How to prepare for the midterm

- The midterm will be based on Chapters 1-5 and sections 6.1-6.7. It will be one-hour, take-home, open-textbook and open-notes exam.
- Read “Review and Summary” after each Chapter. Brush up on topics that are not familiar.
- Make sure you know how to solve HW problems and sample problems. It is useful to review all sample problems, or at least 2.9, 3.4, 3.5, 3.7, 4.2, 4.3, 4.4, 4.5, 5.1, 5.2, 5.4, 5.6, 5.9, 5.10, 6.1, 6.2.
- Review important tables/formulae from the book (such as supports and their reactions) so that you can use them easily.
- Remember, the correct reasoning and an error in computation will get you most of the points. However, the right answer with no explanation will get you no points, unless the problem specifically asks for an answer only.
- Do not forget about the honor code. Carefully read the instructions on the front page of the midterm. You cannot discuss anything about the midterm until after the due date.
- The rest of this handout, is a brief summary of important topics we have learned so far.

Force defined by its magnitude and two points

Often the force is defined by its magnitude $F$ and by two points on its line of action, $M(x_1,y_1,z_1)$ and $N(x_2,y_2,z_2)$

$$\vec{F} = F \vec{\lambda}$$

$$F_x = \frac{Fd_x}{d} \quad F_y = \frac{Fd_y}{d} \quad F_z = \frac{Fd_z}{d}$$
Moment of a Force About a Point

- A force vector is defined by its magnitude and direction. Its effect on the rigid body also depends on its line of action.
- The moment of $F$ about $O$ is defined as $M_O = r \times F$
- The moment vector $M_O$ is perpendicular to the plane containing $O$ and the force $F$.
- Magnitude of $M_O$ measures the tendency of the force to cause rotation of the body about an axis along $M_O$.
  $M_O = rF \sin \theta = Fd$
  The sense of the moment may be determined by the right-hand rule.
- Any force $F'$ that has the same magnitude and direction as $F$, is equivalent if it also has the same line of action and therefore, produces the same moment.

Rectangular Components of the Moment of a Force

The moment of $F$ about $O$,

$$M_O = \bar{r} \times \vec{F} = (\vec{r} \times \vec{F}) \cdot \hat{z}$$

$$\vec{F} = F_x \hat{i} + F_y \hat{j} + F_z \hat{k}$$

$$M_O = M_x \hat{i} + M_y \hat{j} + M_z \hat{k}$$

$$M_x = \begin{vmatrix} i & j & k \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix}$$

$$M_y = \left( yF_z - zF_y \right) \hat{i} + \left( zF_x - xF_z \right) \hat{j} + \left( xF_y - yF_x \right) \hat{k}$$
Vector Mechanics for Engineers: Statics

Scalar Product of Two Vectors: Applications

• Angle between two vectors:
  \[ \vec{P} \cdot \vec{Q} = PQ \cos \theta = P_xQ_x + P_yQ_y + P_zQ_z \]
  \[ \cos \theta = \frac{P_xQ_x + P_yQ_y + P_zQ_z}{PQ} \]

• Projection of a vector on a given axis:
  \[ P_{OL} = P \cos \theta = \text{projection of } P \text{ along } OL \]
  \[ \vec{P} \cdot \vec{O} = PQ \cos \theta \]
  \[ \frac{\vec{P} \cdot \vec{O}}{Q} = P \cos \theta = P_{OL} \]

• For an axis defined by a unit vector:
  \[ P_{OL} = \vec{P} \cdot \hat{a} \]
  \[ = P_x \cos \theta_x + P_y \cos \theta_y + P_z \cos \theta_z \]

Vector Mechanics for Engineers: Statics

Mixed Triple Product of Three Vectors

• Mixed triple product of three vectors,
  \[ \vec{S} \cdot (\vec{P} \times \vec{Q}) = \text{scalar result} \]

• The six mixed triple products formed from \( \vec{S}, \vec{P}, \) and \( \vec{Q} \) have equal magnitudes but not the same sign,
  \[ \vec{S} \cdot (\vec{P} \times \vec{Q}) = \vec{P} \cdot (\vec{Q} \times \vec{S}) = \vec{Q} \cdot (\vec{S} \times \vec{P}) \]
  \[ = -\vec{S} \cdot (\vec{Q} \times \vec{P}) = -\vec{P} \cdot (\vec{S} \times \vec{Q}) = -\vec{Q} \cdot (\vec{P} \times \vec{S}) \]

