Models of Coupled Nanomechanical Oscillators

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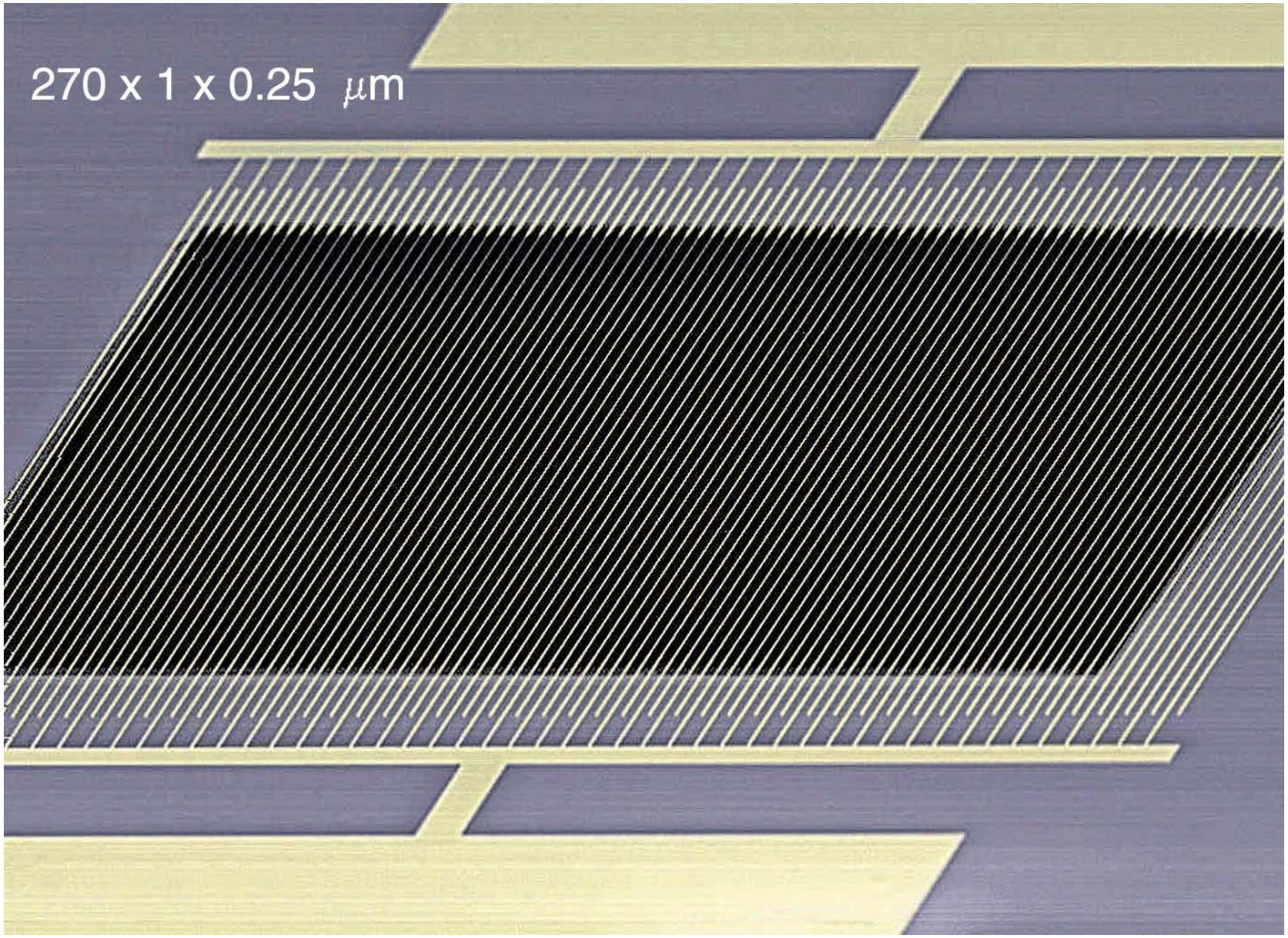
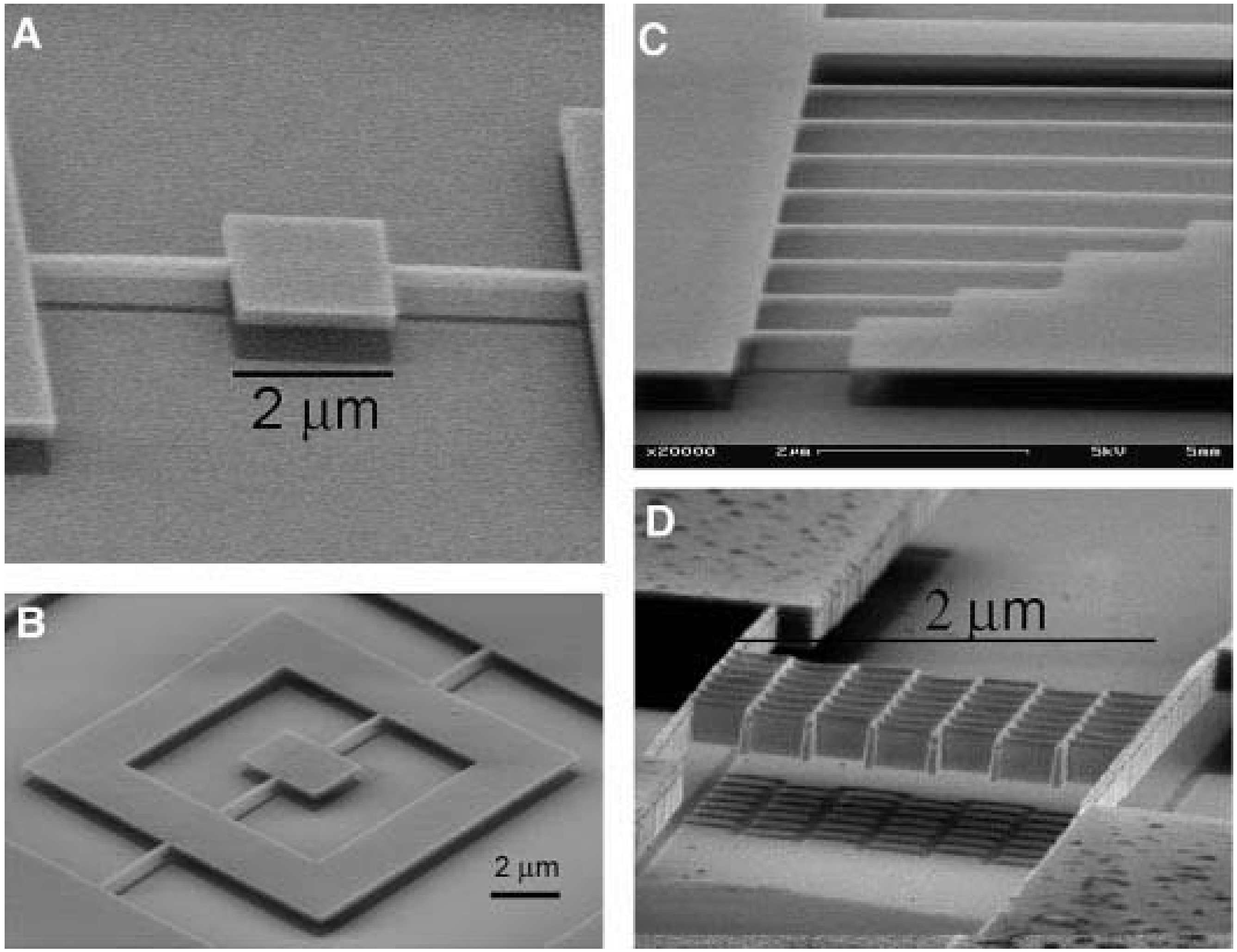
Support: NSF, Nato and EU, BSF, HRL

Outline

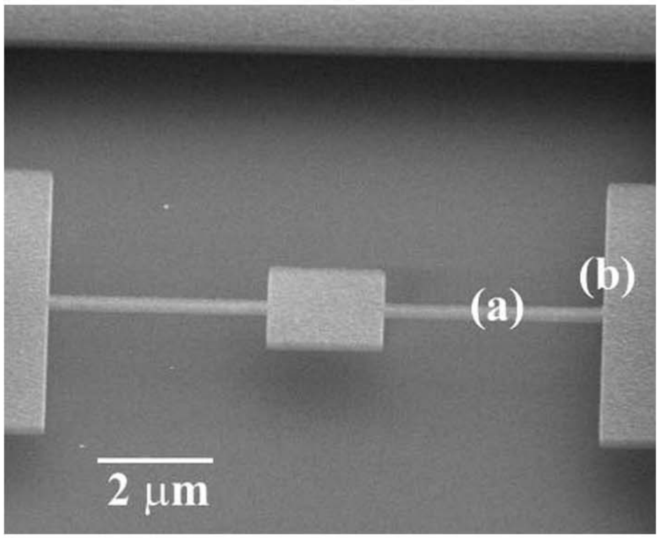
- Motivation: MEMS and NEMS
- Nonlinearity
- Synchronization of arrays of oscillators
- Pattern formation in parametrically driven arrays
- Conclusions

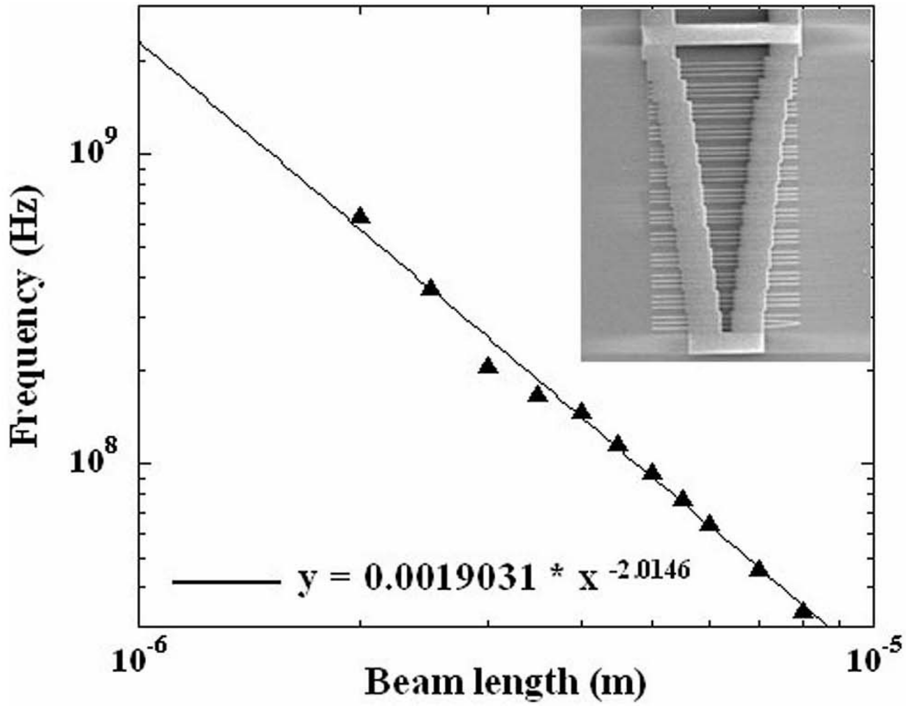


Array of μ-scale oscillators
[From Buks and Roukes J. MEMS. **11**, 802 (2002)]



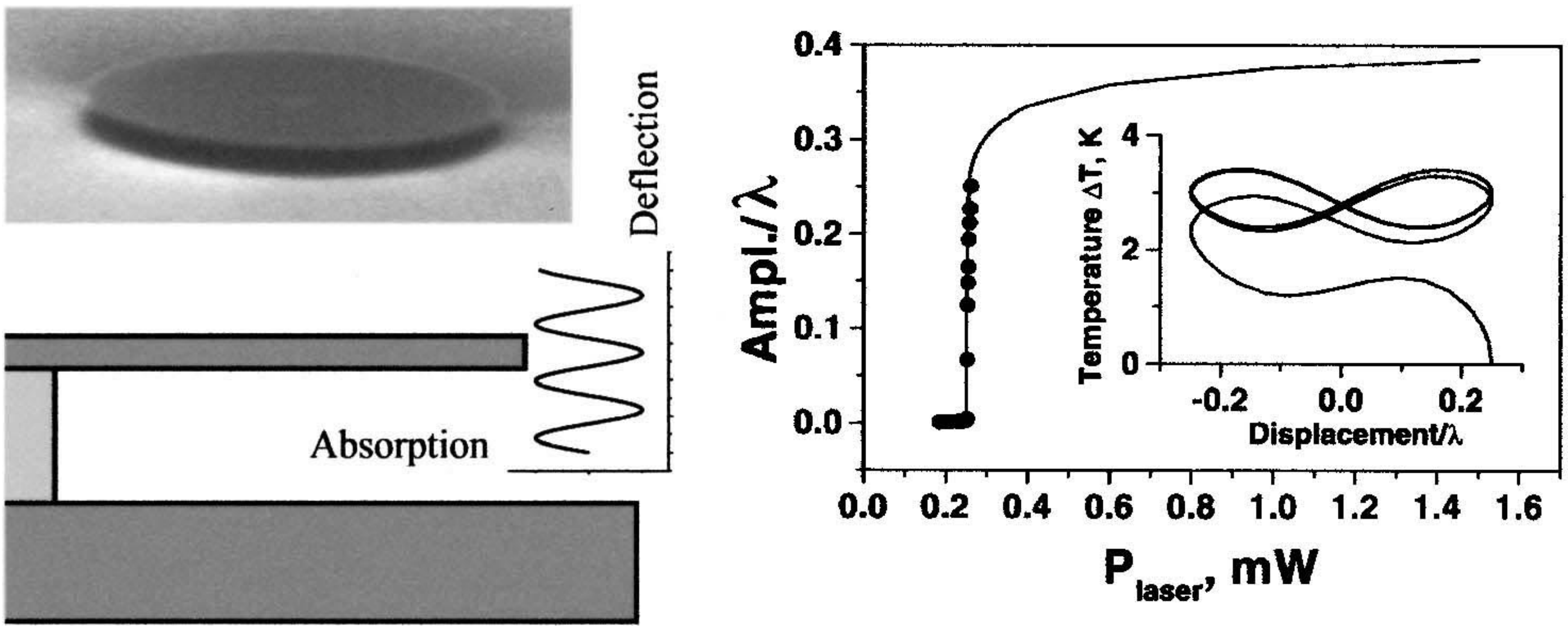
Single crystal silicon [From Craighead, Science **290**, 1532 (2000)]





Diamond Film [From Sekaric et al., Appl. Phys. Lett. **81**, 4445 (2002)]

Self-Oscillations



[Zalalutdinov et al., Appl. Phys. Lett. **79**, 695 (2001)]

MicroElectroMechanical Systems and NEMS

Arrays of tiny mechanical oscillators:

- driven, dissipative $\Rightarrow$ nonequilibrium
- nonlinear
- collective
- noisy
- (potentially) quantum

Goals

- Apply knowledge from nonlinear dynamics, pattern formation etc. to technologically important questions
- Investigate pattern formation and nonlinear dynamics in new regimes
- Study new aspects of old questions

Modelling

$$0 = \ddot{x}_n + x_n$$

Modelling

$$0 = \ddot{x}_n + x_n$$
$$+ \delta_n x_n \qquad \text{with } \delta_n \text{ taken from distribution } g(\delta_n)$$

Modelling

$$
\begin{aligned}
0 = \ddot{x}_n + x_n \\
+ \delta_n x_n \\
+ \sum_m D_{nm}(x_m - x_n)
\end{aligned}
$$

reactive coupling

Modelling

$$0 = \ddot{x}_n + x_n$$

$$+ \delta_n x_n$$

$$+ \sum_m D_{nm}(x_m - x_n)$$

$$- \sum_m \bar{\gamma}_{nm}(\dot{x}_m - \dot{x}_n)$$ linear damping

Modelling

$$
\begin{aligned}
0 = \ddot{x}_n + x_n & \\
& + \delta_n x_n \\
& + \sum_m D_{nm}(x_m - x_n) \\
& - \sum_m \bar{\gamma}_{nm}(\dot{x}_m - \dot{x}_n) \\
& + x_n^3 \qquad \text{nonlinear stiffening}
\end{aligned}
$$

Modelling

$$
\begin{aligned}
0 = \ddot{x}_n + x_n \\
+ \delta_n x_n \\
+ \sum_m D_{nm}(x_m - x_n) \\
- \sum_m \bar{\gamma}_{nm}(\dot{x}_m - \dot{x}_n) \\
+ x_n^3
\end{aligned}
$$

$$+\eta\left[(x_{n+1} - x_n)^2(\dot{x}_{n+1} - \dot{x}_n) - (x_n - x_{n-1})^2(\dot{x}_n - \dot{x}_{n-1})\right]$$ nonlinear damping

Modelling

$$
\begin{aligned}
0 = \ddot{x}_n + x_n & \\
& + \delta_n x_n \\
& + \sum_m D_{nm}(x_m - x_n) \\
& - \sum_m \bar{\gamma}_{nm}(\dot{x}_m - \dot{x}_n) \\
& + x_n^3 \\
+\eta \left[(x_{n+1} - x_n)^2 (\dot{x}_{n+1} - \dot{x}_n) - (x_n - x_{n-1})^2 (\dot{x}_n - \dot{x}_{n-1}) \right] & \\
& - \gamma \dot{x}_n (1 - x_n^2) \qquad \text{energy input}
\end{aligned}
$$

Modelling

$$
\begin{aligned}
0 = \ddot{x}_n + x_n \\
+ \delta_n x_n \\
+ \sum_m D_{nm}(x_m - x_n) \\
- \sum_m \bar{\gamma}_{nm}(\dot{x}_m - \dot{x}_n) \\
+ x_n^3
\end{aligned}
$$

$$
+\eta\left[(x_{n+1} - x_n)^2(\dot{x}_{n+1} - \dot{x}_n) - (x_n - x_{n-1})^2(\dot{x}_n - \dot{x}_{n-1})\right]
$$

$$
+g_P \cos\left[(2 + \delta\omega_P)t\right] x_n \qquad \text{parametric drive}
$$

Modelling

$$
\begin{aligned}
0 = \ddot{x}_n + x_n \\
+ \delta_n x_n \\
+ \sum_m D_{nm}(x_m - x_n) \\
- \sum_m \bar{\gamma}_{nm}(\dot{x}_m - \dot{x}_n) \\
+ x_n^3 \\
+\eta \left[(x_{n+1} - x_n)^2(\dot{x}_{n+1} - \dot{x}_n) - (x_n - x_{n-1})^2(\dot{x}_n - \dot{x}_{n-1})\right] \\
+g_P \cos[(2 + \delta\omega_P)t]\, x_n \\
+ g_D \cos[(1 + \delta\omega_D)t]
\end{aligned}
$$

signal

Modelling

$$
\begin{aligned}
0 = \ddot{x}_n + x_n \\
+ \delta_n x_n \\
+ \sum_m D_{nm}(x_m - x_n) \\
- \sum_m \bar{\gamma}_{nm}(\dot{x}_m - \dot{x}_n) \\
+ x_n^3 \\
+\eta\left[(x_{n+1} - x_n)^2(\dot{x}_{n+1} - \dot{x}_n) - (x_n - x_{n-1})^2(\dot{x}_n - \dot{x}_{n-1})\right] \\
+g_P \cos\left[(2 + \delta\omega_P)t\right] x_n \\
+ g_D \cos\left[(1 + \delta\omega_D)t\right] \\
+ \text{Noise}
\end{aligned}
$$

