Synchronization by Nonlinear Frequency Pulling

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May 2006

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Support: NSF, BSF, NATO and EU

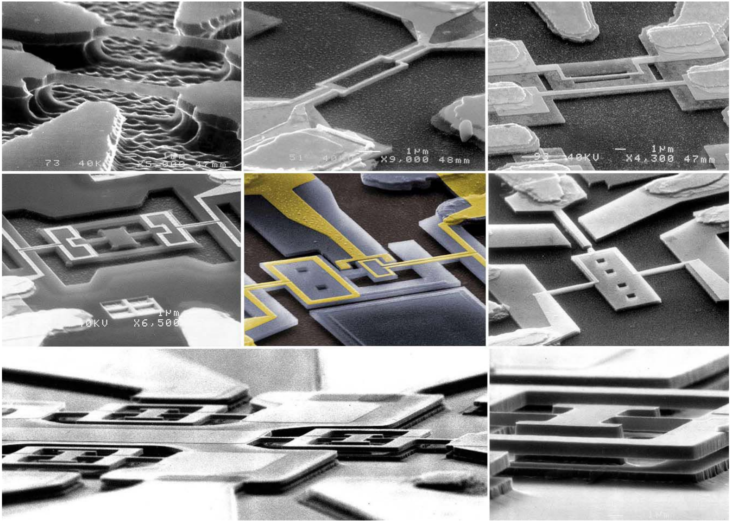
Message

- Arrays of micro- and nano-scale oscillators provide an interesting dynamical and pattern forming system
- Synchronization is an important feature of nonlinear oscillators and may be important technologically in MEMS/NEMS
- Synchronization in MEMS/NEMS motivates a new analysis of synchronization involving nonlinear frequency pulling and reactive coupling

Outline

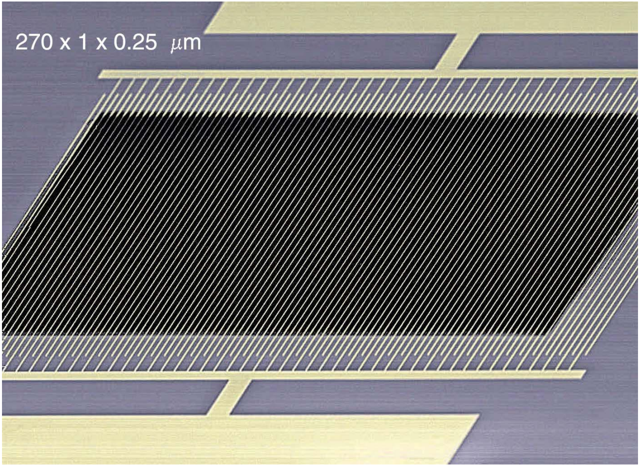
Outline

Variety of Devices



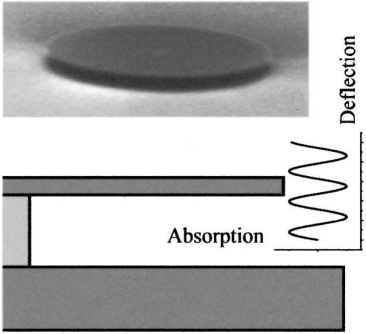
[From M. R. Roukes, Caltech]

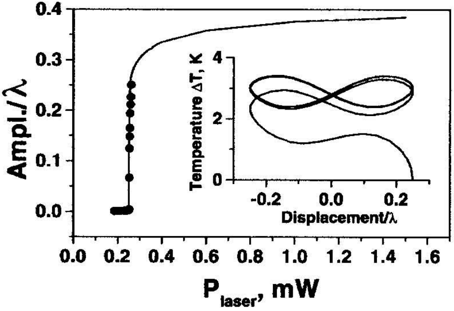
Array of μ-scale Oscillators



[From Buks and Roukes J. MEMS. **11**, 802 (2002)]

Self-Oscillations





[Zalalutdinov et al., Appl. Phys. Lett. **79**, 695 (2001)]

MicroElectroMechanical Systems and NEMS

Tiny mechanical oscillators:

- driven, dissipative $\Rightarrow$ nonequilibrium
- nonlinear
- collective (arrays)
- noisy
- (potentially) quantum

