Pattern Formation and Spatiotemporal Chaos in Systems Far from Equilibrium

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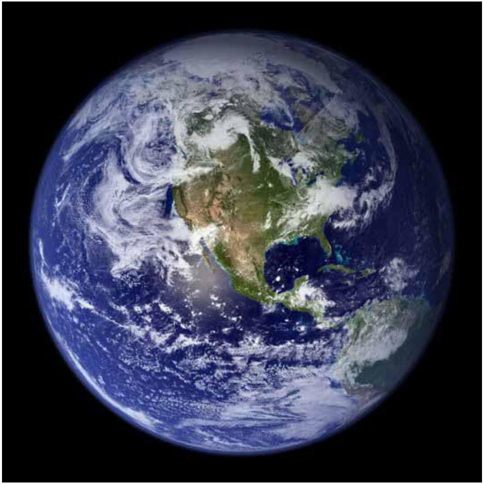
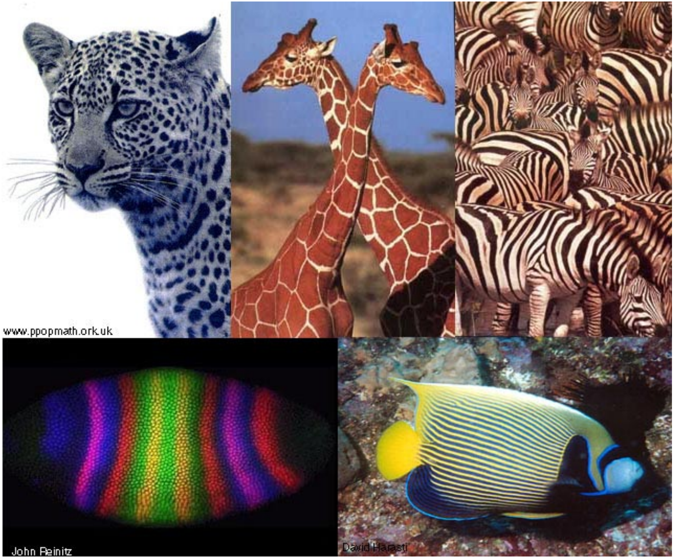
Outline

An Open System



Patterns in Geophysics

Patterns in Biology



The spontaneous formation of spatial structure in open systems driven far from equilibrium

Origins of Pattern Formation

- Fluid Instabilities

  1900 Bénard's experiments on convection in a dish of fluid heated from below and with a free surface

  1916 Rayleigh's theory explaining the formation of convection rolls and cells in a layer of fluid with rigid top and bottom plates and heated from below

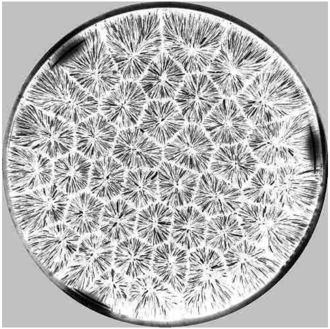
  ...

- Chemical Instabilities

  1952 Turing's suggestion that instabilities in chemical reaction and diffusion equations might explain morphogenesis

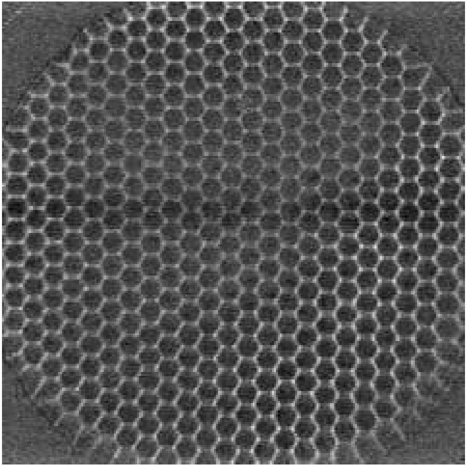
  1950+ Belousov and Zhabotinskii work on chemical reactions showing oscillations and waves

Bénard's Experiments



(From the website of Carsten Jäger)
Movie

Ideal Hexagonal Pattern



From the website of Michael Schatz

Rayleigh's Stability Analysis

Rayleigh made two simplifications:

- In the present problem the case is much more complicated, unless we arbitrarily limit it to two dimensions. The cells of Bénard are then reduced to infinitely long strips, and when there is instability we may ask for what wavelength (width of strip) the instability is greatest.