Modelling

$$0 = \ddot{x}_n + x_n$$
$$+ \delta_n x_n$$
$$+ \sum_m D_{nm}(x_m - x_n)$$
$$- \sum_m \bar{\gamma}_{nm}(\dot{x}_m - \dot{x}_n)$$
$$+ x_n^3$$
$$+\eta\left[(x_{n+1} - x_n)^2(\dot{x}_{n+1} - \dot{x}_n) - (x_n - x_{n-1})^2(\dot{x}_n - \dot{x}_{n-1})\right]$$

$$+g_P \cos\left[(2 + \delta\omega_P)t\right] x_n$$
$$+ g_D \cos\left[(1 + \delta\omega_D)t\right]$$
$$+ \text{Noise}$$

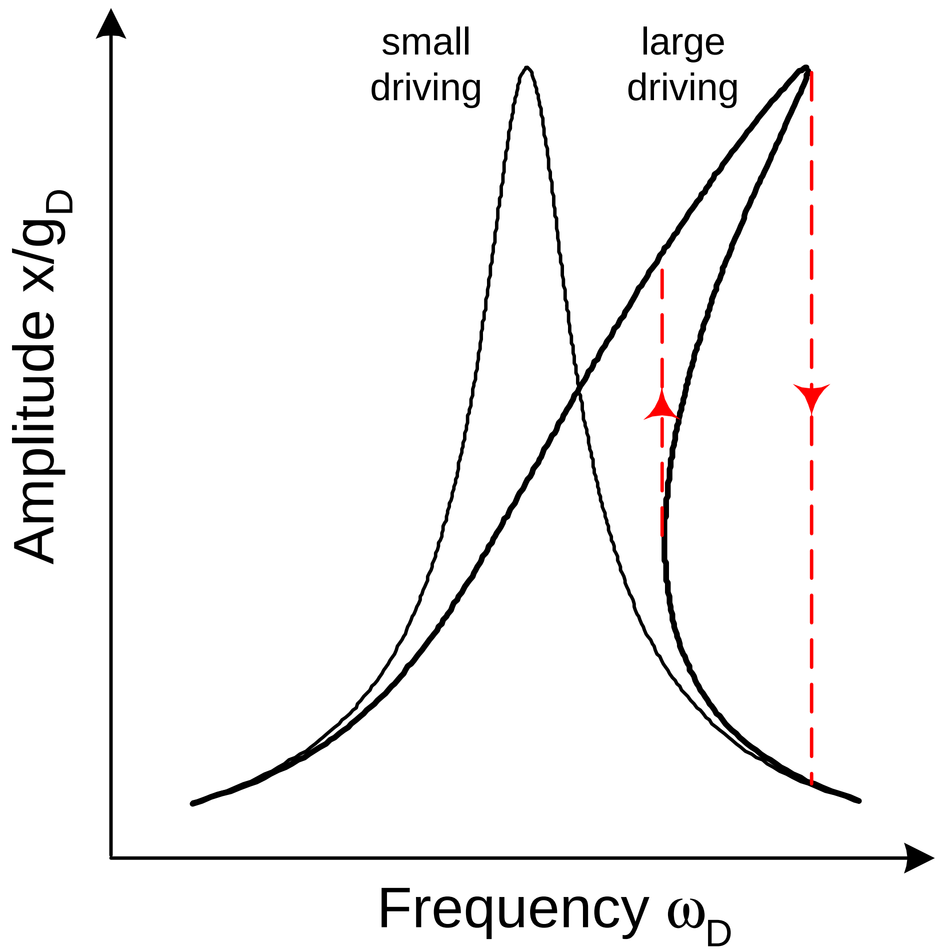
Theoretical approach

- Oscillators at frequency unity + small corrections
- Assume dispersion, coupling, damping, driving, noise, and nonlinear terms are small.
- Introduce small parameter ε with ε^p characterizing the size of these various terms.
- Then with the "slow" time scale $T = \varepsilon t$

$$x_n(t) = \varepsilon^{1/2}\left[A_n(T)e^{it} + c.c.\right] + \varepsilon x_n^{(1)}(t) + \ldots$$

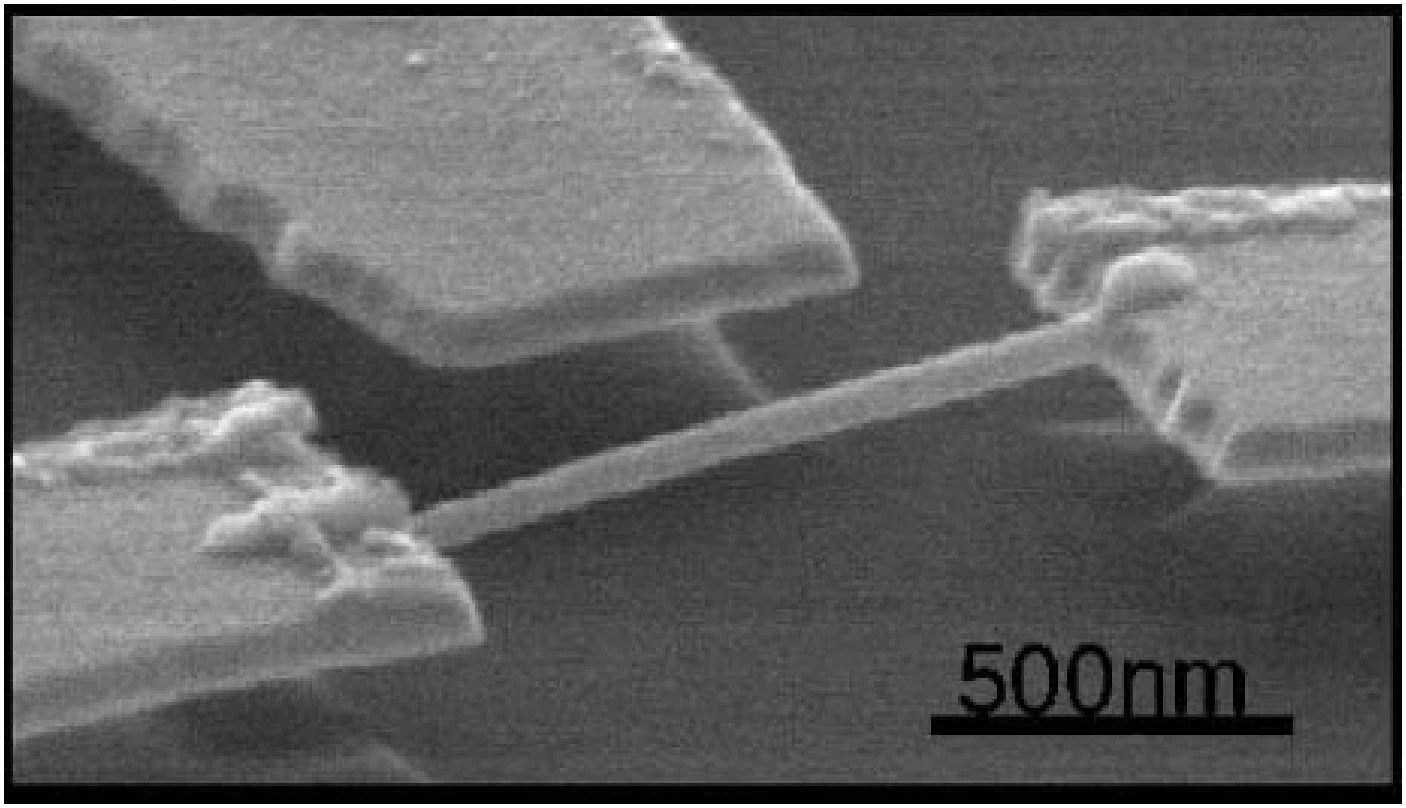
derive equations for $dA_n/dT = \cdots$.

Nonlinearity: Frequency pulling



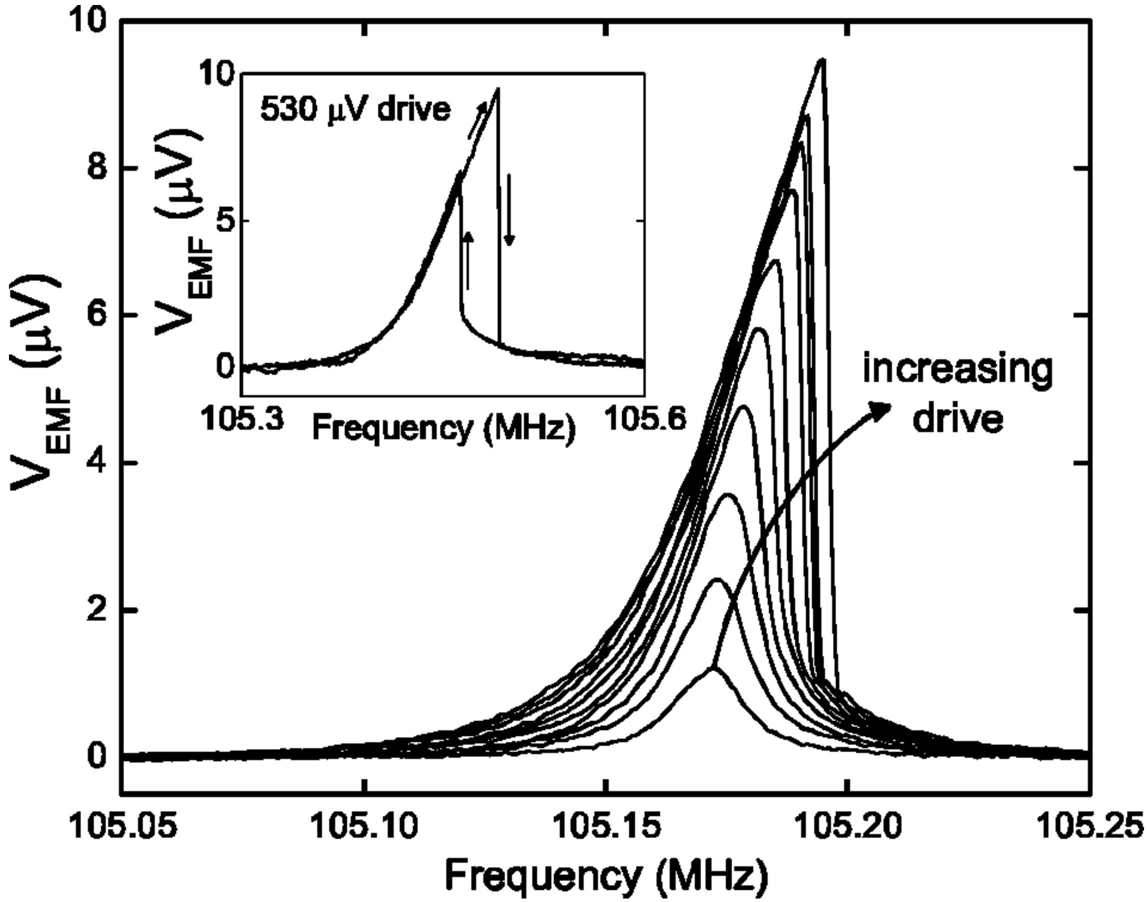
$$\ddot{x}_n + \gamma \dot{x}_n + x_n + x_n^3 = g_D \cos(\omega_D t)$$

Experiment



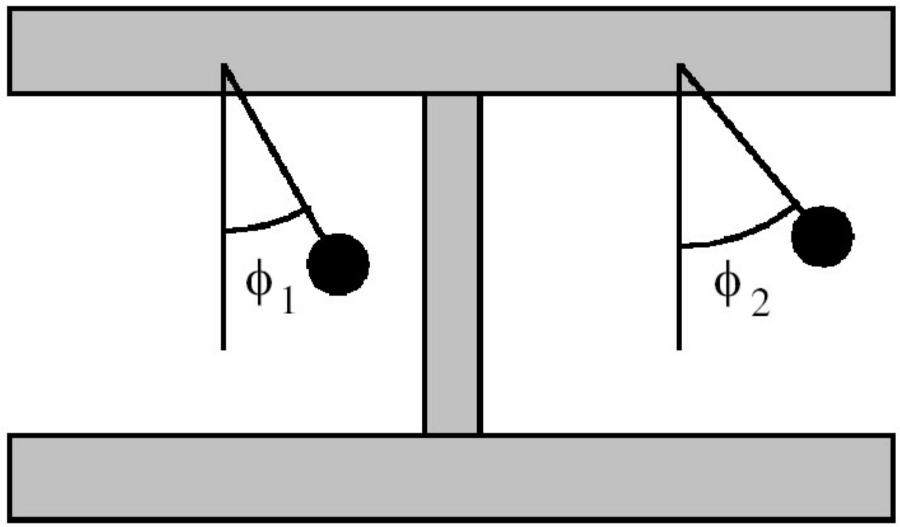
Platinum Wire [Husain et al., Appl. Phys. Lett. **83**, 1240 (2003)]

Results


10
8
6
4
2
0
V_{EMF} (μV)
105.05
105.10
105.15
105.20
105.25
Frequency (MHz)
increasing drive
530 μV drive
10
5
0
V_{EMF} (μV)
105.3
105.6
Frequency (MHz)

Synchronization

Huygen’s Clocks (1665)



From: Bennett, Schatz, Rockwood, and Wiesenfeld (Proc. Roy. Soc. Lond. 2002)

Paradigm I: Synchronization occurs through dissipation acting on the phase differences

- Huygen's clocks (cf. Bennett, Schatz, Rockwood, and Wiesenfeld)
- Winfree-Kuramoto phase equation

$$\dot{\theta}_n = \omega_n - \sum_m K_{nm} \sin(\theta_n - \theta_{n+m})$$

  with ω_n taken from distribution $g(\omega)$. Continuum limit (short range coupling)

$$\dot{\theta} = \omega(x) + K\nabla^2\theta + O(\nabla(\nabla\theta)^3)$$

  —phase diffusion, not propagation (eg. no $(\nabla\theta)^2$ term)

- Aronson, Ermentrout and Kopell analysis of two coupled oscillators
- Matthews, Mirollo and Strogatz magnitude-phase model

Synchronization in MEMS $\Rightarrow$ alternative mechanism

Paradigm II: Synchronization occurs by nonlinear frequency pulling and reactive coupling

MEMS equation

$$0 = \ddot{x}_n + (1 + \omega_n)x_n - \nu(1 - x_n^2)\dot{x}_n + ax_n^3 + \sum_m D_{nm}(x_m - x_n)$$

leads to

$$\dot{A}_n = i(\omega_n - \alpha\,|A_n|^2)A_n + (1 - |A_n|^2)A_n + i\sum_m \beta_{mn}(A_m - A_n)$$

with $a \Rightarrow \alpha,\ D \Rightarrow \beta$.

(cf. *Synchronization* by Pikovsky, Rosenblum, and Kurths)

Synchronization in MEMS $\Rightarrow$ alternative mechanism

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MEMS equation

$$0 = \ddot{x}_n + (1+\omega_n)x_n - \nu(1-x_n^2)\dot{x}_n + ax_n^3 + \sum_m D_{nm}(x_m - x_n)$$

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with $a \Rightarrow \alpha,\ D \Rightarrow \beta$.

(cf. *Synchronization* by Pikovsky, Rosenblum, and Kurths)

Analyze mean field version (all-to-all coupling): $\beta_{mn} \to \beta/N$

Definitions of Synchronization

1. Order parameter

$$\Psi = N^{-1} \sum_n A_n = N^{-1} \sum_n r_n \, e^{i\theta_n} = R \, e^{i\Theta}$$

   Synchronization occurs if $R \neq 0$

2. Full locking: $\dot{\theta}_n = \Omega$ for all the oscillators

3. Partial frequency locking

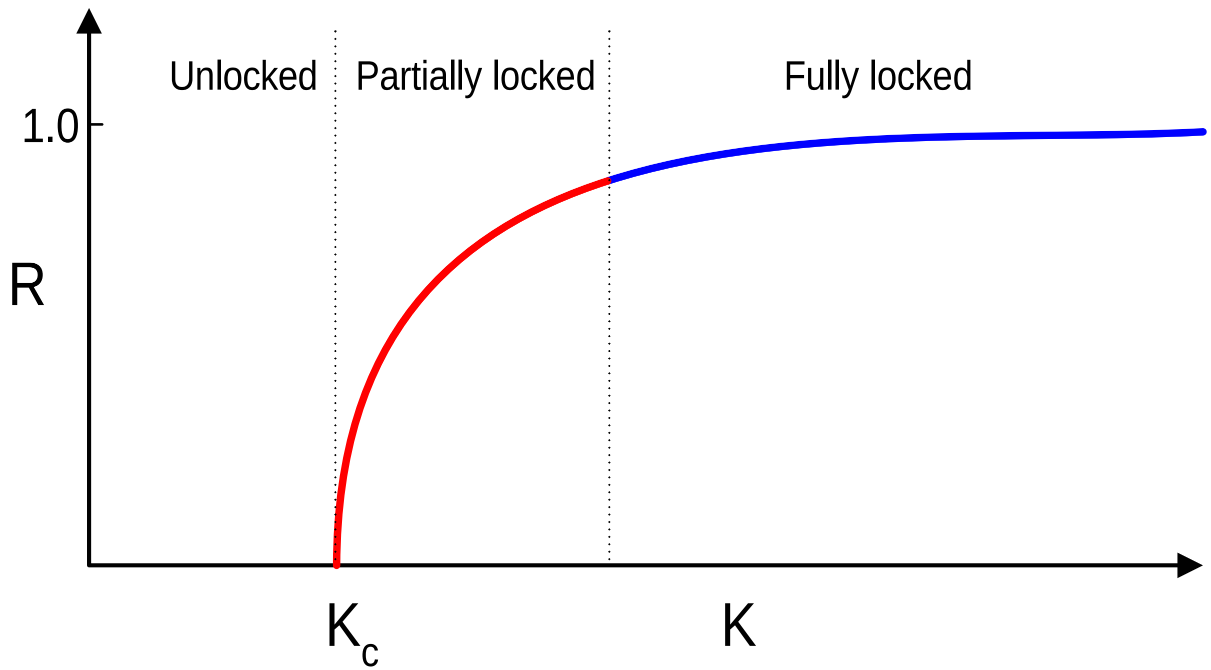
$$\bar{\omega}_n = \lim_{T \to \infty} \frac{\theta_n(T) - \theta_n(0)}{T}$$

   and then $\bar{\omega}_n = \Omega$ for some $O(N)$ subset of oscillators

4. …

Results for the mean field phase model (Kuramoto 1975)

$$\dot{\theta}_n = \omega_n - \frac{K}{N} \sum_m \sin(\theta_n - \theta_{n+m})$$



Calculations [MCC, Zumdieck, Lifshitz, and Rogers (2004)]

- Analytics
  - ◇ Linear instability of unsynchronized $R = 0$ state (for Lorentzian, triangular, top-hat $g(\omega)$)
  - ◇ Linear instability of fully locked state
- Numerical simulations of amplitude-phase model for up to 10000 oscillators with all-to-all coupling

Analytics: Basics

- Label the oscillators by bare frequency $\omega = \omega_n$
- Write equations in magnitude-phase form

$$d_t\bar{\theta} = \bar{\omega} + \alpha(1 - r^2) + \frac{\beta R}{r}\cos\bar{\theta}$$
$$d_t r = (1 - r^2)r + \beta R \sin\bar{\theta}$$

where $\bar{\theta} = \theta - \Theta$ is the oscillator phase relative to the order parameter and $\bar{\omega} = \omega - \alpha - \beta - \Omega$

- For large α, narrow distribution $d_t r \to 0$, $r \simeq 1$ so that

$$r^2 \simeq 1 + \beta R \sin\bar{\theta}$$

Then

$$d_t\bar{\theta} \simeq \bar{\omega} - \alpha\beta R \sin\bar{\theta}$$

Kuramoto equation $\Rightarrow$ synchronization for $\alpha\beta > 2(\pi g(0))^{-1}$

Onset from unsynchronized state (cf. Matthews et al., 1991)

Introduce distribution $\rho(r, \bar{\theta}, \bar{\omega}, t)$

Self-consistency condition

$$R = \left\langle r e^{i\bar{\theta}} \right\rangle = \int d\bar{\omega}\bar{g}(\bar{\omega}) \int r\, dr\, d\bar{\theta}\, \rho(r, \bar{\theta}, \bar{\omega}, t) r e^{i\bar{\theta}}.$$

where $\bar{g}(\bar{\omega})$ is the distribution of oscillator frequencies expressed in terms of the shifted frequency $\bar{\omega}$.

