

There are a number of reported successful heists of around \$ 20 million. The largest foiled heist, of a truck carrying money between two Federal Reserve banks, was \$ 50 million (in Florida, 2003), so our estimate seems to be about right.

Now you want to take your money and retire to a nice tropical island with no extradition treaties. So you might consider how much money can be carried on, say, a Boeing 777 aircraft. A 777 carries 400 people $\times$ 100 kg, including the luggage, which amounts to 4 $\times 10^7$ g. So, it can carry $4 \times 10^7 \times \$20$ = \$800 million in \$ 20's,or \$ 40 million in \$ 1's.

Would it be better to carry gold than bills? Gold is \$ 425/oz, 1 oz = 28 g, which means that gold is worth \$ 15/g. So, \$ 20's are better!

A Boeing 777 costs \$70 million, so if you have only \$ 1's, you might as well just steal an empty airplane and resell it when you get to that nice place with no extradition treaty with the US.

1.2 Dying in an auto accident

How would you estimate the probability that you will die in a traffic accident? One way is to collect all the police accident reports filed in your state or country for a few years, and compare to the number of licenced drivers. But that is a huge amount of work and in many fields not practical since no one may have collected the data, or the data may not be reliable or complete.

Another, much more convenient way to do this is by the "*sampling*" method. This works well if your sample is really typical, and not too small. After you get an answer, but before you put a lot of trust in it, it is wise to think for a while about how typical your sample really is, whether there are selection effects in your memory or source of information, and to check the statistics to get an idea of what is a reasonable range of means, given your sample size.

1. How many people do you know?

2. How many people do you know who have died in car crashes?

3. How long have you been paying attention to this sort of thing?

$\frac{2}{1 \times 3}$ is an estimate of the probability of dying in a car crash per unit time.

Two estimates

1. In my high school, there were 800 kids among whom two were killed in car accidents during the 4 years I was there.

$$\frac{2}{800} \frac{1}{4\text{yrs}} = \frac{1}{1600\text{yrs}} \tag{4}$$

These were rural teens, in the days before seatbelt laws were enforced, so one might hope the average rate now is lower.

2. Caltech faculty

There are about 270 faculty members at Caltech. None of them so far as I know died in a car accident in last 20 years, but three of their family members died in the same period. Therefore, including their spouses and the children in the total sample,

$$\frac{3}{270 \times 4} \frac{1}{20\text{years}} \simeq \frac{1}{7000\text{years}} \tag{5}$$

This was in an urban environment, including a population of adult drivers, with the seatbelt laws enforced. So one might hope that this sample is more representative of the death rate than a sample of rural teenagers.

Aside on statistics The number of 'hits' in our samples are quite low (2 in the first estimate, 3 in the second). So we suspect that those estimates are quite susceptible to Poisson fluctuations. We can see how much by using Bayes' theorem to estimate the probability of getting $x = k$ ($k = 2$ or 3 in the two samples) in drawing from a Poisson distribution with mean $m = \mu$:

$$P(m = \mu | x = k) = \frac{P(m = \mu)P(x = k | m = \mu)}{\sum_j P(m = \mu_j)P(x = k | m = \mu_j)} \ . \tag{6}$$

If we take a uniform prior ($P(m = \mu)$ independent of μ for μ taking on continuous values between 0 and infinity), and using Poisson statistics $P(x = k | m = \mu) = \mu^k \exp(-\mu)/k!$, we get

$$P(m = \mu | x = k) = \frac{P(x = k | m = \mu)}{\sum_j P(x = k | m = \mu_j)} = \frac{\mu^k \exp(-\mu)}{\int_0^\infty \mu_j^k \exp(-\mu_j)\, d\mu_j} = \mu^k \exp(-\mu)/k! \tag{7}$$

Integrating this over various ranges of μ, we find that for $k = 2$ (highschool estimate), the most probable $\mu = 2$, the median $\mu = 2.7$, and 90% of the time $0.9 < \mu < 6.5$. For $k = 3$ (Caltech estimate), the most probable value is $\mu = 3$, the median $\mu = 3.7$, and 90% of the time $1.4 < \mu < 8$. So the 90% sampling error in our small-sample estimates is a factor of about 2.5 either way, good enough for order of magnitude but not more.

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Reality check   Let us take estimate 2. It gives a ~ 1 % probability over a 70-year lifetime that someone will die in a traffic accident. Let's compare this to reported complete sample numbers. In the U.S. the average traffic fatality rate is reported to be $\sim 15/100,000/\text{yr}$. Multiplying the rate by 70 years, we get a 1 % probability of death by car in a lifetime. In Wyoming and Mississippi the average is 30/100,000/yr. Those states do not have seatbelt laws and consequently have far more deaths in single vehicle rollovers and ejections than other states.

1.3   Caltech budget vs. tuition

What fraction of the Caltech budget is due to the tuition you pay?