- …and we have to consider boundary conditions. Those have been chosen which are simplest from the mathematical point of view, and they deviate from those obtaining in Bénard's experiment, where, indeed, the conditions are different at the two boundaries.

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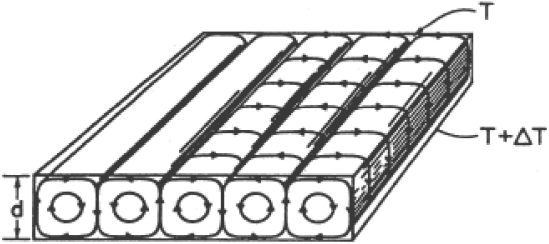
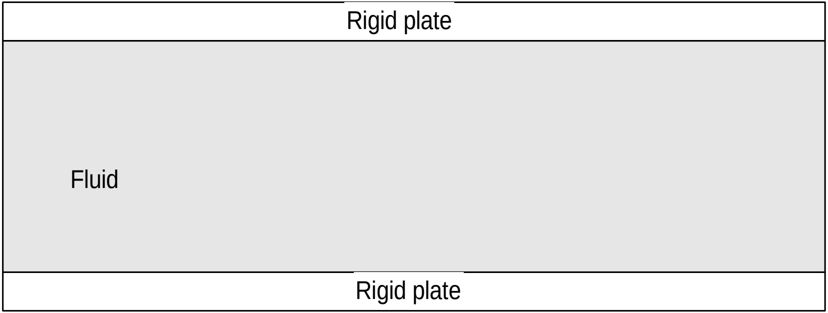
Rayleigh and his Solution



Schematic of Instability



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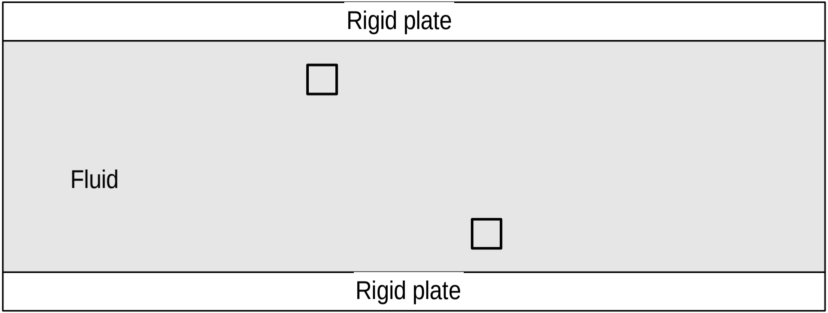

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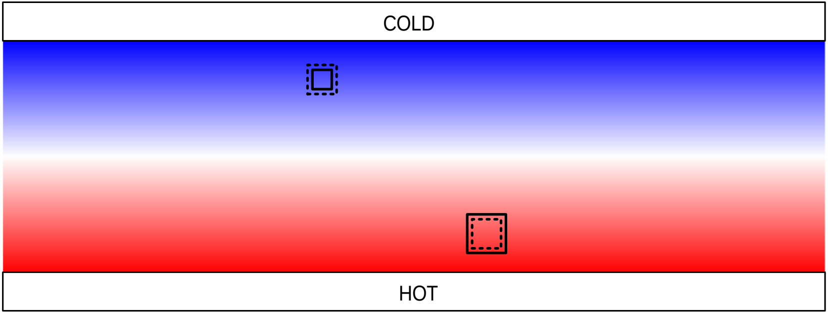