Imaginary (transverse) part

$$\int d\bar{\omega}\bar{g}(\bar{\omega}) \int r\, dr\, d\bar{\theta}\, \rho(r, \bar{\theta}, \bar{\omega}, t) r \sin\bar{\theta} = 0.$$

Real (longitudinal) part

$$\int d\bar{\omega}\bar{g}(\bar{\omega}) \int r\, dr\, d\bar{\theta}\, \rho(r, \bar{\theta}, \bar{\omega}, t) r \cos\bar{\theta} = R.$$

- Ansatz

$$\rho(r,\theta,\bar{\omega},t) = (2\pi r)^{-1}\delta[r-1-\varepsilon r_1(\bar{\theta},\bar{\omega},t)][1+\varepsilon f_1(\bar{\theta},\bar{\omega},t)]$$
$$R = \varepsilon R_1(t)$$

with $r_1(\bar{\theta},\bar{\omega},t),\ f_1(\bar{\theta},\bar{\omega},t),\ R_1(t) \propto e^{\lambda t}$

- Substitute into equation of motion

$$r_1 = \beta R_1\left[\frac{(\lambda+2)}{\bar{\omega}^2+(\lambda+2)^2}\sin\bar{\theta} - \frac{\bar{\omega}}{\bar{\omega}^2+(\lambda+2)^2}\cos\bar{\theta}\right]$$
$$f_1 = \beta R_1[\cdots]$$

- Insert ρ into self-consistency condition
- $\lambda \to 0^+$ gives onset condition $\beta_c(\alpha)$ and order parameter frequency Ω

Full locking

$$d_t\bar{\theta} = \bar{\omega} + \alpha(1 - r^2) + \frac{\beta R}{r}\cos\bar{\theta}$$

$$d_t r = (1 - r^2)r + \beta R \sin\bar{\theta}$$

Locking assumption

$$d_t r = 0, d_t\bar{\theta} = 0$$

Analysis in terms of locking force $F(\bar{\theta})$

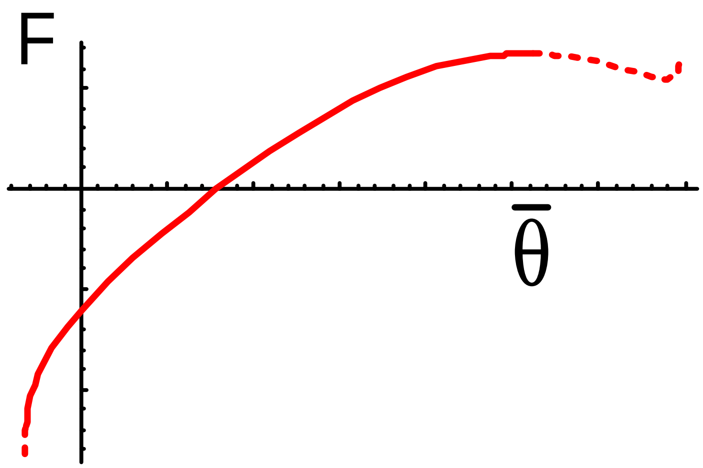
$$\bar{\omega} = F(\bar{\theta}) = \frac{\beta R}{r}(\alpha \sin\bar{\theta} - \cos\bar{\theta})$$