• Evaluating the mixed triple product,
  \[ \vec{S} \cdot (\vec{P} \times \vec{Q}) = S_x(P_yQ_z - P_zQ_y) + S_y(P_zQ_x - P_xQ_z) \]
  \[ + S_z(P_xQ_y - P_yQ_x) \]
  \[ = \begin{vmatrix} S_x & S_y & S_z \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{vmatrix} \]
Moment of a Force About a Given Axis

- Moment $M_O$ of a force $F$ applied at the point $A$ about a point $O$,
  $$
  M_O = \vec{r} \times \vec{F}
  $$

- Scalar moment $M_{OL}$ about an axis $OL$ is the projection of the moment vector $M_O$ onto the axis,
  $$
  M_{OL} = \lambda \cdot M_O = \lambda \cdot (\vec{r} \times \vec{F})
  $$

- The moment $M_{OL}$ of $F$ about the axis $OL$ measures the tendency of the force $F$ to impart a rigid body rotation about the axis $OL$.

Moment of a Couple

- Two forces $F$ and $-F$ having the same magnitude, parallel lines of action, and opposite sense are said to form a couple.

- Moment of the couple,
  $$
  \vec{M} = \vec{r}_A \times \vec{F} + \vec{r}_B \times (-\vec{F})
  = (\vec{r}_A - \vec{r}_B) \times \vec{F}
  = \vec{r} \times \vec{F}
  $$
  $$
  M = rF \sin \theta = Fd
  $$

- The moment vector of the couple is independent of the choice of the origin of the coordinate axes, i.e., it is a free vector that can be applied at any point with the same effect.
Two couples will have equal moments if

- \( F_1d_1 = F_2d_2 \)
- the two couples lie in parallel planes, and
- the two couples have the same sense or the tendency to cause rotation in the same direction.

Two systems of forces are equivalent if we can transform one of them into the other by one or more of the following operations:

- replacing two forces acting on the same particle by their resultant;
- resolving the force into components (including attaching two equal and opposite forces to the same particle);
- moving the force along its line of action.

Two couples that have the same moments are equivalent.
System of Forces: Reduction to a Force and Couple

- A system of forces may be replaced by a collection of force-couple systems acting a given point $O$.
- The force and couple vectors may be combined into a resultant force vector and a resultant couple vector,
  \[ \vec{R} = \sum \vec{F}, \quad \vec{M}_O = \sum (\vec{r} \times \vec{F}) \]
- The force-couple system at $O$ may be moved to $O'$ with the addition of the moment of $\vec{R}$ about $O'$,
  \[ \vec{M}_{O'} = \vec{M}_O + \vec{s} \times \vec{R} \]
- Two systems of forces are equivalent if they can be reduced to the same force-couple system.

Equilibrium of a Rigid Body in Two Dimensions

- For all forces and moments acting on a two-dimensional structure,
  \[ F_x = 0 \quad M_x = M_y = 0 \quad M_z = M_O \]
- Equations of equilibrium become
  \[ \sum F_x = 0 \quad \sum F_y = 0 \quad \sum M_A = 0 \]
  where $A$ is any point in the plane of the structure.
- The 3 equations can be solved for no more than 3 unknowns.
- The 3 equations can not be augmented with additional equations, but they can be replaced
  \[ \sum F_x = 0 \quad \sum M_A = 0 \quad \sum M_B = 0 \]
Reactions at Supports and Connections for a Two-Dimensional Structure

- Reactions equivalent to a force with known line of action.

- Reactions equivalent to a force of unknown direction and magnitude.

- Reactions equivalent to a force of unknown direction and magnitude and a couple of unknown magnitude.
• Six scalar equations are required to express the conditions for the equilibrium of a rigid body in the general three-dimensional case.

\[ \sum F_x = 0 \quad \sum F_y = 0 \quad \sum F_z = 0 \]

\[ \sum M_x = 0 \quad \sum M_y = 0 \quad \sum M_z = 0 \]

• These equations can be solved for no more than 6 unknowns which generally represent reactions at supports or connections.