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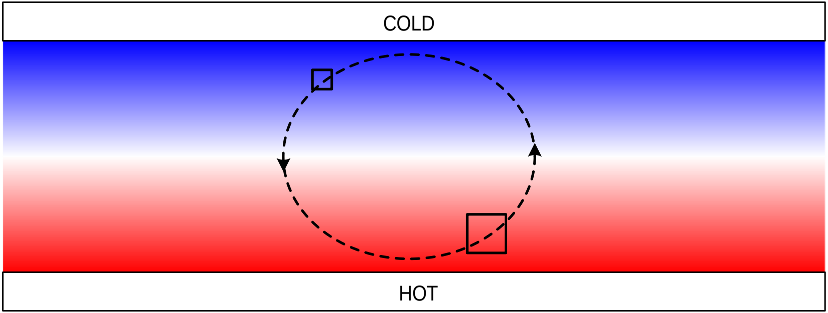
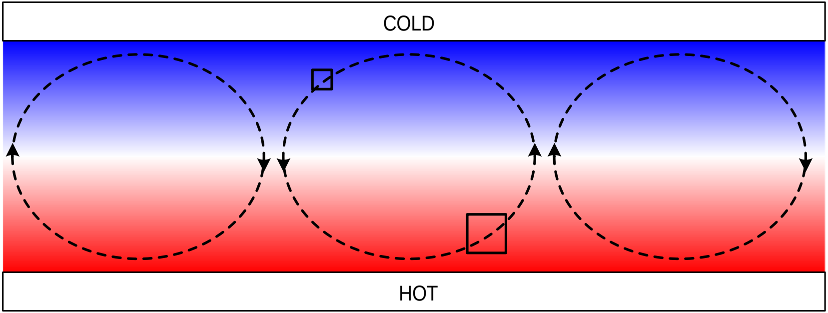


Schematic of Instability



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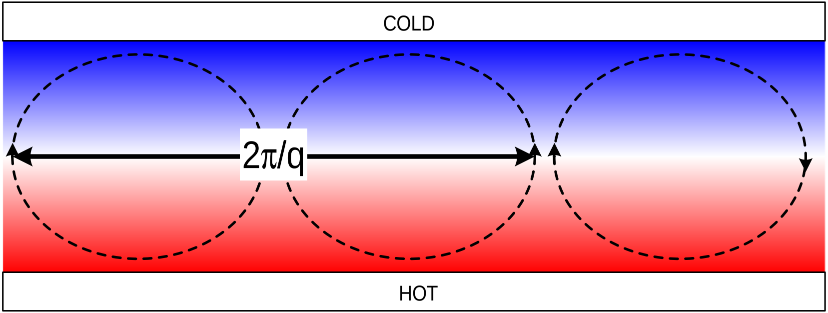

Rayleigh's Solution

Linear stability analysis:

- linear instability towards two dimensional mode with wave number q
- exponential time dependence with growth/decay rate $\sigma(q)$

Two important parameters:

- Rayleigh number $R \propto \Delta T$ (ratio of buoyancy to dissipative forces)
- Prandtl number $\mathcal{P}$, a property of the fluid (ratio of viscous and thermal diffusivities). For a gas $\mathcal{P} \sim 1$, for oil $\mathcal{P} \sim 10^2$, for mercury $\mathcal{P} \sim 10^{-2}$.

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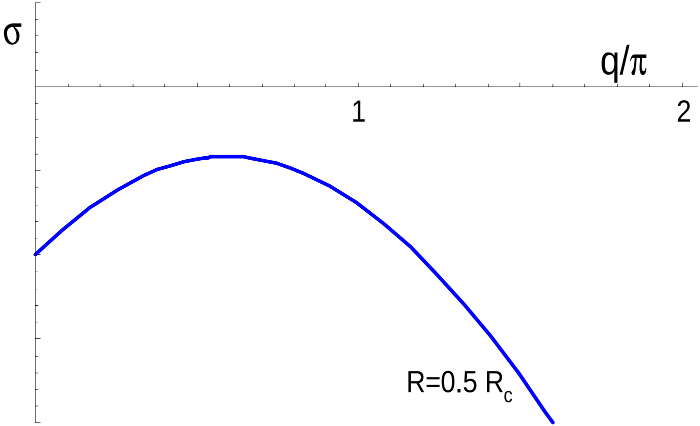
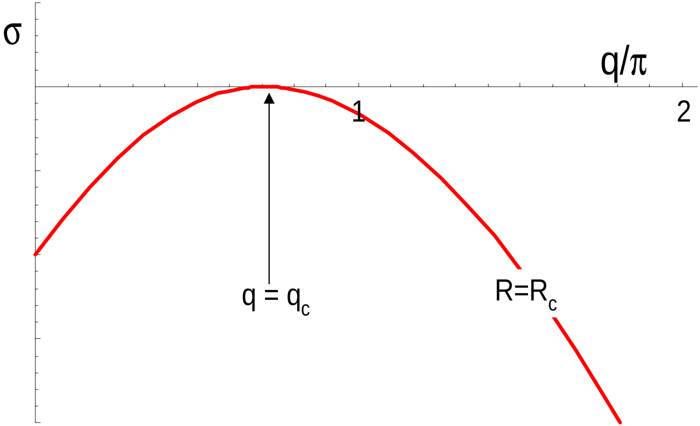
Rayleigh's Growth Rate