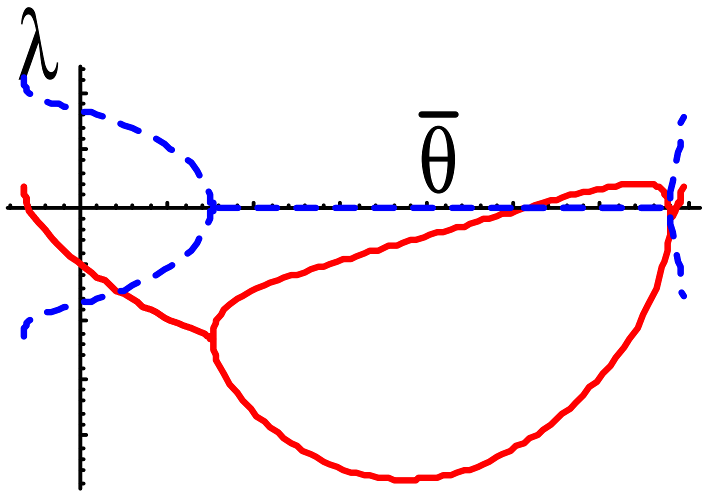
with

$$(1 - r^2)r = -\beta R \sin\bar{\theta}$$

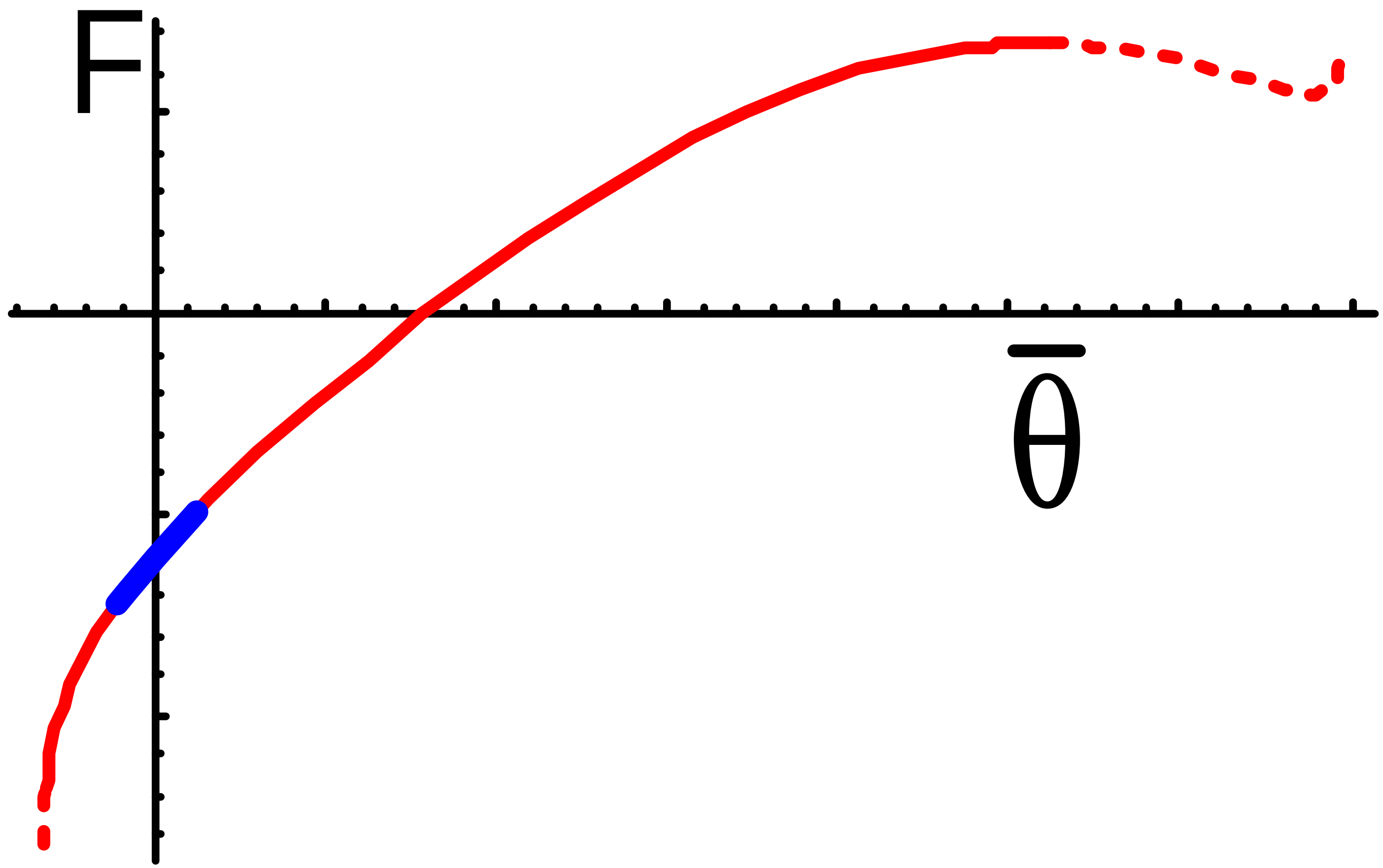
Example

$$\alpha = 1.0,\ \beta R = 1.2$$

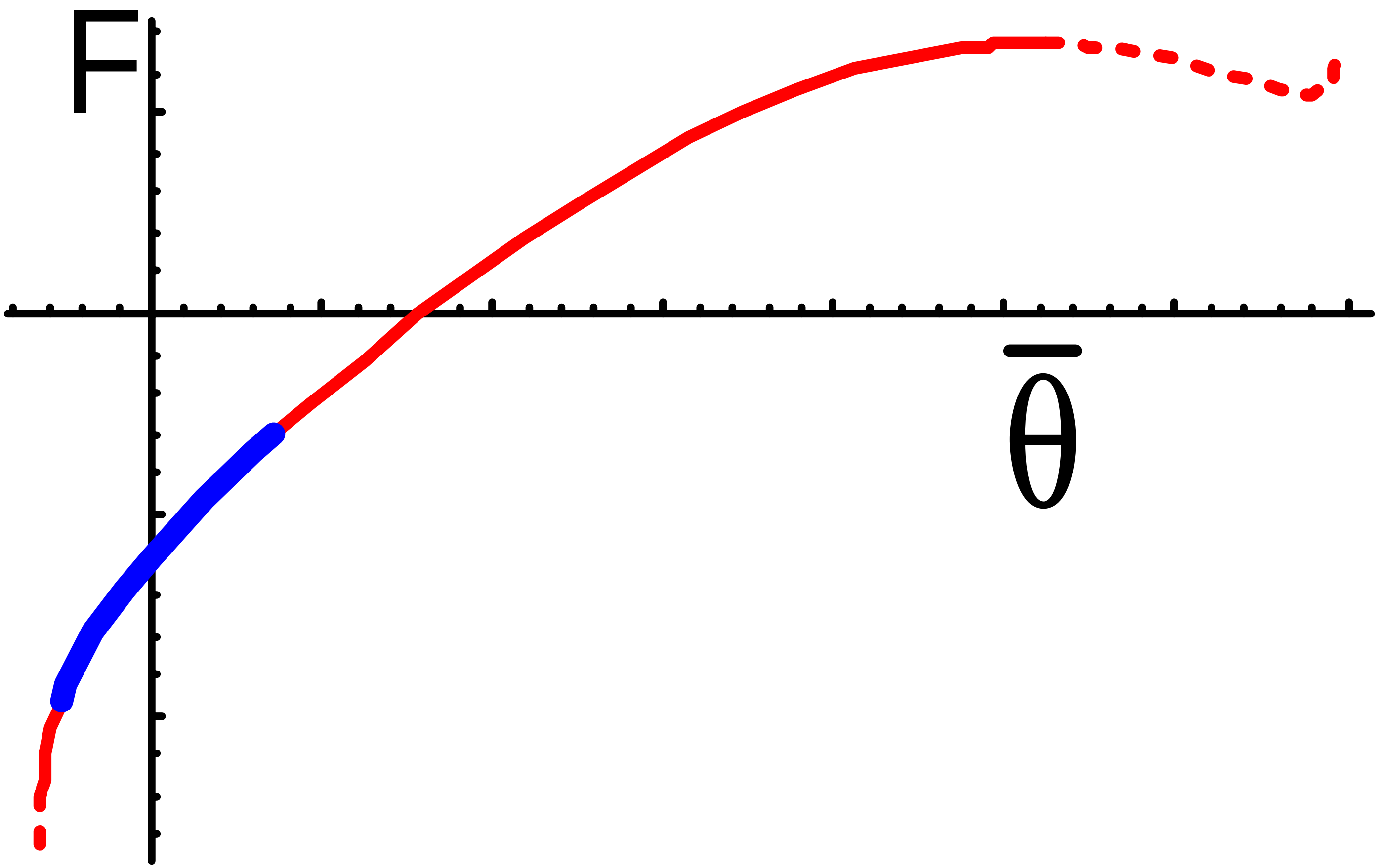




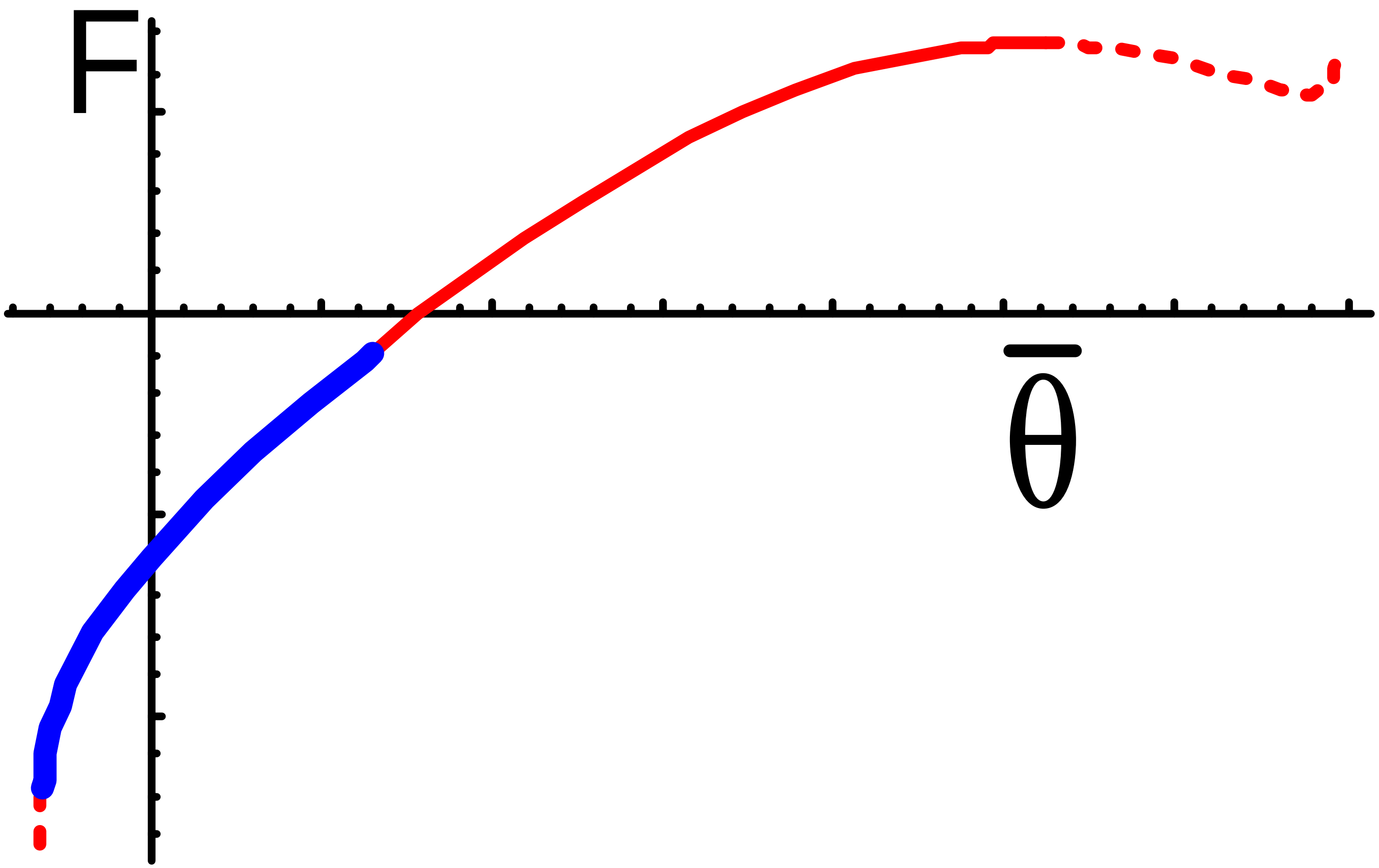
Solution for narrow distribution



Solution for wider distribution



Critical distribution width

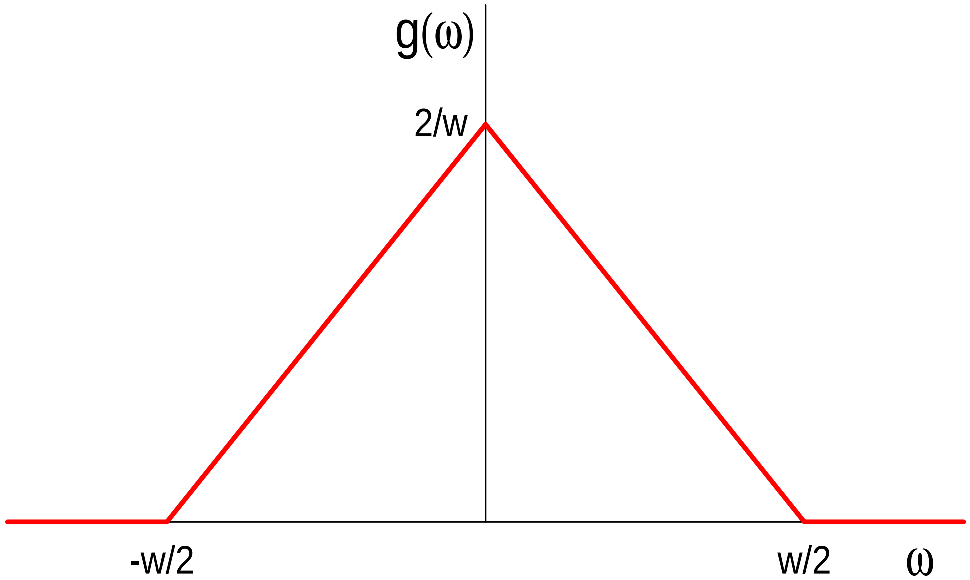


Simulations

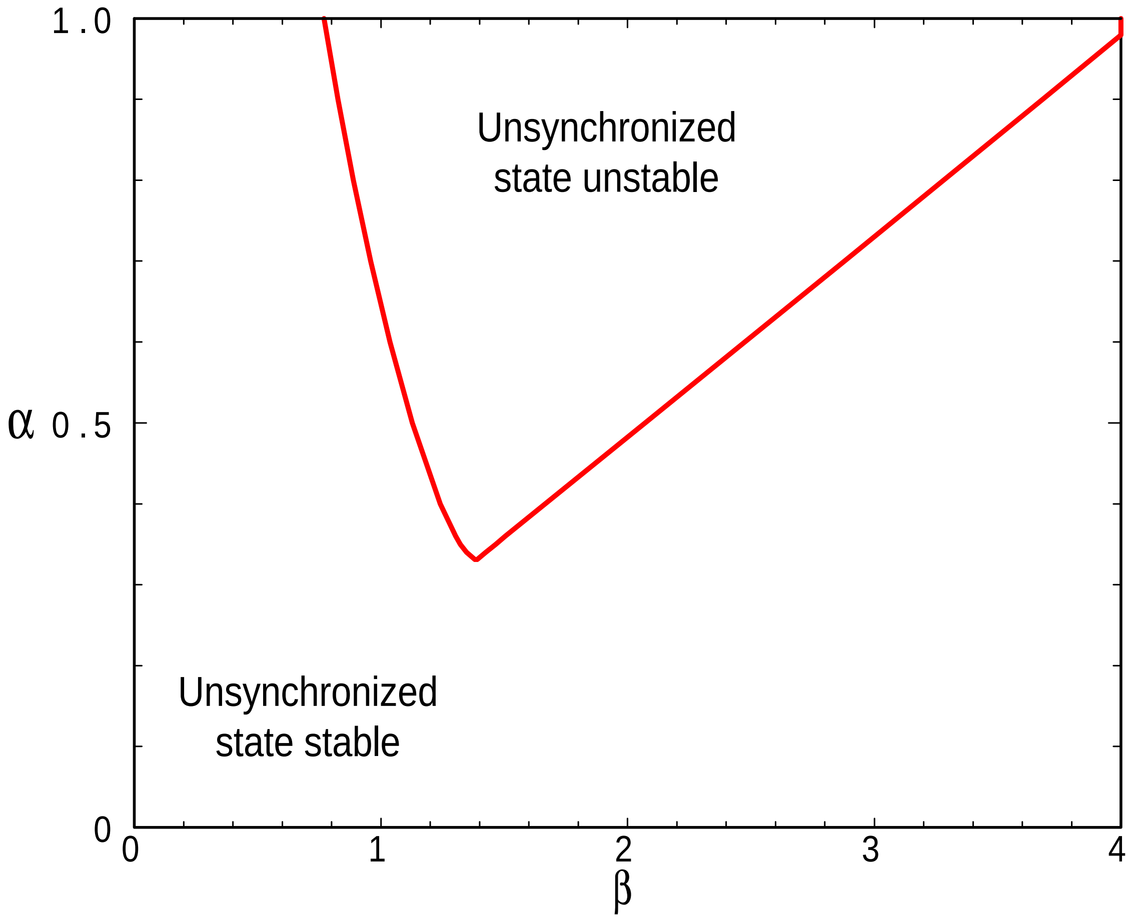
Results

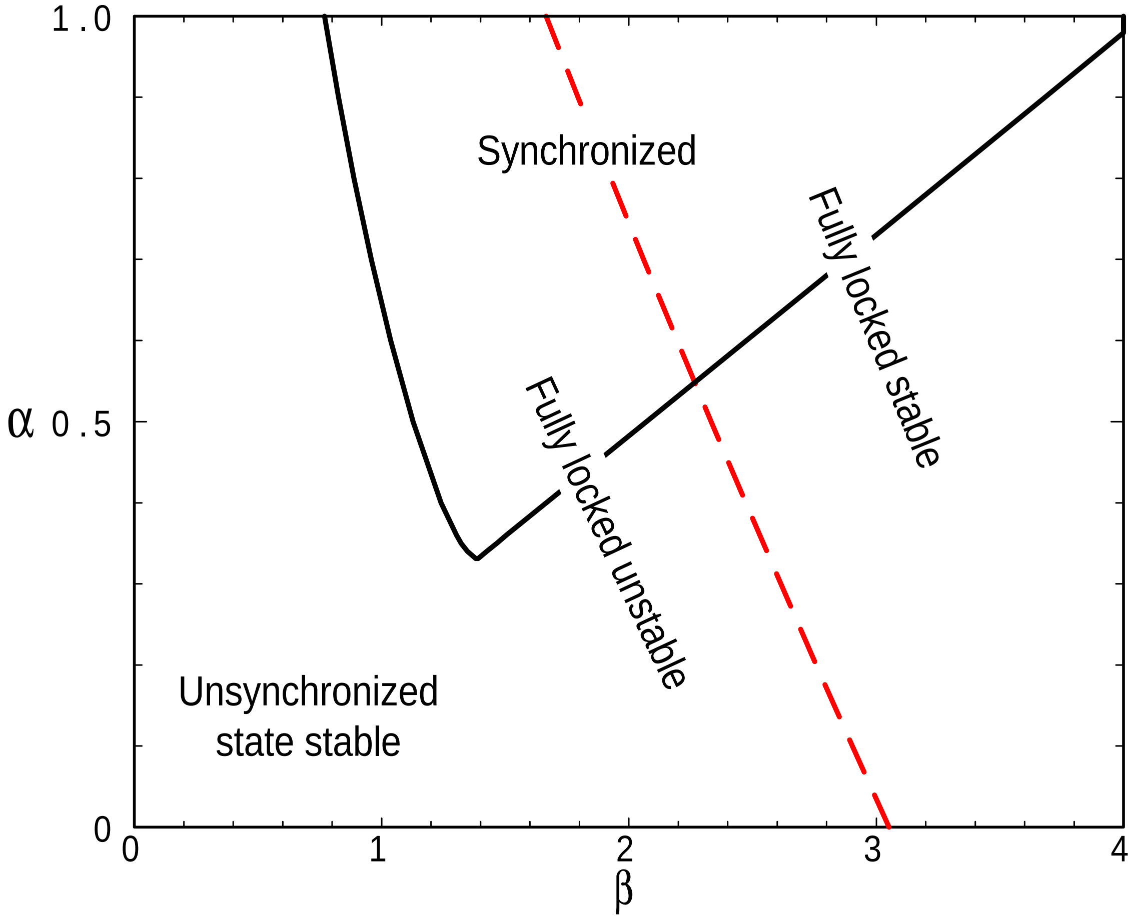
- Order parameter frequency $\Omega = \dot{\Theta}$ not trivially given by $g(\omega)$
- For fixed $\alpha > \alpha_{\text{min}}$ there are two values of β giving linear instability
- Large β instability may be to a synchronized state with no frequency locking
- Linear instability of fully locked state may be through stationary or Hopf bifurcation
- No “amplitude death” as in Matthews et al.
- Complicated phase diagram with regions of coexisting states
- Hysteresis common on parameter sweeps

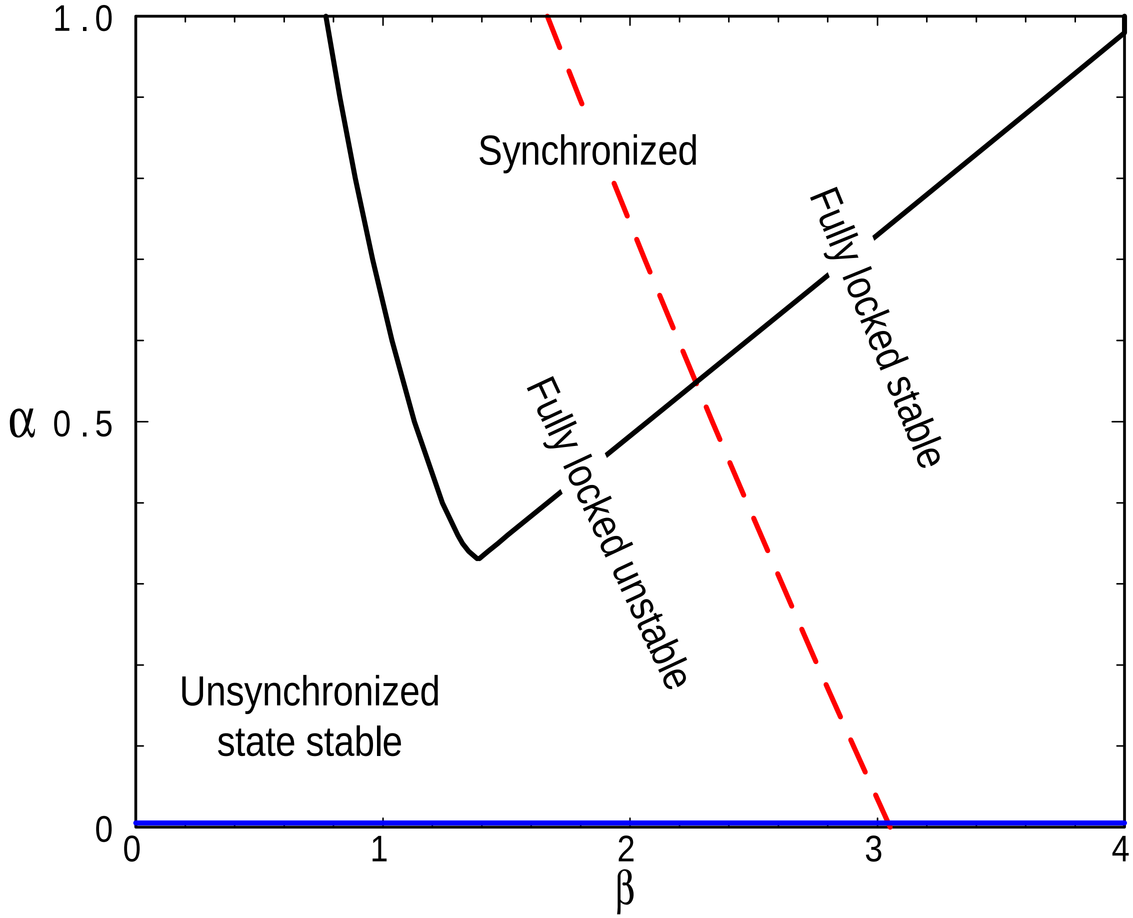
Results for a triangular distribution

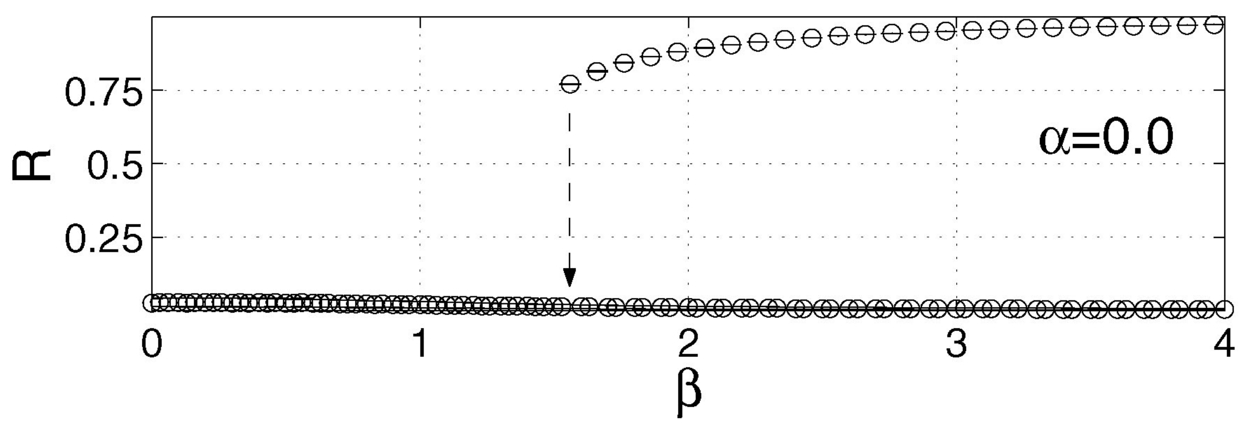


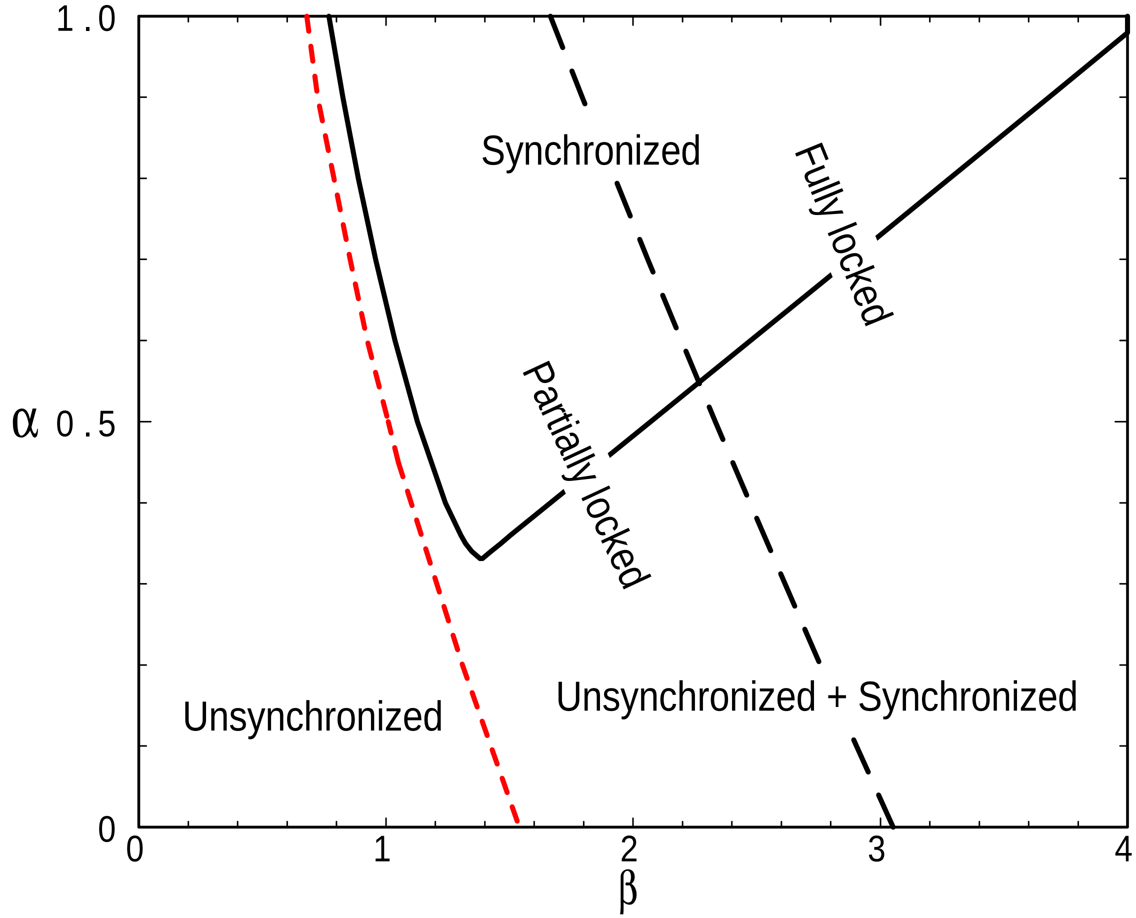
Show results for $w = 2 \ldots$

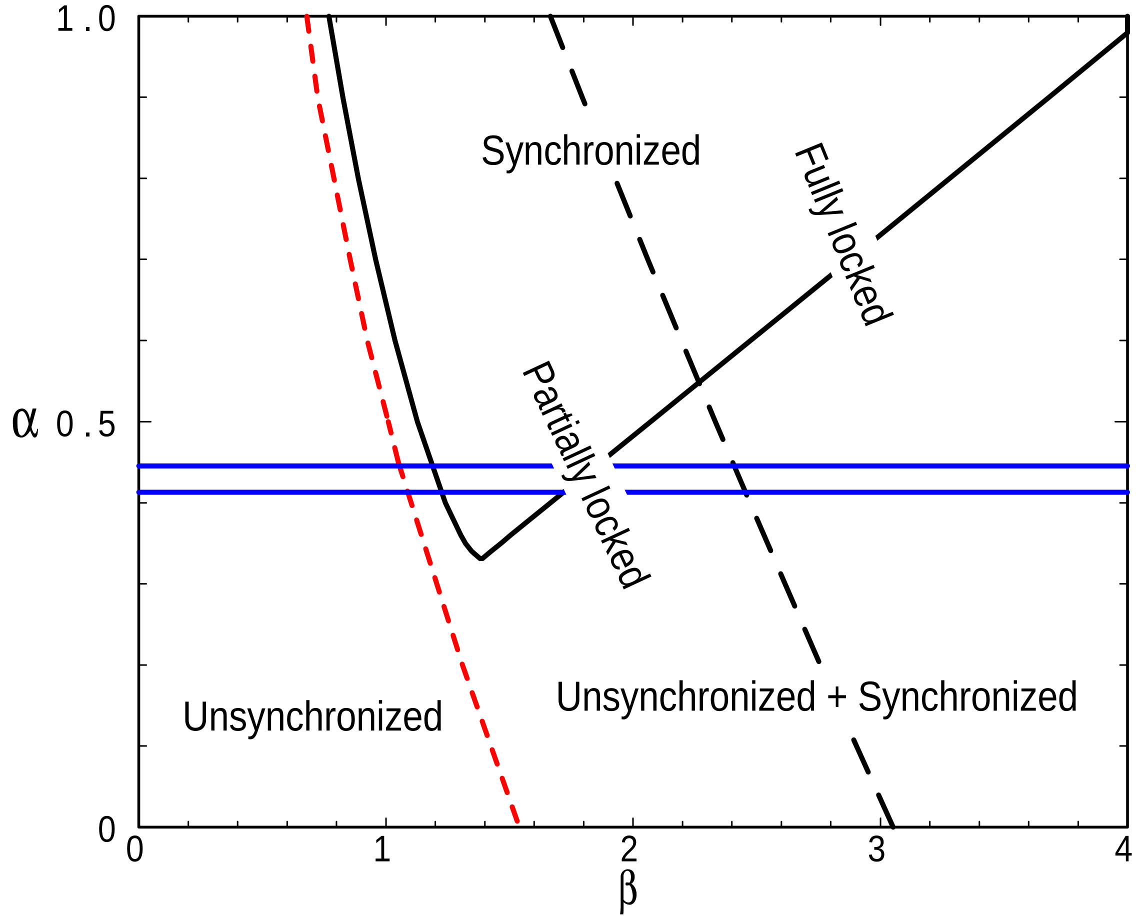

1.0
α 0.5
0
0
1
2
3
4
β
Unsynchronized
state unstable
Unsynchronized
state stable