• The scalar equations are conveniently obtained by applying the vector forms of the conditions for equilibrium,

\[ \sum \vec{F} = 0 \quad \sum \vec{M}_O = \sum (\vec{r} \times \vec{F}) = 0 \]
Reactions at Supports and Connections for a Three-Dimensional Structure


Vector Mechanics for Engineers: Statics

Centroids and First Moments of Areas and Lines

First moments of the area \( A \):
\[
Q_x = \int x \, dA, \quad Q_y = \int y \, dA
\]
Coordinates \( \bar{x}, \bar{y} \) of the centroid of the area \( A \):
\[
\bar{x} = \frac{\int x \, dA}{A}, \quad \bar{y} = \frac{\int y \, dA}{A}
\]

Similarly for a line:
\[
\bar{x}L = \int x \, dL, \quad \bar{y}L = \int y \, dL
\]
\[
\bar{W} = \int x \, dW, \quad \bar{(\gamma a) L} = \int x(\gamma a) \, dL
\]
First Moments of Areas and Lines

- An area is symmetric with respect to an axis $BB'$ if for every point $P$ there exists a point $P'$ such that $PP'$ is perpendicular to $BB'$ and is divided into two equal parts by $BB'$.

- The first moment of an area with respect to a line of symmetry is zero.

- If an area possesses a line of symmetry, its centroid lies on that axis.

- If an area possesses two lines of symmetry, its centroid lies at their intersection.

- An area is symmetric with respect to a center $O$ if for every element $dA$ at $(x,y)$ there exists an area $dA'$ of equal area at $(-x,-y)$.

- The centroid of the area coincides with the center of symmetry.

Centroids of Common Shapes of Areas

<table>
<thead>
<tr>
<th>Shape</th>
<th>$x$</th>
<th>$y$</th>
<th>Area</th>
</tr>
</thead>
<tbody>
<tr>
<td>Triangle</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>Quarter-circle area</td>
<td>$\frac{1}{8}$</td>
<td>$\frac{1}{8}$</td>
<td>$\frac{1}{8}$</td>
</tr>
<tr>
<td>Semicircle area</td>
<td>$\frac{1}{4}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{8}$</td>
</tr>
<tr>
<td>Quadrilateral area</td>
<td>$\frac{1}{4}$</td>
<td>$\frac{1}{4}$</td>
<td>$\frac{1}{4}$</td>
</tr>
<tr>
<td>Trapezoid area</td>
<td>$\frac{1}{4}$</td>
<td>$\frac{1}{4}$</td>
<td>$\frac{1}{4}$</td>
</tr>
<tr>
<td>Parabola area</td>
<td>$\frac{1}{3}$</td>
<td>0</td>
<td>$\frac{1}{3}$</td>
</tr>
<tr>
<td>Parabolic spindle</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>Circular sector</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
</tr>
</tbody>
</table>
Centroids of Common Shapes of Lines

<table>
<thead>
<tr>
<th>Shape</th>
<th>x</th>
<th>y</th>
<th>Length</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quarter-circular arc</td>
<td>( \frac{b}{2} )</td>
<td>( \frac{b}{2} )</td>
<td>( \frac{b}{2} )</td>
</tr>
<tr>
<td>Semi-circular arc</td>
<td>( \frac{a}{2} )</td>
<td>( \frac{d}{2} )</td>
<td>( a )</td>
</tr>
<tr>
<td>Area of circle</td>
<td>( \frac{r}{2} )</td>
<td>0</td>
<td>( 2\pi r )</td>
</tr>
</tbody>
</table>

Composite Plates and Areas

- Composite plates
  \[ \bar{x} \sum W = \sum xW \]
  \[ \bar{y} \sum W = \sum yW \]

- Composite area
  \[ \bar{X} \sum A = \sum xA \]
  \[ \bar{Y} \sum A = \sum yA \]
Center of Gravity of a 2D Body

- Center of gravity of a plate

Plate of uniform thickness $t$, specific weight $\gamma$, and area $A$:

$W = \gamma t A$;

$\bar{x}(\gamma t A) = \int x(\gamma t) \, dA$;

$\bar{x}A = \int x \, dA = Q_x$

= first moment of the area $A$

with respect to the $y$ axis

$\bar{y}A = \int y \, dA = Q_y$

---

Theorems of Pappus-Guldinus

- Surface of revolution is generated by rotating a plane curve about a fixed axis.