$$R_c = \frac{27\pi^4}{4}, \; q_c = \frac{\pi}{\sqrt{2}}$$

Rayleigh's Growth Rate



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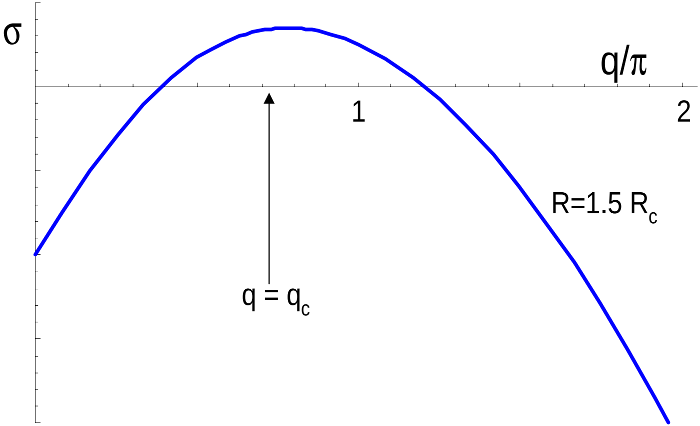

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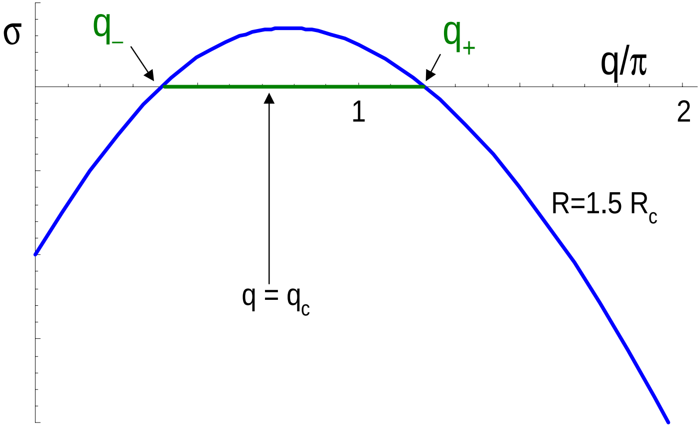

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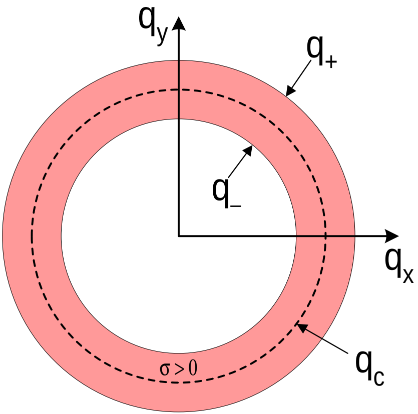
Outline

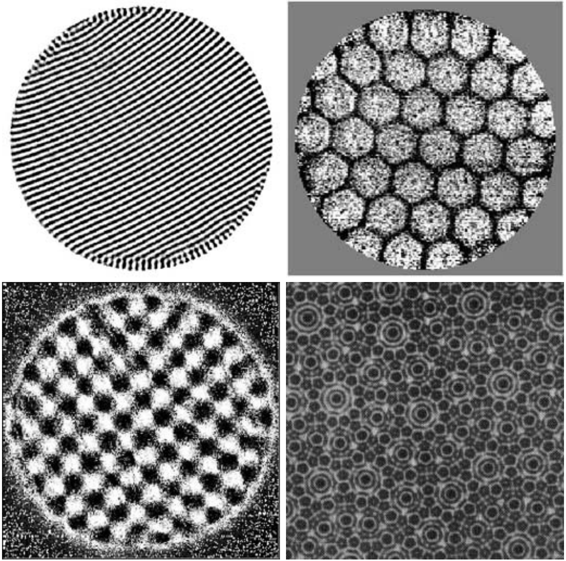
1. Introduction
2. Rayleigh-Bénard Convection
3. Pattern Formation
   - Basic Question
   - Nonlinearity
4. Spatiotemporal Chaos
   - What is it?
   - Spatiotemporal Chaos in Rayleigh-Bénard Convection
   - Theory, Experiment, and Simulation
5. Conclusions

Pattern Formation: the Question



What spatial structures can be formed from the growth and saturation of the unstable modes?