1.0
α 0.5
0
0
1
2
3
4
β
Synchronized
Fully locked stable
Fully locked unstable
Unsynchronized
state stable

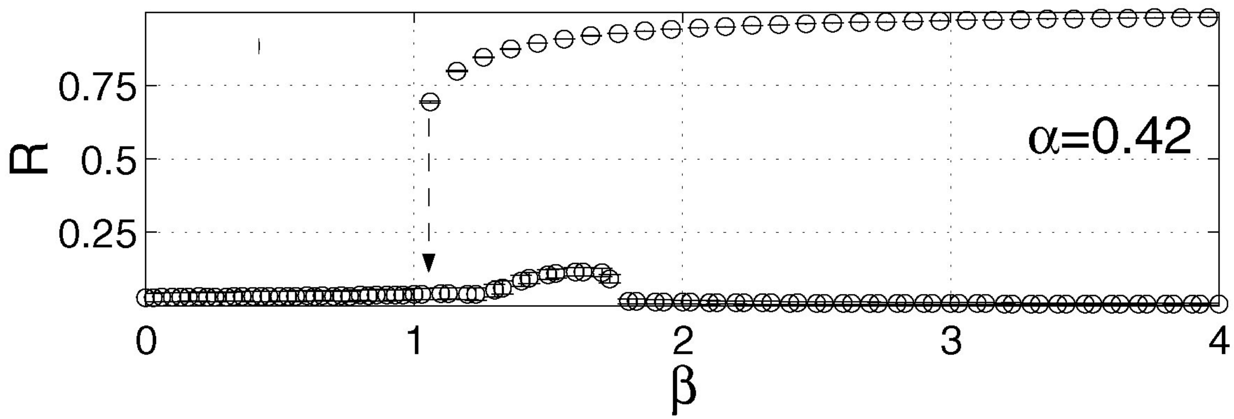

1.0
α 0.5
0
0
1
2
3
4
β
Synchronized
Fully locked stable
Fully locked unstable
Unsynchronized
state stable


R
0.75
0.5
0.25
0
1
2
3
4
β
α=0.0

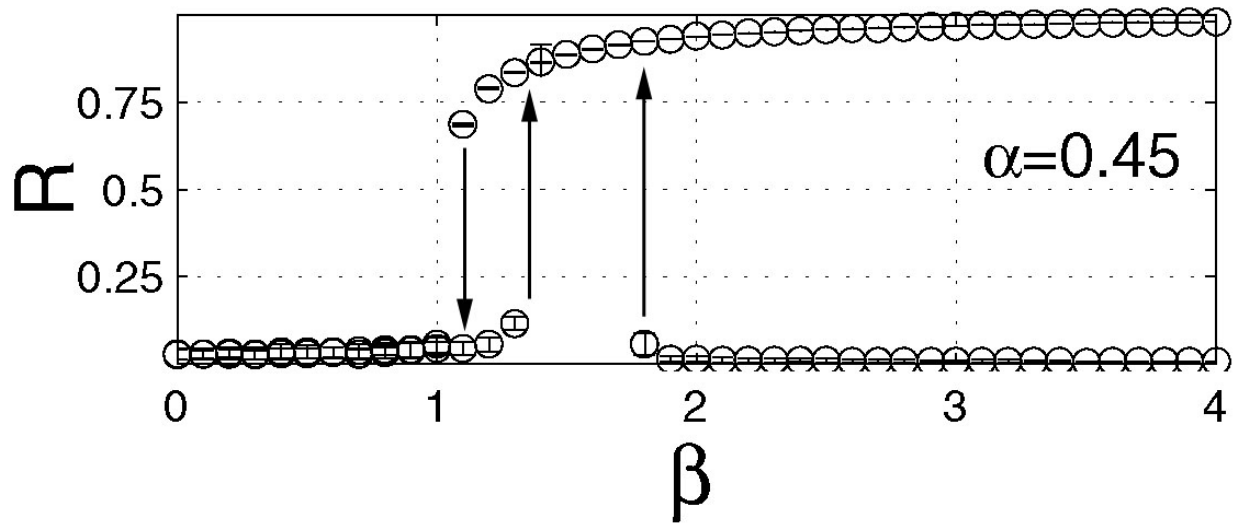

1.0
α 0.5
0
0
1
2
3
4
β
Synchronized
Fully locked
Partially locked
Unsynchronized
Unsynchronized + Synchronized



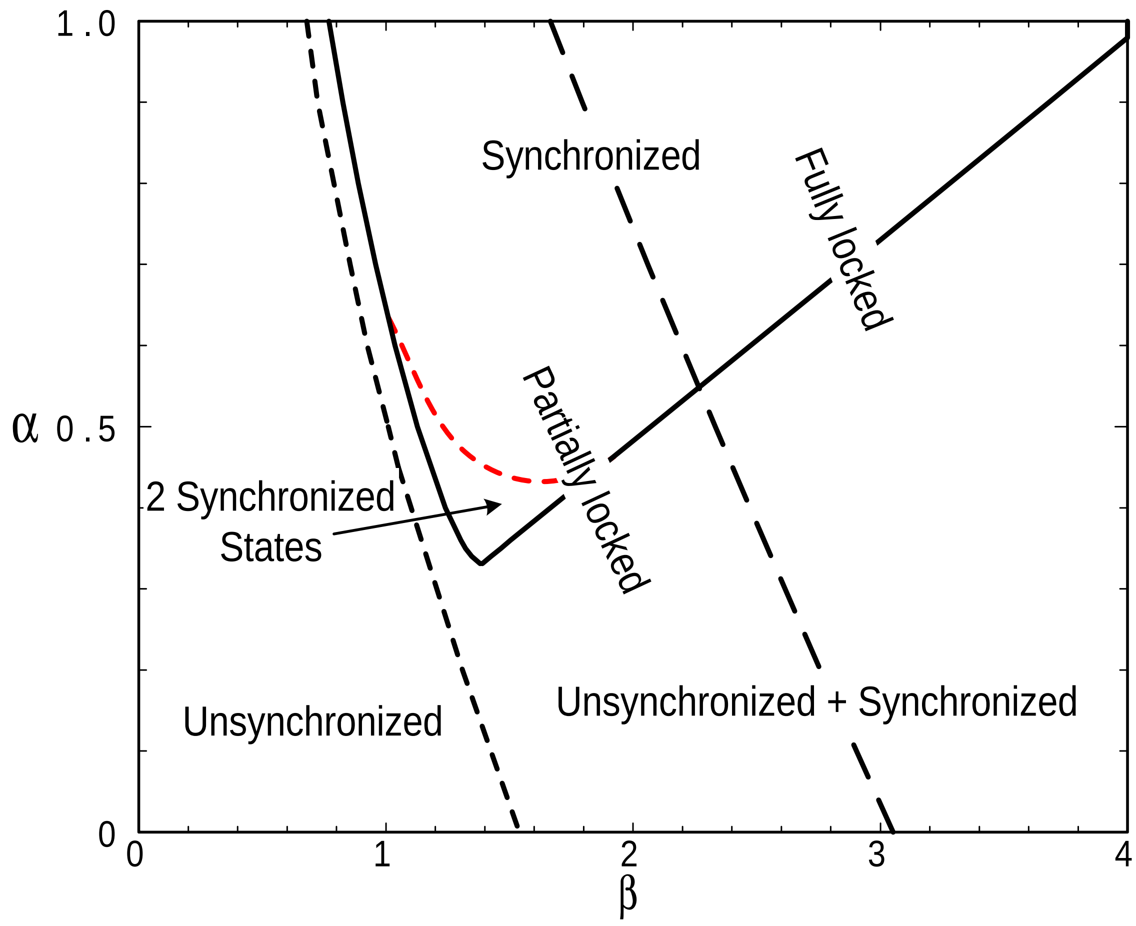
1.0
α 0.5
0
0
1
2
3
4
β
Synchronized
Fully locked
Partially locked
Unsynchronized
Unsynchronized + Synchronized

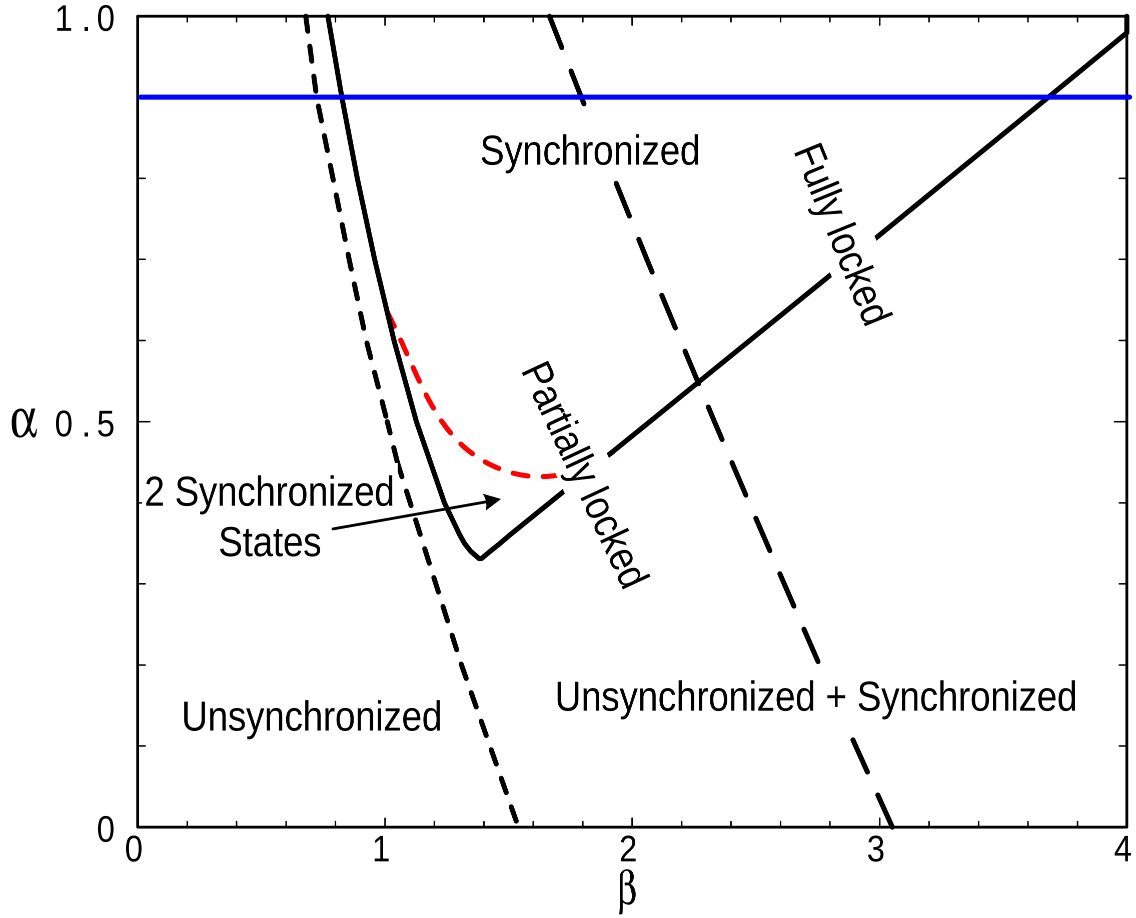


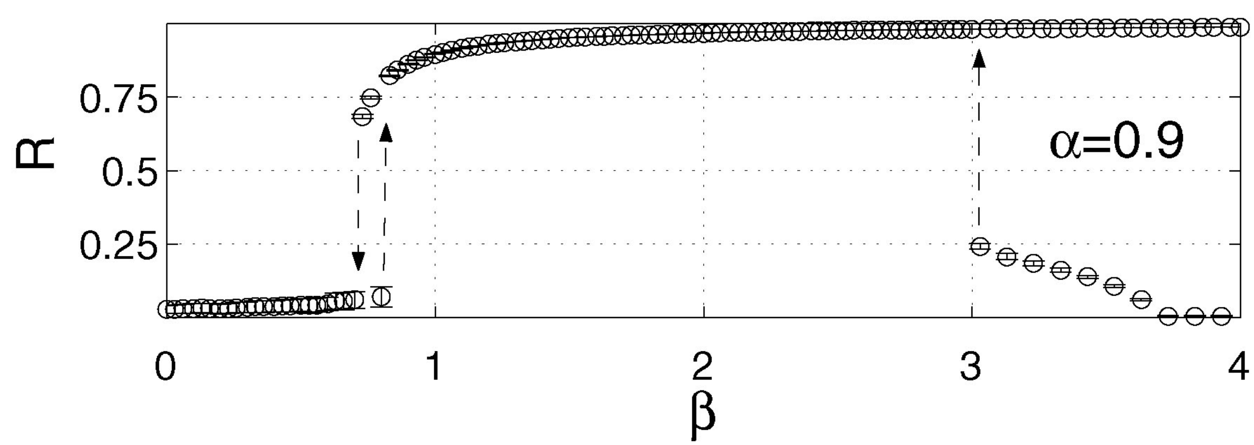
0.75
R
0.5
0.25
α=0.42
0
1
2
3
4
β

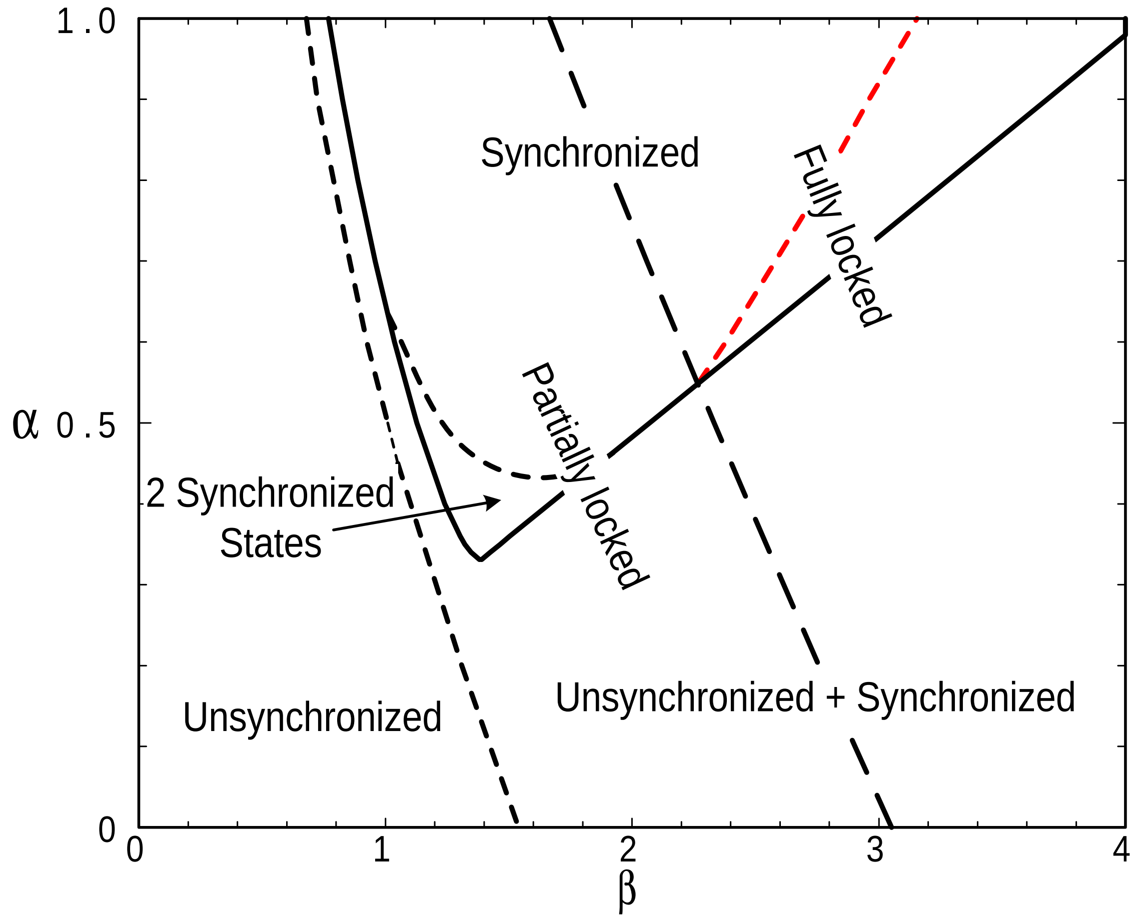


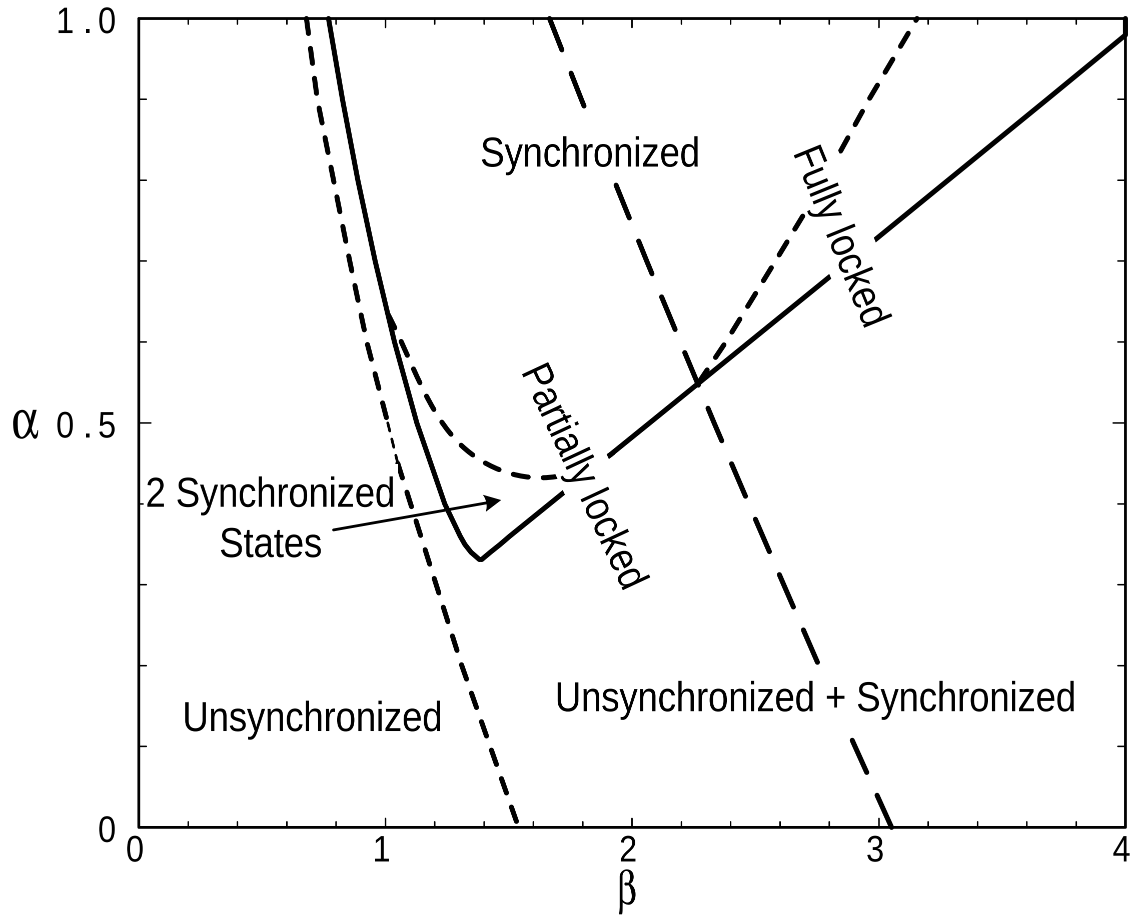
0.75
R
0.5
0.25
α=0.45
0
1
2
3
4
β


1.0
α 0.5
0
0
1
2
3
4
β
Synchronized
Fully locked
Partially locked
2 Synchronized
States
Unsynchronized
Unsynchronized + Synchronized