- Area of a surface of revolution is equal to the length of the generating curve times the distance traveled by the centroid through the rotation.

$$A = 2\pi \bar{y}L$$
Theorems of Pappus-Guldinus

- Body of revolution is generated by rotating a plane area about a fixed axis.

- Volume of a body of revolution is equal to the generating area times the distance traveled by the centroid through the rotation.

\[ V = 2\pi yA \]

Distributed Loads on Beams

- A distributed load is represented by plotting the load per unit length, \( w \) (N/m). The total load is equal to the area under the load curve.

\[ W = \int_0^L wdx = \int dA = A \]

- A distributed load can be replaced by a concentrated load with a magnitude equal to the area under the load curve and a line of action passing through the area centroid.

\[ (OP)W = \int xdw \]

\[ (OP)A = \int_0^L xdA = \pi A \]
Center of Gravity of a 3D Body: Centroid of a Volume

- Center of gravity \( G \)
  
  \[ -W \vec{j} = \sum (-\Delta W \vec{j}) \]

- Results are independent of body orientation,
  
  \[ \bar{x}W = \int xdW \quad \bar{y}W = \int ydW \quad \bar{z}W = \int zdW \]

- For homogeneous bodies,
  
  \[ W = \gamma V \quad dW = \gamma dV \]
  
  \[ \bar{x}V = \int xdV \quad \bar{y}V = \int ydV \quad \bar{z}V = \int zdV \]

Definition of a Truss

- A truss consists of straight members connected at joints. No member is continuous through a joint.

- Most structures are made of several trusses joined together to form a space framework. Each truss carries those loads which act in its plane and may be treated as a two-dimensional structure.

- Bolted or welded connections are assumed to be pinned together. Forces acting at the member ends reduce to a single force and no couple. Only two-force members are considered.

- When forces tend to pull the member apart, it is in tension. When the forces tend to compress the member, it is in compression.
Simple Trusses

- A rigid truss will not collapse under the application of a load.
- A simple truss is constructed by successively adding two members and one connection to the basic triangular truss.
- In a simple truss, \( m = 2n - 3 \) where \( m \) is the total number of members and \( n \) is the number of joints.

Analysis of Trusses by the Method of Joints

- Dismember the truss and create a freebody diagram for each member and pin.
- The two forces exerted on each member are equal, have the same line of action, and opposite sense.
- Forces exerted by a member on the pins or joints at its ends are directed along the member and equal and opposite.
- Conditions of equilibrium on the pins provide \( 2n \) equations for \( 2n \) unknowns. For a simple truss, \( 2n = m + 3 \). May solve for \( m \) member forces and 3 reaction forces at the supports.
- Conditions for equilibrium for the entire truss provide 3 additional equations which are not independent of the pin equations.
**Joints Under Special Loading Conditions**

- Forces in opposite members intersecting in two straight lines at a joint are equal.
- The forces in two opposite members are equal when a load is aligned with a third member. The third member force is equal to the load (including zero load).
- The forces in two members connected at a joint are equal if the members are aligned and zero otherwise.
- Recognition of joints under special loading conditions simplifies a truss analysis.

**Space Trusses**

- An *elementary space truss* consists of 6 members connected at 4 joints to form a tetrahedron.
- A *simple space truss* is formed and can be extended when 3 new members and 1 joint are added at the same time.
- In a simple space truss, \( m = 3n - 6 \) where \( m \) is the number of members and \( n \) is the number of joints.
- Conditions of equilibrium for the joints provide \( 3n \) equations. For a simple truss, \( 3n = m + 6 \) and the equations can be solved for \( m \) member forces and 6 support reactions.
- Equilibrium for the entire truss provides 6 additional equations which are not independent of the joint equations.
• When the force in only one member or the forces in a very few members are desired, the method of sections works well.

• To determine the force in member $BD$, pass a section through the truss as shown and create a free body diagram for the left side.

• With only three members cut by the section, the equations for static equilibrium may be applied to determine the unknown member forces, including $F_{BD}$. 

![Diagram of a truss with forces and sections](image-url)