Ideal Patterns from Experiment



More Patterns From Experiment

From the website of EberhardBodenschatz

What is Hard about Pattern Formation?

Why are we still working on this 50 or 100 years later?

The analysis of Rayleigh, Taylor, and Turing was largely linear, and gave interesting, but in the end unphysical solutions.

To understand the resulting patterns we need to understand nonlinearity.

> *One would like to be able to follow this more general [nonlinear] process mathematically also. The difficulties are, however, such that one cannot hope to have any very embracing theory of such processes, beyond the statement of the equations. It might be possible, however, to treat a few particular cases in detail with the aid of a digital computer. [Turing]*

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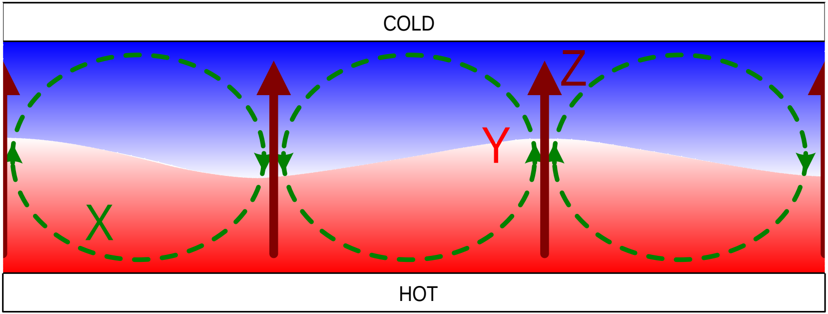
Outline

Lorenz Model (1963)



Equations of Motion

$$\begin{aligned}\dot{X} &= -\mathcal{P}(X - Y)\\ \dot{Y} &= rX - Y - XZ\\ \dot{Z} &= b(XY - Z)\end{aligned}$$

(where $\dot{X} = dX/dt$, etc.).

The equations give the "velocity" $\mathbf{V} = (\dot{X}, \dot{Y}, \dot{Z})$ of the point $\mathbf{X} = (X, Y, Z)$ in the *phase space*

$r = R/R_c$, $b = 8/3$ and $\mathcal{P}$ is the Prandtl number.
Lorenz used $\mathcal{P} = 10$ and $r = 27$.

Linear Equations

$$\begin{aligned}\dot{X} &= -\mathcal{P}(X - Y)\\ \dot{Y} &= rX - Y\\ \dot{Z} &= -bZ\end{aligned}$$

Solution ($\mathcal{P} = 1$)

$$X = a_1 e^{(\sqrt{r}-1)t} + a_2 e^{-(\sqrt{r}+1)t}$$

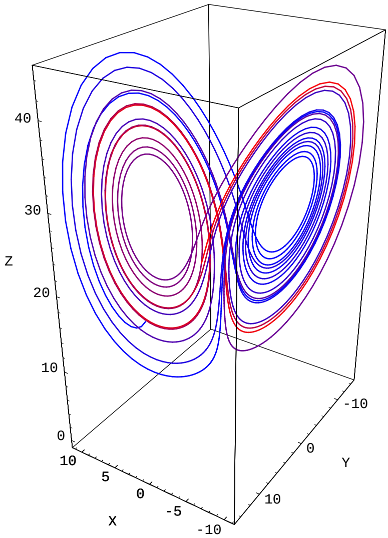
$$Y = a_1 \sqrt{r} e^{(\sqrt{r}-1)t} - a_2 \sqrt{r} e^{-(\sqrt{r}+1)t}$$

$$Z = a_3 e^{-bt}$$

with a_1, a_2, a_3 determined by initial conditions.

This is exactly Rayleigh's solution with an instability at $r = 1$

Nonlinear Equations



Outline

Spatiotemporal Chaos

- Definitions
  - dynamics, disordered in time and space, of a large, uniform system
  - collective motion of many chaotic elements
  - breakdown of pattern to dynamics
- Natural examples:
  - atmosphere and ocean (weather, climate etc.)
  - arrays of nanomechanical oscillators
  - heart fibrillation

Cultured monolayers of cardiac tissue (from Gil Bub, McGill)