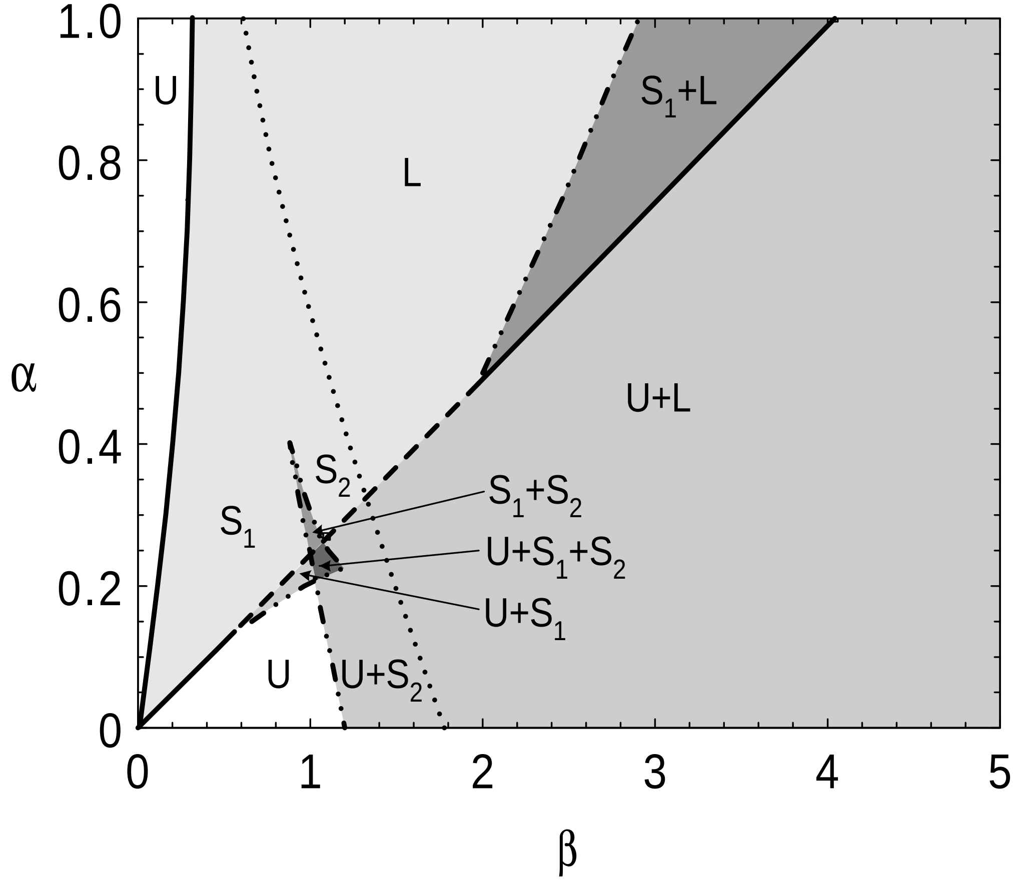

1.0
α 0.5
0
0
1
2
3
4
β
Synchronized
Fully locked
Partially locked
2 Synchronized
States
Unsynchronized
Unsynchronized + Synchronized


0.75
0.5
0.25
R
α=0.9
0
1
2
3
4
β

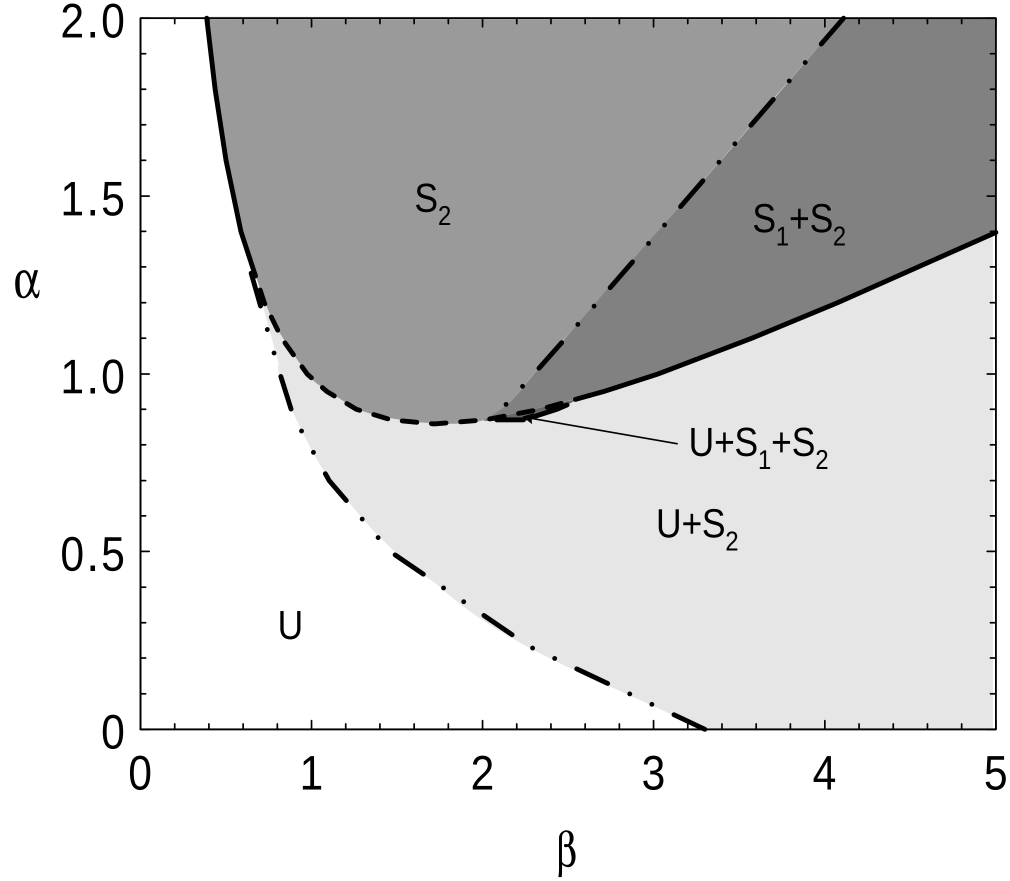

1.0
α 0.5
0
0
1
2
3
4
β
Synchronized
Fully locked
Partially locked
2 Synchronized
States
Unsynchronized
Unsynchronized + Synchronized


1.0
α 0.5
0
0
1
2
3
4
β
Synchronized
Fully locked
Partially locked
2 Synchronized
States
Unsynchronized
Unsynchronized + Synchronized