Let's estimate the Caltech budget. We guess that the largest fraction is salaries rather than external purchases:

300 professors at $ 150 k	$ 45 M
1500 Ph.D. staff/postdocs at $ 45 k	$ 67 M
1200 graduate students at $ 24 k	$ 30 M
2000 staff (technicians, secretaries, janitors, gardeners, etc.) at $ 40 k	$ 80 M
Total	$ 222 M/yr

There are about 800 undergraduate students and the tuition is $ 33 k/y. If they all paid the full amount, it would give $ 27 M, but the actual amount is only about $\frac{1}{3}$ of that, since about $\frac{2}{3}$ is financial aid.

To this one should add costs of electricity, heating, water, paint, furniture, books and journals, scientific equipment, etc. One should also add retirement and medical benefits. We won't estimate those here. But one very visible additional item are the buildings.

How much are the buildings worth and what is the cost to maintain them? The new astronomy building, Cahill, is to cost about $50 M and have 100,000 square feet, which gives $ 500/sf. Let us estimate how much square feet of buildings we have.

$ 800 people $\times$ 200 sf/person $= 10^6$ sf $\times$ $ 500/sf $=$ $ 500 M in buildings. There is depreciation/maintainence cost for buildings. If we amortize their costs over 20 years, we guess Caltech is spending a minimum of $\rightarrow$ $ 25 M/yr to maintain/rehab the existing buildings.

Reality check: The actual 2003 – 4 report:

Academic support	$ 214 M
Organized research	$ 189 M
Caltech research	$ 37 M
Off-campus research	$ 33 M
	$ 470 M

Thus we see that our salary estimates account for about half of the budget, in line with the usual rule of thumb that if buildings are 'free', overhead is about equal to direct salary costs.

Sources:

Federal grants	$ 260 M	(direct and overhead)
Gifts & endowment	$ 190 M	
Net tuition (graduate & undergraduate)	$ 17 M	(cf. $ 54 M without the aid)
	$ 467 M	

The reported value of the buildings (including lab equipment and furnishings) is $ 650 million. So our guesstimate wasn't far off.

Conclusion   The tuition you pay is almost in the noise, but we love you for yourselves, not for your money. At least not your present money; the Development office's secret dream is that one of you will be the next Arnold Beckman or Gordon Moore (Moore's latest $ 600 M gift represents about 35 years worth of total tuition income!). Note that because of this donation dependence, the small number of alumni, and Pareto's law, Caltech's budget is quite susceptible to Poisson fluctuations.

1.4   Hurricane evacuation

Can one evacuate the Florida coast with a couple of days notice before a hurricane? First, let us specify the question a little.

1. How many people live within 10 km of the coast?

   The population is more dense near the beach, and so we estimate that about 4 million people live within 10 km of the coast.

2. If we have 2 days of warning, can one move 4 million people in 2 days?

   Note that most of them probably have to be moved $\gg$ 10 km to be able to find housing. Not much point marching them 10km inland to stand in the wind and rain for several days.

Assuming that no one would like to go out in boats, we are left with two modes of transportation. First, let us consider airplanes. In a major airport, one plane can take off about every 2 minutes. If a plane carries 200 people,

$$\Rightarrow 200 \times \frac{30}{\text{hr}} \times 48\text{hrs} = 300,000\text{people} \ll 4\text{million} \tag{8}$$

So, most people have to be moved out by cars. Say roads have 10 lanes each direction. If the incoming lanes are reversed for the emergency, and everyone follows the DMV's recommended 2-second rule for maintaining a safe distance to the next car, at speed V

$$10\text{lanes} \times 2 = 20\text{lanes} \times \frac{V\text{mph} \times 4\text{people/car}}{2\text{s} \times V\text{mph}} \times 48\text{h} = 7\text{million} \tag{9}$$

With an orderly movement over two days, this suggests one could *just* do it. But you know that everyone will not be orderly nor agree to space their departures evenly over the 48 hours, etc. So it is a daunting, but probably possible task.

Notice that the traffic speed V cancelled out of the above calculation, since the flux of cars per lane (speed V of cars divided by distance between cars)

$$\frac{V\,\mathrm{mph}}{\frac{2}{3600}\mathrm{h}\cdot V\,\mathrm{mph}} = 1800\mathrm{h}^{-1} \text{ per lane} \tag{10}$$

independent of the car speed V. This is part of the explanation for why freeways can sustain those mysterious 'shock waves' of traffic jam, where you roar in at 60mph, crawl at 10 mph for several minutes, and then the jam mysteriously ends and you roar off at 60mph at the end. There are several speed states with virtually the same flux of cars, and small perturbations can trigger transitions from one state to the other.