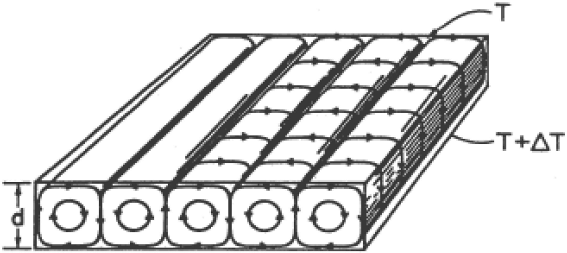
Outline

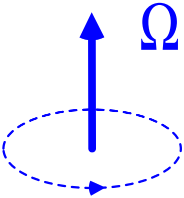
Spiral Chaos in Rayleigh-Bénard Convection

…and from experiment

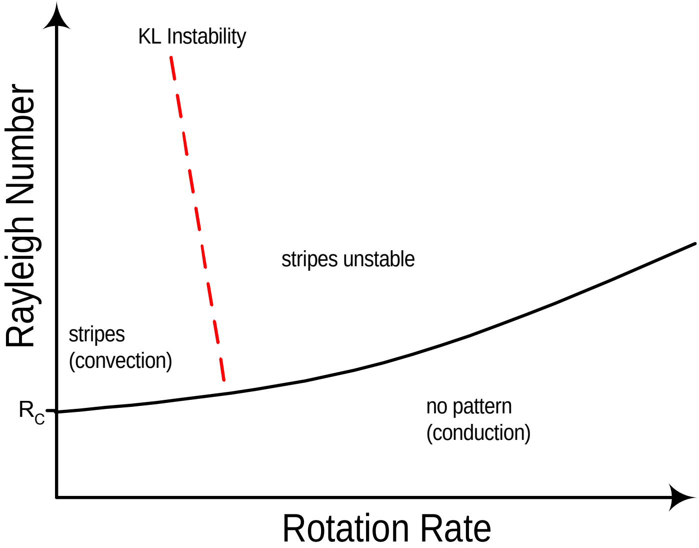
Domain Chaos in Rayleigh-Bénard Convection

Rotating Rayleigh-Bénard Convection





Spiral and Domain Chaos in Rayleigh-Bénard Convection



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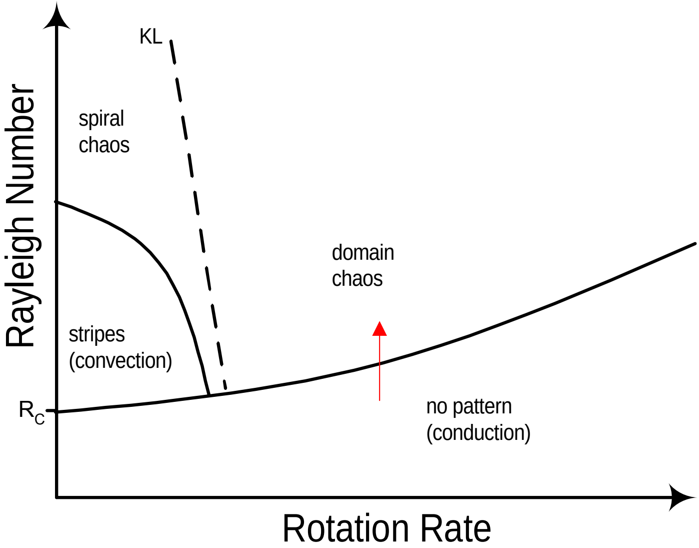
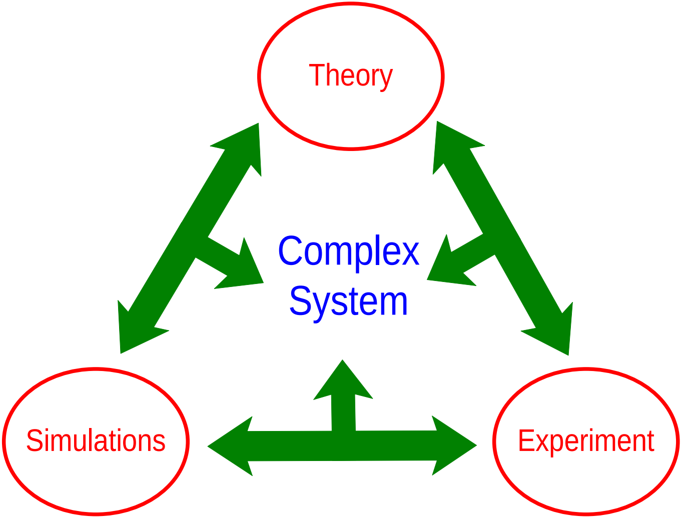