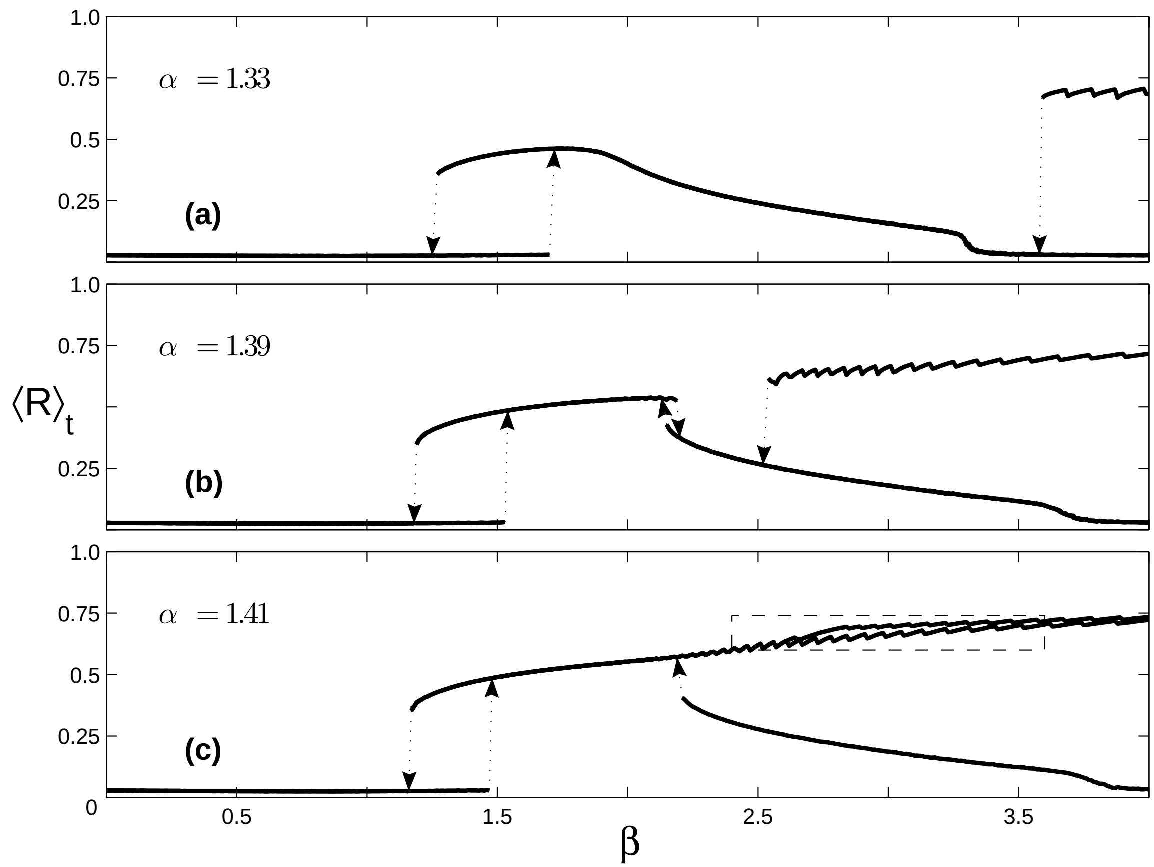
Top-Hat $g(0) = 1$



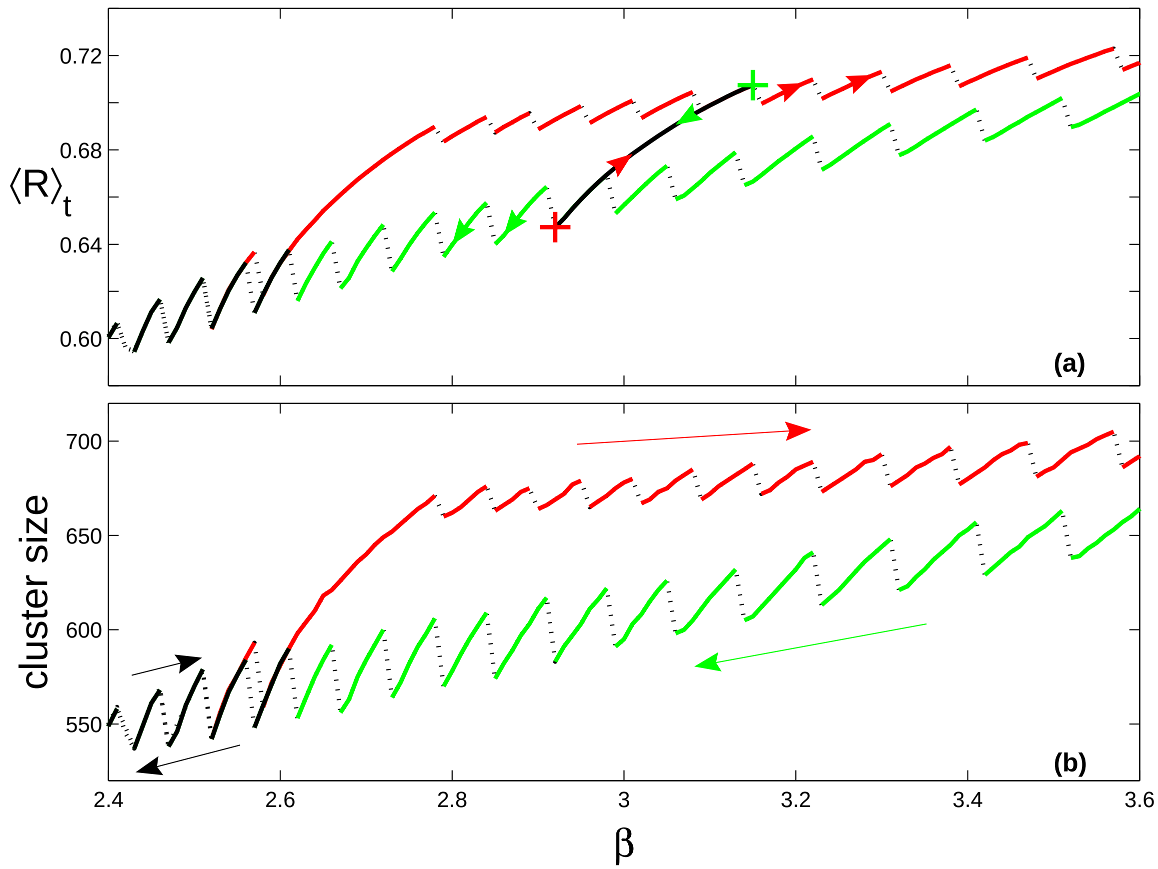
Lorentzian $g(0) = 1$



Wider Lorentzian $g(0) = 0.5$



Large amplitude synchronized state

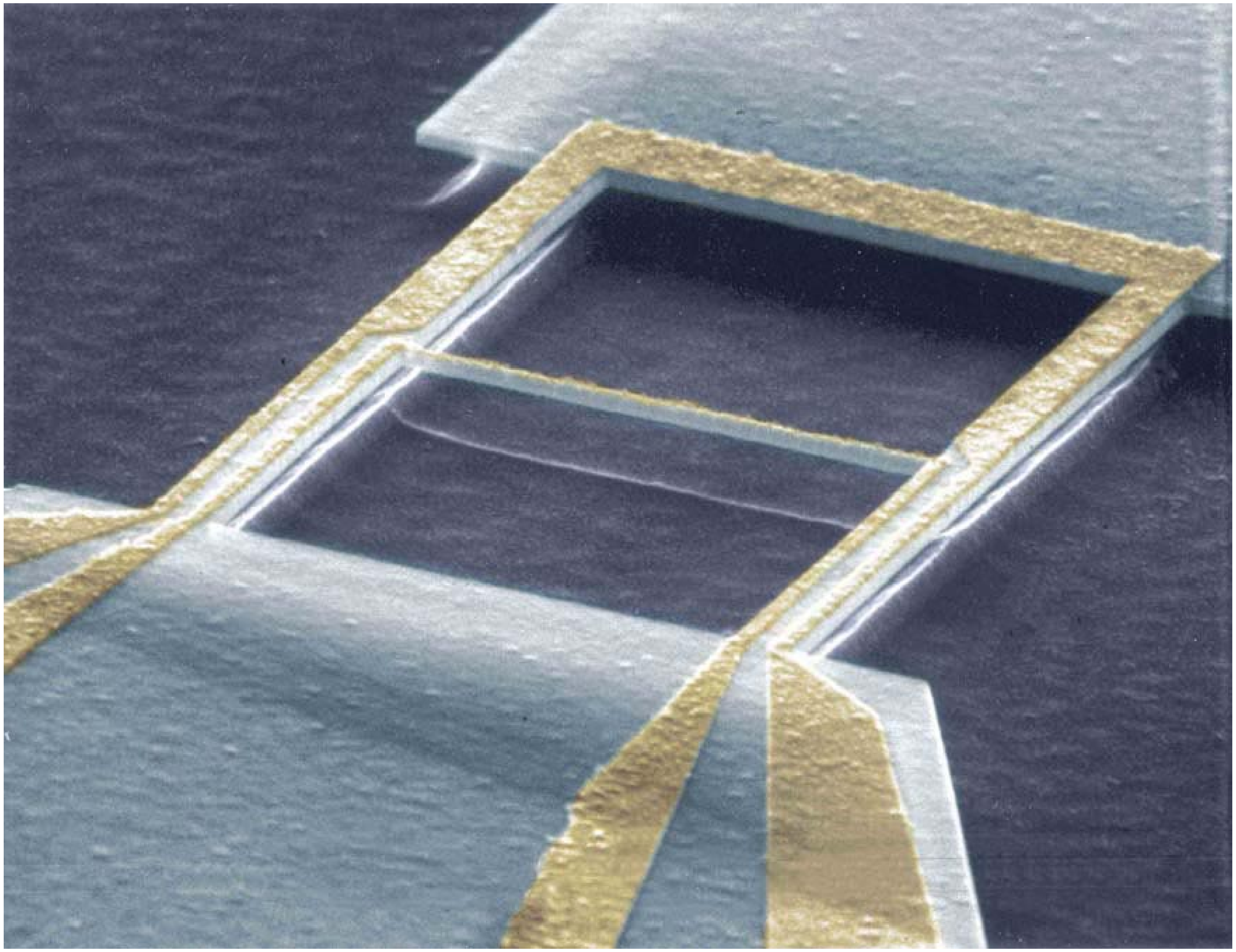


Parametric drive in MEMS

$$\ddot{x} + \gamma \dot{x} + (1 + g_P \cos \omega_P t)x + x^3 = 0$$

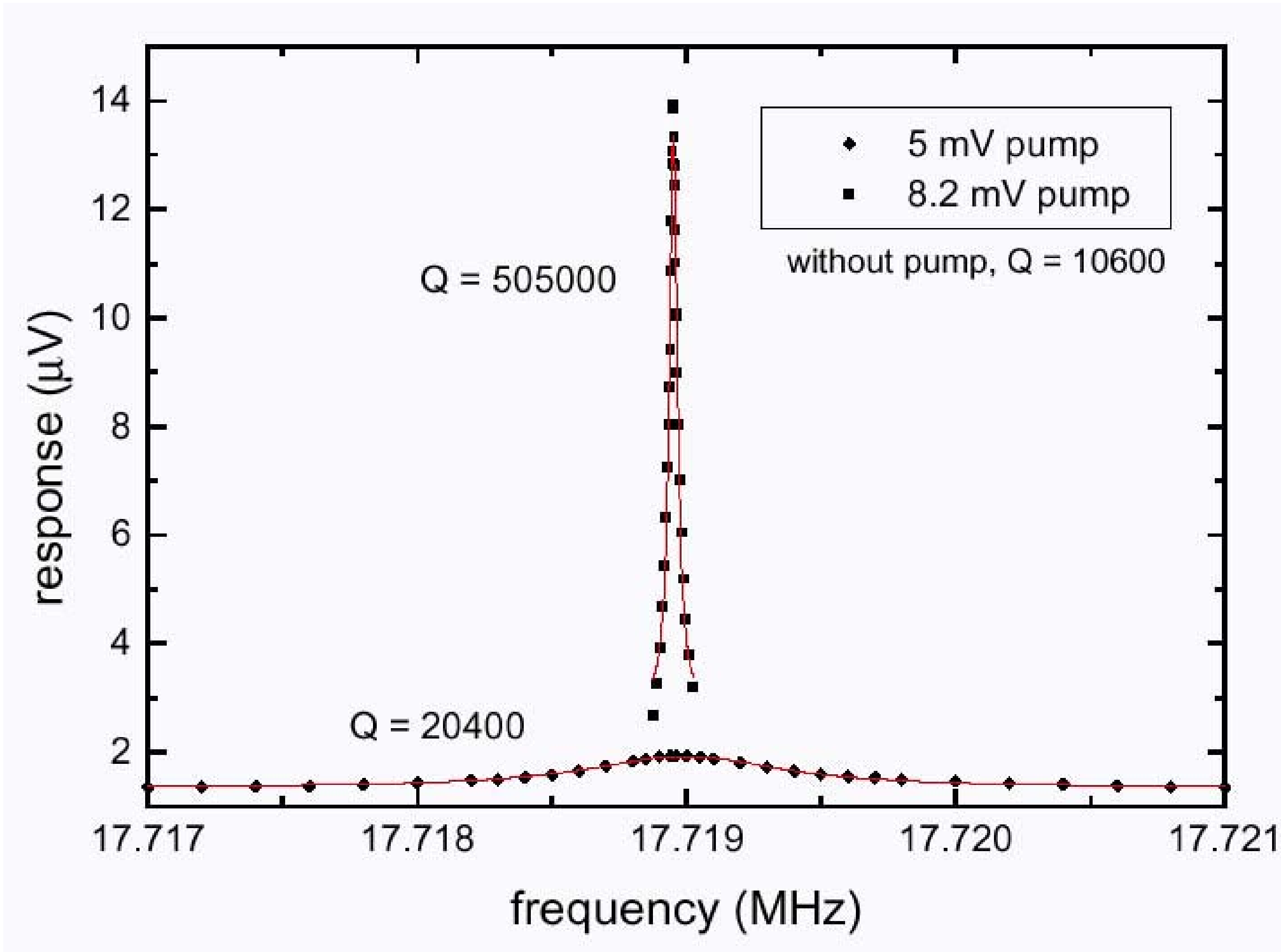
- oscillation of *parameter* of equation—here the spring constant
- $x = 0$ remains a solution in the absence of noise
- parametric drive decreases effective dissipation (for one quadrature of oscillations)
  - ◇ amplification for small drive amplitudes
  - ◇ instability for large enough drive amplitudes
- strongest response for $\omega_p = 2$

MEMS Elastic parametric drive



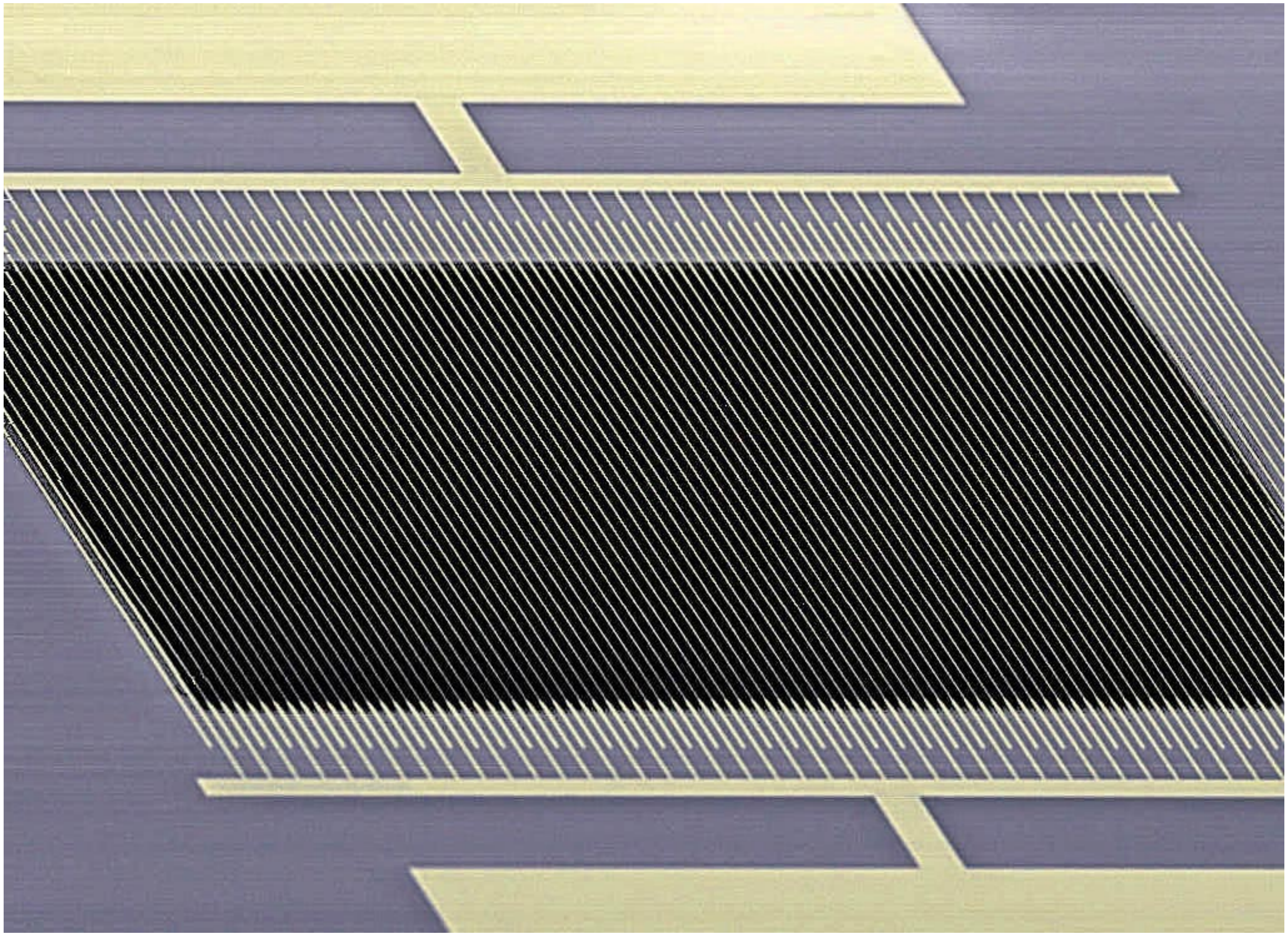
[Harrington and Roukes]

Amplification



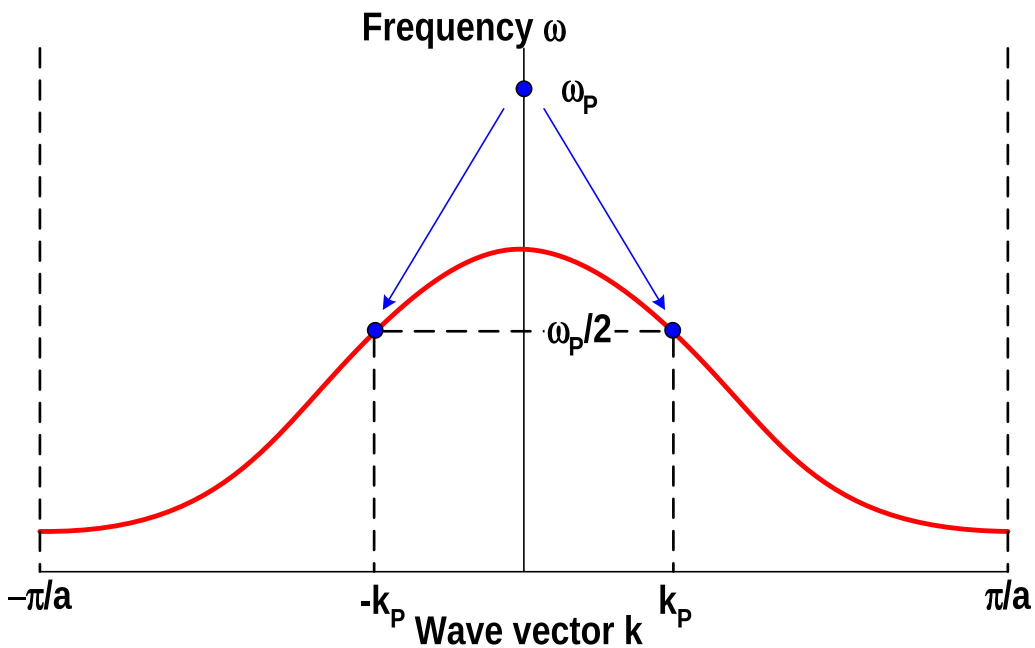
[Harrington and Roukes]

Parametric instability in arrays of oscillators



[Buks and Roukes, 2001]

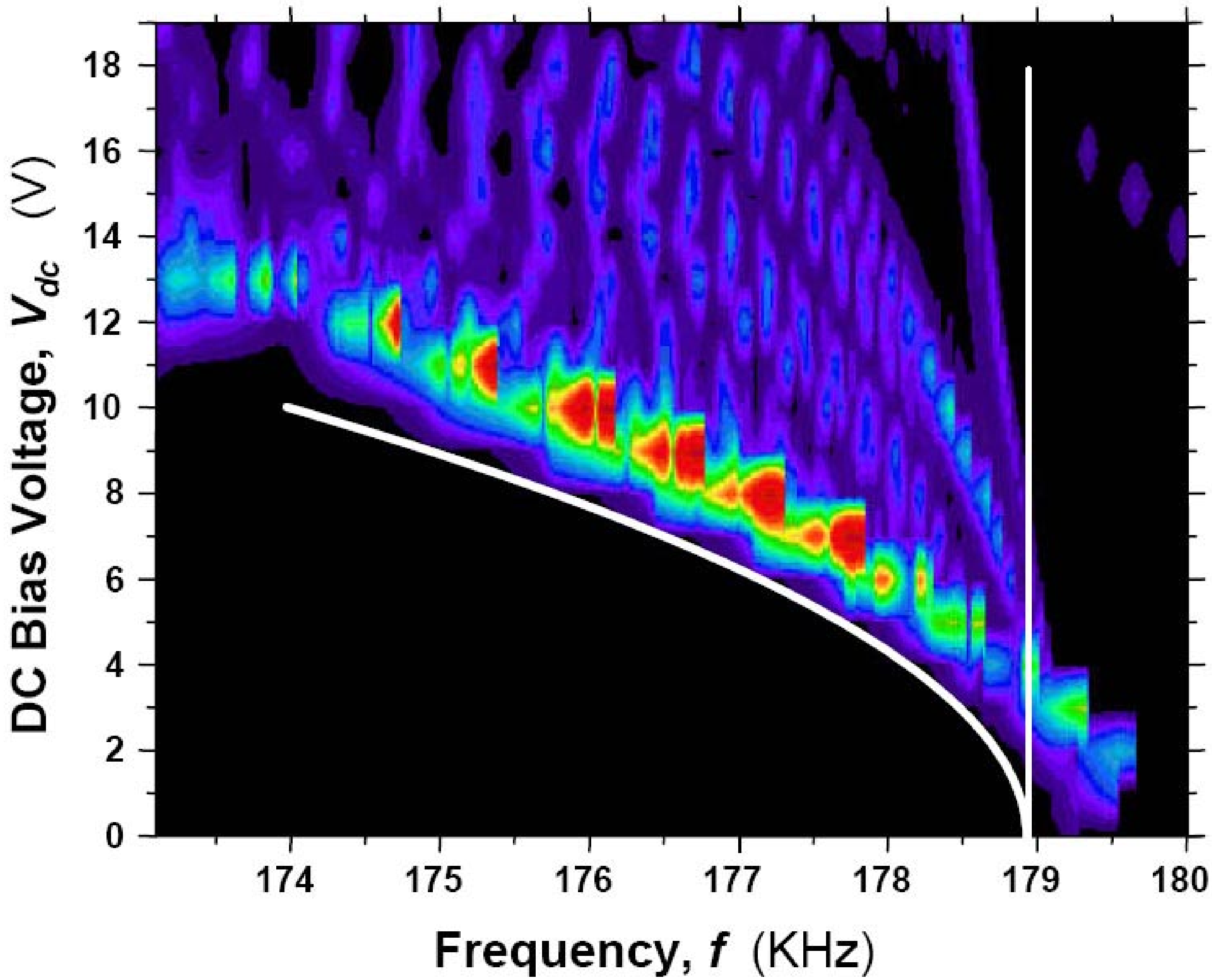
Simple intuition



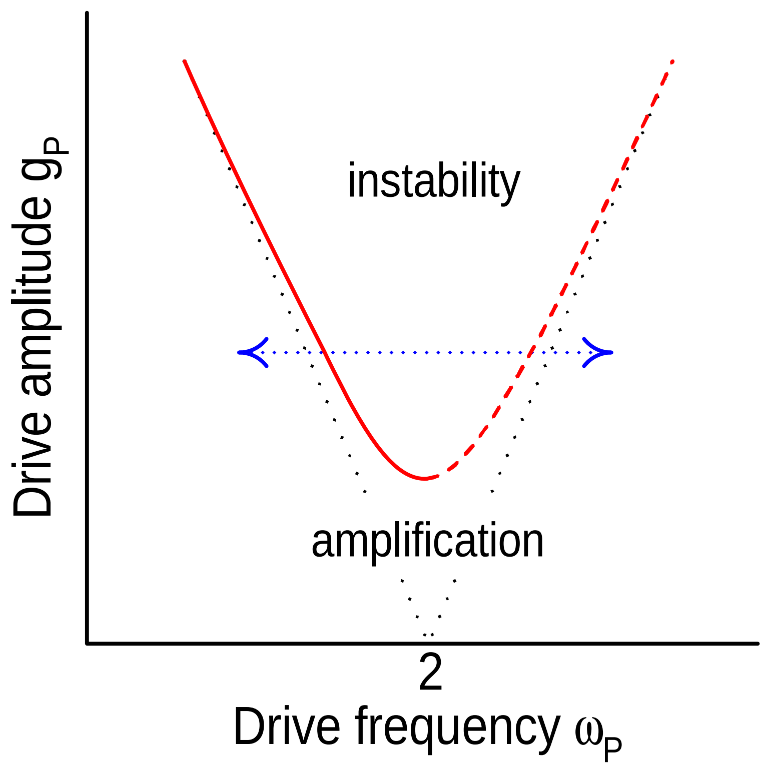
Above the parametric instability nonlinearity is essential to understand the oscillations.

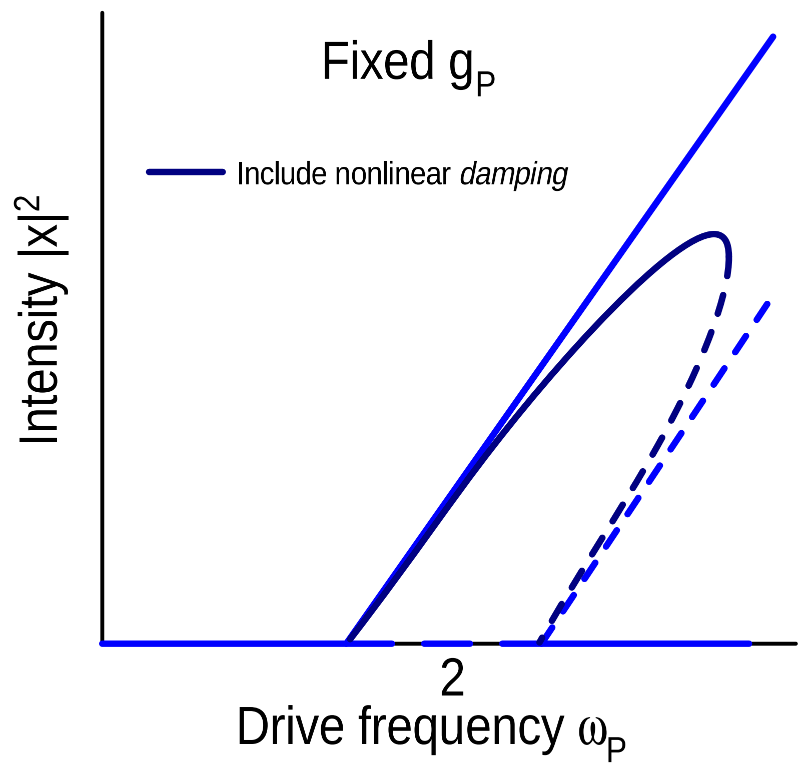
- Mode Competition
- Pattern formation

Experimental results



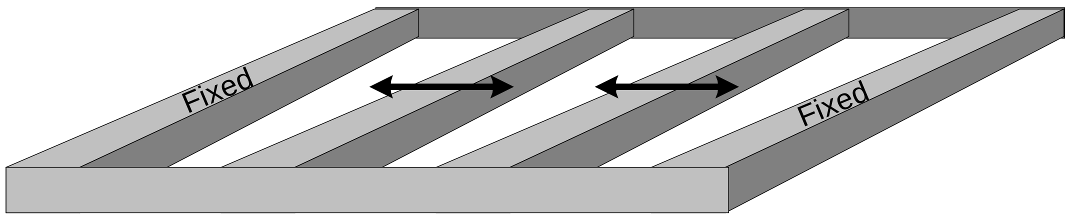
One beam theory





$$2i\omega\frac{dA}{dT} - \frac{h}{2}A^* e^{i\Omega T} + i\omega\gamma A + 3|A|^2 A + i\omega\eta |A|^2 A = 0, \qquad A(T) \Rightarrow a e^{i\frac{\Omega}{2}T}$$