Walking would work, too, if we needed to move the people only 10 km inland. The same two second rule at 4mph walking speed gives a spacing of 4 people every 10 feet (3 meters) per lane. So you could evacuate everyone in a nice uncrowded 10km stroll down the highways and byways. But they'd probably need to go farther to find housing, and they'd probably want to carry suitcases, infants, pets etc, which would make walking more problematic.

2 Scaling

Quite often one can avoid multiplying lots of numbers (which may or may not be known) in an estimate by simply *scaling* from a known situation to an unknown one.

2.1 Estimating escape velocities from planets and moons

$$g = \frac{GM}{R^2} \tag{11}$$

Since $M \propto \rho R^3$,

$$g = \text{constant} \cdot \rho R \tag{12}$$

and

$$v_{\mathrm{esc}} = \sqrt{\frac{2GM}{R}} = \sqrt{2gR} \propto \sqrt{\rho} \cdot R \tag{13}$$

$G = 6.67 \times 10^{-8}$ in cgs. So if you remember g and the Earth's radius[1] you can estimate its escape velocity. To get the escape velocity from other solar system bodies, you can remember that they are all about the same (rock, ice) density within a factor of a few, so escape velocity will be roughly proportional to the radius.

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[1] There are many ways to reverse engineer the earth's radius if you forget the number: The meter was originally defined (in 1791) as 10^{-7} of the earth's pole to equator distance, i.e. about $10^{-7}\pi R_{\oplus}/2$. So $R_{\oplus} \simeq 2 \times 10^7/\pi = 6.4 \times 10^6$ meters. The nautical mile was defined (in 1670) to subtend 1 arcminute$=(1/60)(\pi/180) \simeq 1/3400$ radian on the surface of the earth (near the English Channel, if you want to get picky about oblateness), so the radius of the earth is $\simeq 3400$ nautical miles. A nautical mile is a bit longer than a statute mile (1.852km instead of 1.61km). A more approximate method is to recall that the USA covers 3 times zones (3/24 of a circle of longitude) and is about 4000km (2500 miles) across at latitude 40 degrees. So $2\pi R_{\oplus}\cos(40°)(3/24) = 4000$km, which also gives a pretty good value for $R_{\oplus}$. You should be able to figure out several other ways to estimate it using miscellaneous facts you know.

However, even if we do not remember numbers, we can determine relative numbers by scaling: e.g. the gravity or escape velocity of the moon relative to those of earth:

$$g_{\text{moon}} = \frac{\rho_{\text{moon}}}{\rho_\oplus} \frac{R_{\text{moon}}}{R_\oplus} g_\oplus \tag{14}$$

$$v_{\text{esc,moon}} = \sqrt{\frac{\rho_{\text{moon}}}{\rho_\oplus}} \frac{R_{\text{moon}}}{R_\oplus} v_{\text{esc},\oplus} \tag{15}$$

Since

$$R_\oplus = 6400\text{km} \tag{16}$$

and

$$R_{\text{moon}} = 1700\text{km} = \frac{1}{4}R_\oplus \ , \tag{17}$$

it follows that if earth and moon had the same density we'd expect the moon's escape velocity and surface gravity to be about a quarter those of the earth. Actually the moon is mostly rock, without the iron core of the earth, so it has a somewhat lower density $\bar{\rho}_{\text{moon}} = 3.3\text{g cm}^{-3}$ compared to earth ($\bar{\rho}_\oplus = 5.5\text{g cm}^{-3}$). Thus the exact scalings of equations 14 and 15 give $g_{\text{moon}} = (1/6)g_\oplus$ and $v_{\text{esc,moon}} = (1/5)v_{\text{esc},\oplus}$

2.2   Jumping off an asteroid

We can use scaling to answer a related question: what is the largest asteroid from whose gravitational pull you can escape just by jumping? We'll here ignore interesting questions of friction and muscle tension in low gravity environments, and assume that the jump in question is vertical, giving you an initial kinetic energy equal to a fixed fraction of the energy stored in your flexed muscles. In this approximation your initial velocity v_i should be independent of the size of the body you are jumping on. We scale this to earth, by supposing that jumping on earth you can raise your center of mass by a distance h, say $h \sim 30\text{cm}$. The initial velocity is given by

$$v_i^2/2 = g_\oplus h = G4\pi/3\rho_\oplus R_\oplus \ . \tag{18}$$

To escape from an asteroid of mass m, radius r and density ρ requires

$$v_i^2/2 > Gm/r = G4\pi/3\rho r^2 \ . \tag{19}$$

Substituting the right side of equation (18) into the left side of (19) gives

$$r^2 < R_\oplus h(\rho_\oplus/\rho) \ . \tag{20}$$

Since asteroids have about the same density as the earth, we get the elegant result that the maximum radius of an asteroid you can jump off is about the geometric mean of the radius of the earth and the height you can jump on earth, i.e. about $\sqrt{6.4 \times 10^6\text{m} \times 0.3\text{m}} \simeq 1\text{km}$.