Outline

Theory, Experiment, and Simulation



Summary of Results

- Theory predicts

Length scale	$\xi \sim \varepsilon^{-1/2}$
Time scale	$\tau \sim \varepsilon^{-1}$
Velocity scale	$v \sim \varepsilon^{1/2}$

  with $\varepsilon = (R - R_c(\Omega))/R_c(\Omega)$
- Numerical Tests
  - model equations ✓
  - full fluid dynamic simulations ✓
- Experiment × (but now we understand why)

Theory for Domain Chaos: Amplitudes

- Rayleigh, Turing etc., found solutions

$$u = u_0 e^{\sigma t} \cos(qx) \ldots = A(t) \cos(qx) \ldots$$

so that in the linear approximation and for R near R_c

$$\frac{dA}{dt} = \sigma A \qquad \text{with} \qquad \sigma \propto \varepsilon = \frac{R - R_c}{R_c}$$

- Use ε as small parameter in expansion about threshold
- Nonlinear saturation

$$\frac{dA}{dt} = (\varepsilon - A^2)A$$

- Spatial variation

$$\frac{\partial A}{\partial t} = \varepsilon A - A^3 + \frac{\partial^2 A}{\partial x^2}$$

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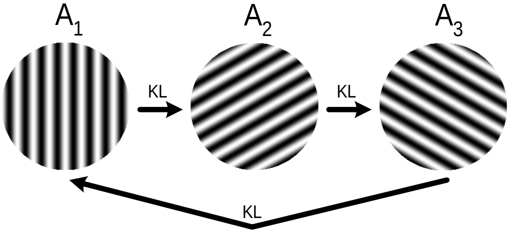
Three Amplitudes + Rotation + Spatial Variation

Tu and MCC (1992)



$$\partial A_1/\partial t = \varepsilon A_1 - A_1(A_1^2 + g_+ A_2^2 + g_- A_3^2) \; + \; \partial^2 A_1/\partial x_1^2$$
$$\partial A_2/\partial t = \varepsilon A_2 - A_2(A_2^2 + g_+ A_3^2 + g_- A_1^2) \; + \; \partial^2 A_2/\partial x_2^2$$
$$\partial A_3/\partial t = \varepsilon A_3 - A_3(A_3^2 + g_+ A_1^2 + g_- A_2^2) \; + \; \partial^2 A_3/\partial x_3^2$$

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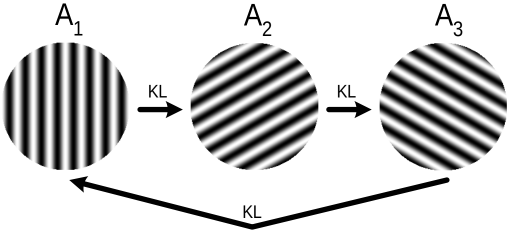

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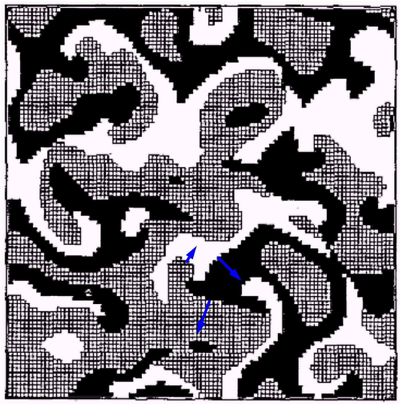
$$\partial A_3/\partial t = \varepsilon A_3 - A_3(A_3^2 + g_+ A_1^2 + g_- A_2^2) \; + \; \partial^2 A_3/\partial x_3^2$$

gives chaos!

Simulations of Amplitude Equations

Tu and MCC (1992)

Length scale	$\xi \sim \varepsilon^{-1/2}$
Time scale	$\tau \sim \varepsilon^{-1}$
Velocity scale	$\upsilon \sim \varepsilon^{1/2}$



Grey: A_1 largest; White: A_2 largest; Black: A_3 largest

Model Equations

MCC, Meiron, and Tu (1994)

Real field of two spatial dimensions $\psi(x, y; t)$

$$\frac{\partial \psi}{\partial t} = \varepsilon \psi + (\nabla^2 + 1)^2 \psi - \psi^3 \qquad \text{gives stripes}$$

Model Equations

MCC, Meiron, and Tu (1994)

Real field of two spatial dimensions $\psi(x, y; t)$

$$\frac{\partial \psi}{\partial t} = \varepsilon\psi + (\nabla^2 + 1)^2\psi - \psi^3 + g_2\hat{\mathbf{z}}\cdot\nabla \times [(\nabla\psi)^2\nabla\psi] + g_3\nabla\cdot[(\nabla\psi)^2\nabla\psi]$$

gives chaos!