Many beam theory

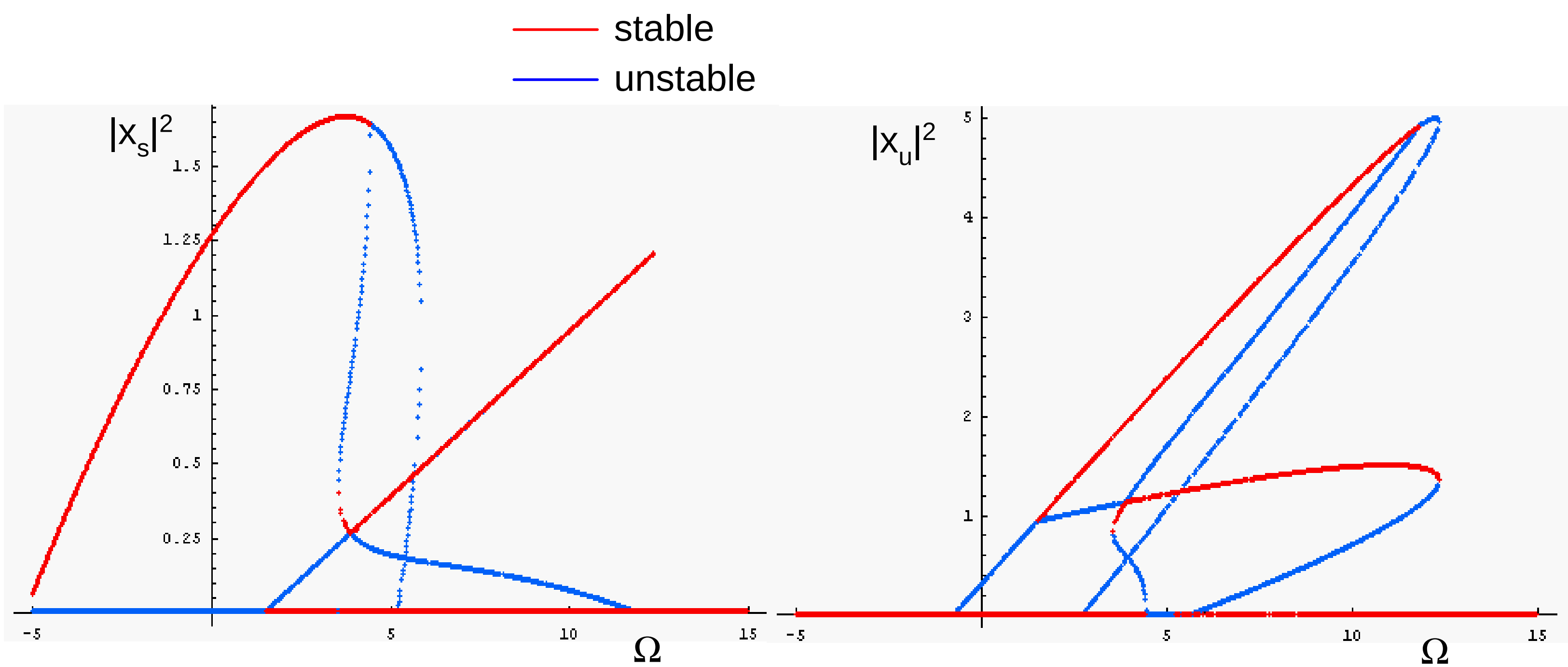


$$
\begin{aligned}
0 = \ddot{x}_n + x_n + x_n^3 \\
+ \Delta^2(1 + g_P \cos[(2 + \varepsilon\Omega_P)t])(x_{n+1} - 2x_n + x_{n-1}) \\
- \gamma(\dot{x}_{n+1} - 2\dot{x}_n + \dot{x}_{n-1}) \\
+ \eta\left[(x_{n+1} - x_n)^2(\dot{x}_{n+1} - \dot{x}_n) - (x_n - x_{n-1})^2(\dot{x}_n - \dot{x}_{n-1})\right]
\end{aligned}
$$

Local Duffing (elasticity) + Electrostatic Coupling (dc and modulated) + Dissipation (currents) + Nonlinear Damping (also currents)

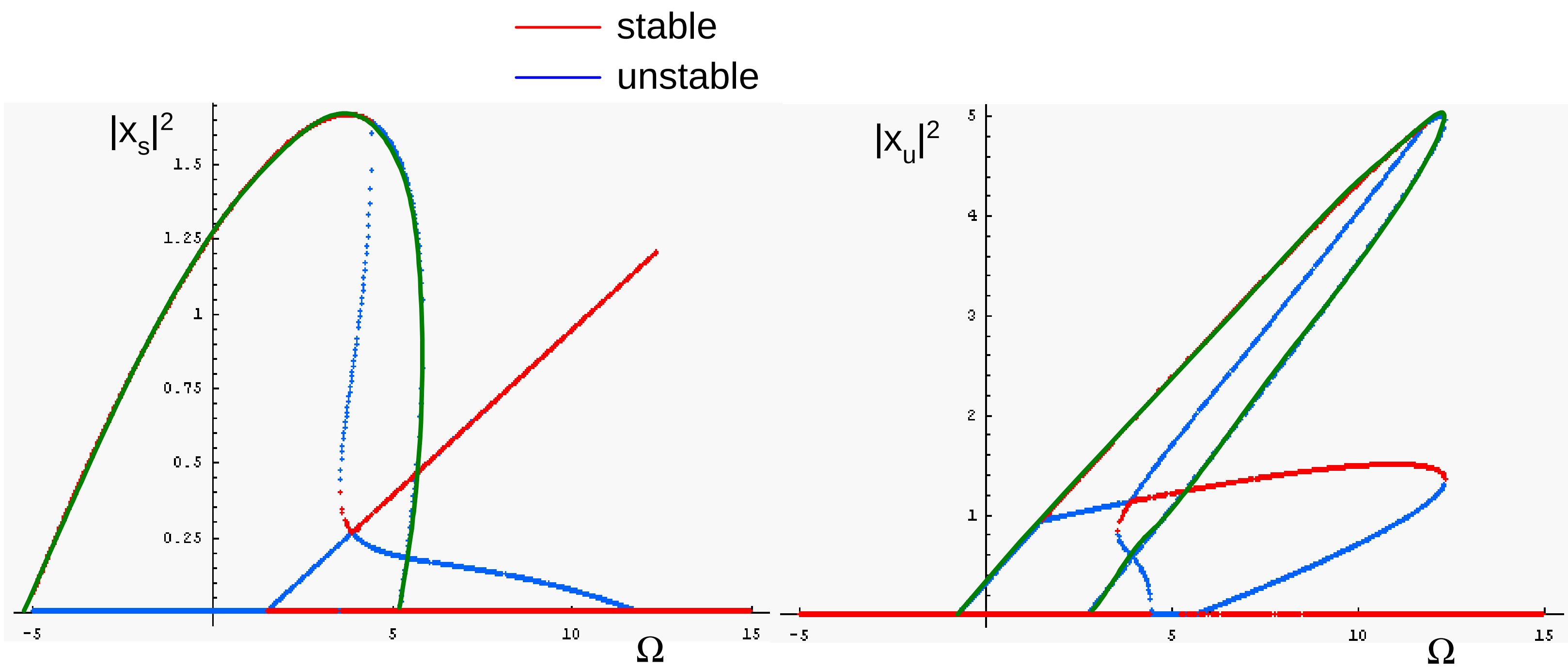
[Lifshitz and MCC Phys. Rev. B67, 134302 (2003)]

2 beam periodic solutions



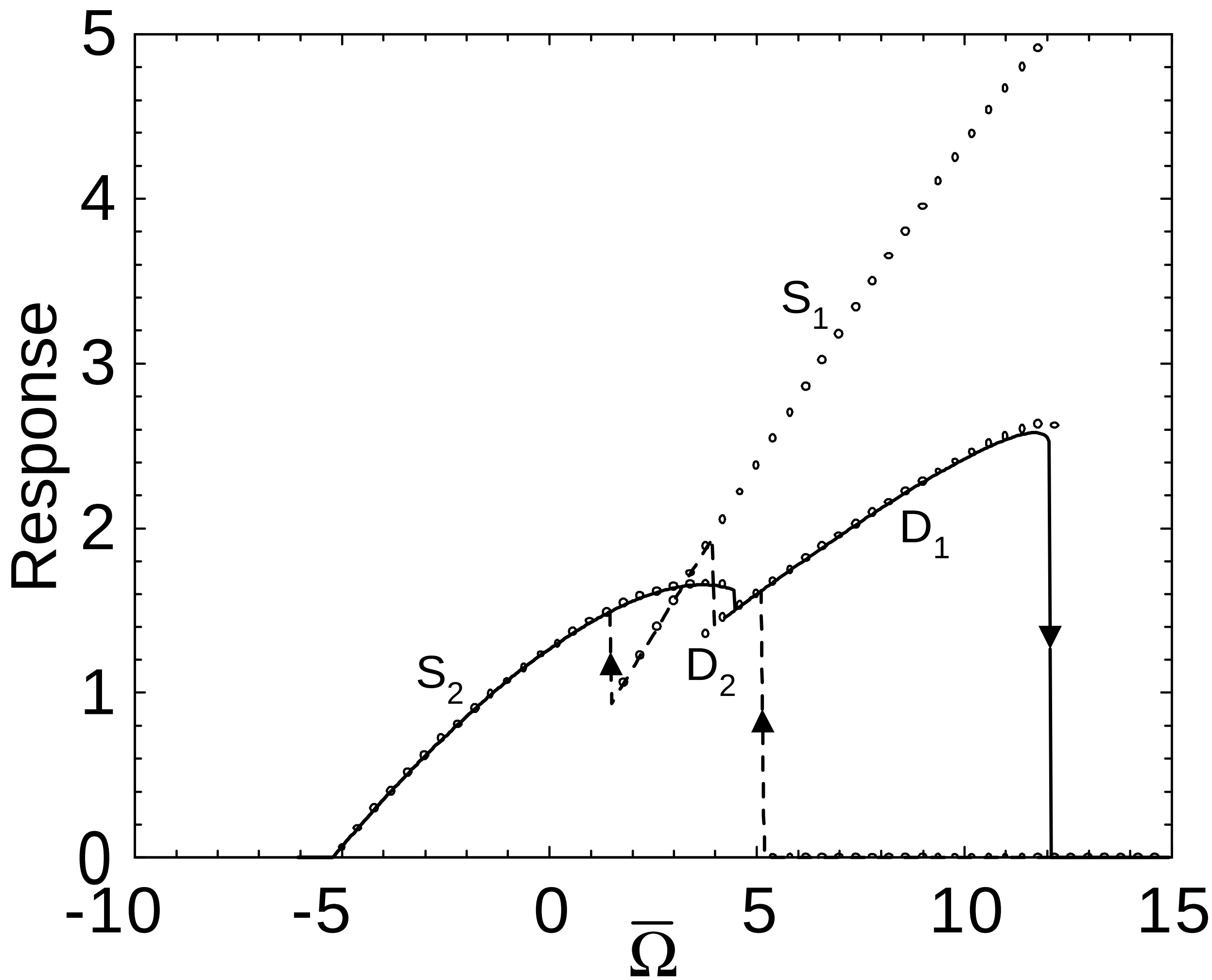
Intensity of symmetric mode $|x_s|^2$ and antisymmetric mode $|x_u|^2$ as frequency is scanned.

2 beam periodic solutions

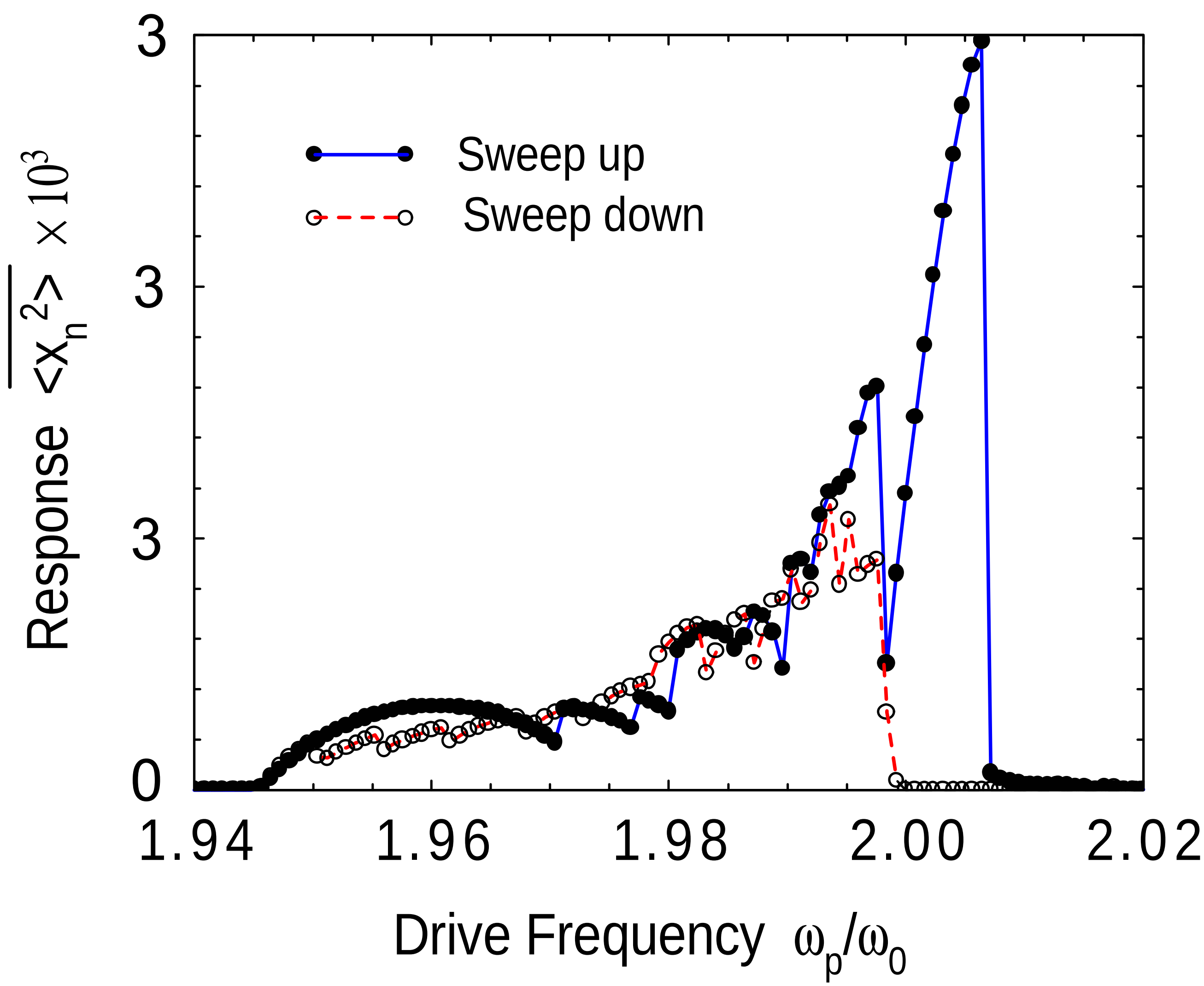


The green lines correspond to a single excited mode, the remainder to coupled modes.

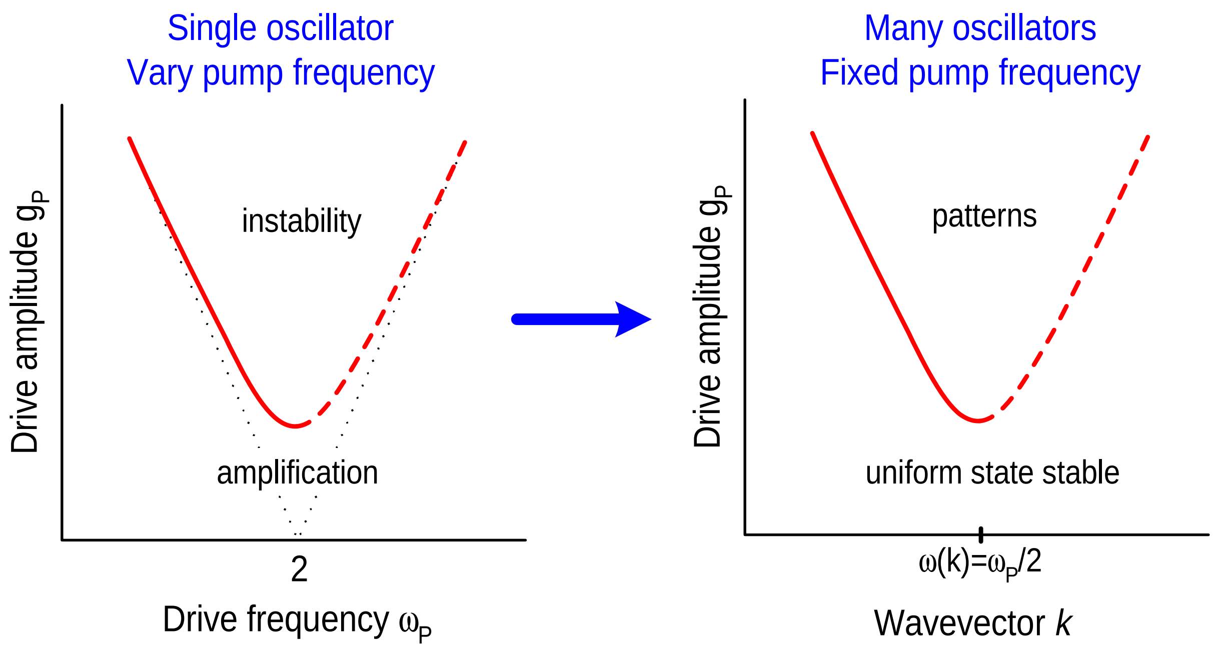
Hysteresis for two beams



Simulations of 67 Beams



Many beams



Continuum approximation: new amplitude equation

[Bromberg, MCC and Lifshitz (preprint, 2005)]

$$\frac{\partial A}{\partial T} = A + \frac{\partial^2 A}{\partial X^2} + i\frac{2}{3}\left(4\,|A|^2\,\frac{\partial A}{\partial X} + A^2\frac{\partial A^*}{\partial X}\right) - 2\,|A|^2\,A - |A|^4\,A$$

Conclusions

- I've described models of nonlinear oscillators motivated by considerations of arrays of nanomechanical devices.
- Collective effects
  - ◇ Synchronization due to nonlinear frequency pulling and reactive coupling
  - ◇ Parametrically driven arrays
- As devices get smaller (e.g. carbon nanotubes) thermal fluctuations and quantum effects will become important
  - ◇ Noise induced transitions between driven (nonequilibrium) states
    - ⋆ Single nonlinear oscillator [Aldridge and Cleland, Phys. Rev. Lett. **94**, 156403 (2005)]
    - ⋆ Collective states in arrays of oscillators
  - ◇ Measurement of discrete levels in quantum harmonic oscillator [Santamore, Doherty, and MCC, Phys. Rev. **B70**, 144301 (2004)]