Stripes Orientations Domain Walls

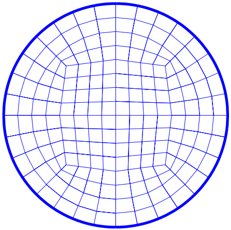
Full Fluid Simulations

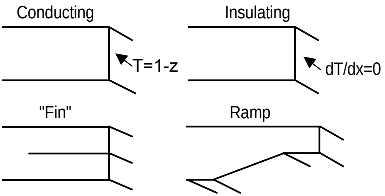
MCC, Greenside, Fischer et al.

With modern supercomputers we can now simulate actual experiments

Spectral Element Method

- Accurate simulation of long-time dynamics
- Exponential convergence in space, third order in time
- Efficient parallel algorithm, unstructured mesh
- Arbitrary geometries, realistic boundary conditions





Fluid Simulations Complement Experiments

- Knowledge of full flow field and other diagnostics (e.g. total heat flow)
- No experimental/measurement noise (roundoff "noise" very small)
- Measure quantities inaccessible to experiment e.g. Lyapunov exponents and vectors
- Readily tune parameters
- Turn on and off particular features of the physics (e.g. centrifugal effects, mean flow, realistic v. periodic boundary conditions)

Full Fluid Dynamic Simulations

Scheel, Caltech thesis (2006)

Periodic Boundaries

Realistic Boundaries

Lyapunov Exponent

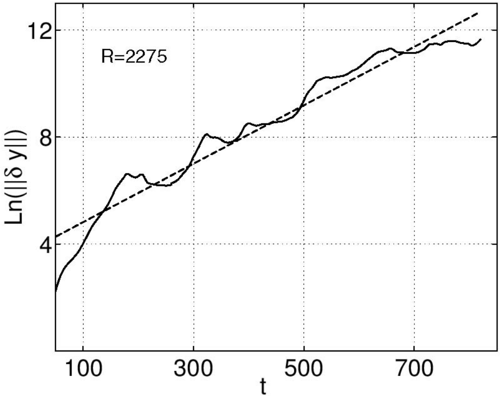
Jayaraman et al. (2005)

Temperature

Temperature Perturbation

Lyapunov Exponent

Jayaraman et al. (2005)

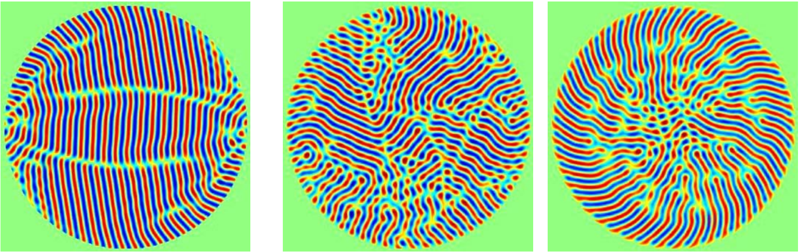


Aspect ratio $\Gamma = 40$, Prandtl number $\sigma = 0.93$, rotation rate $\Omega = 40$

Importance of Centrifugal Force…

Becker, Scheels, MCC, Ahlers (2006)

Aspect ratio $\Gamma = 20$, $\varepsilon \simeq 1.05$, $\Omega = 17.6$



Centrifugal force 0 Centrifugal force x4 Centrifugal force x10

Conclusions

I have described the study of pattern formation and spatiotemporal chaos in open systems driven far from equilibrium.

- Linear stability analysis gives an understanding of the origin of the pattern and the basic length scale
- Nonlinearity is a vital ingredient, and makes the problem difficult
- There has been significant progress in understanding patterns, although many questions remain
- I illustrated the approaches we use by describing attempts to reach a *quantitative* understanding of spatiotemporal chaos in rotating convection experiments
- Close interaction between experiment, theory, and numerical simulation is important to understand complex systems