Goals

- Apply knowledge from statistical mechanics, nonlinear dynamics, pattern formation etc. to technologically important questions
- Investigate stochastic and nonlinear dynamics, and pattern formation in new regimes

Modelling High Q Oscillators

$$
\begin{aligned}
0 = \ddot{x}_n + x_n \\
+ \gamma \dot{x}_n \qquad & \text{linear damping} \\
+ \delta_n x_n \qquad & \text{with } \delta_n \text{ taken from distribution } g(\delta_n) \\
+ \sum_m D_{nm}(x_m - x_n) \qquad & \text{reactive coupling} \\
+ x_n^3 \qquad & \text{nonlinear stiffening} \\
- \gamma_D \dot{x}_n (1 - x_n^2) \qquad & \text{energy input} \\
+ 2 g_D \cos[(1 + \delta\omega_D) t] \qquad & \text{signal} \\
+ \text{Noise}
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Outline

1. Introduction
   - Motivation: MEMS and NEMS
   - Nonlinearity
   - Huygen's Clocks and the Phase Model
2. Synchronization via Nonlinear Frequency Pulling
   - Model
   - Analytic Calculations
   - Numerical Simulations
   - Results
   - Conclusions

Example: Nonlinearity in a Duffing Oscillator

Single driven, damped anharmonic oscillator:

$$\ddot{x} + \gamma \dot{x} + x + x^3 = 2 g_D \cos(\omega_D t)$$

Parameters:

γ	damping
g_D	drive strength
ω_D	drive frequency

Spring gets *stiffer* with increasing displacement.

Duffing Oscillator Near Resonance

We can calculate behavior close to the sinusoidal oscillation $\propto e^{it}$:

- oscillator driving near resonance $\omega_D \simeq 1$
- small damping
- small driving g_D of oscillation implies the effect of the nonlinearity will be small

To implement these "smallnesses" write

$$\omega_D = 1 + \varepsilon \Omega_D$$
$$g_D = \varepsilon^{3/2} g$$
$$\gamma = \varepsilon \Gamma$$

with $\varepsilon \ll 1$ and g, Γ, Ω_D considered to be of order unity.
(For these scalings the different effects that perturb the oscillator away from $e^{\pm it}$ are comparable. If there is a different scaling of the small parameters, one or more effects may not be important in the dynamics.)

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Amplitude Theory

Introduce the WKB-like *ansatz* for the displacement

$$x(t) = \varepsilon^{1/2} A(T) e^{it} + \text{c.c.} + \varepsilon^{3/2} x_1(t) + \cdots$$

- $A(T)$ is a *complex* amplitude that gives the slow modulation
- $T = \varepsilon t$ is a *slow* time variable:

$$\frac{d}{dt} A = \varepsilon A'(T) \ll 1$$

- $x_1(t)$ and $\cdots$ give corrections to the ansatz that are required to be small

Substitute into the equation of motion using

$$\dot{x} = \varepsilon^{1/2}(iA + \varepsilon A')e^{it} + \text{c.c.} + \varepsilon^{3/2}\dot{x}_1 + \cdots$$

$$\ddot{x} = \varepsilon^{1/2}(-A + 2i\varepsilon A' + \varepsilon^2 A'')e^{it} + \text{c.c.} + \varepsilon^{3/2}\ddot{x}_1 + \cdots$$

and collect terms to give at $O(\varepsilon^{3/2})$

$$\ddot{x}_1 + x_1 = (-2iA' - i\Gamma A - 3|A|^2 A + g e^{i\Omega_D T})e^{it} - A^3 e^{3it} + \text{c.c.} + \cdots$$

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Nonlinear Resonance

For x_1 to be small, the resonant driving terms on the right hand side must be zero. This gives

$$\frac{d}{dT}A = -\frac{\Gamma}{2}A + i\frac{3}{2}|A|^2 A - i\frac{g}{2}e^{i\Omega_D T}$$

After transients the solution is $A = ae^{i\Omega_D T}$ with

$$|a|^2 = \frac{(g/2)^2}{(\Omega_D - \frac{3}{2}|a|^2)^2 + (\Gamma/2)^2}$$

or

$$|x|^2 = \frac{(g_D/2)^2}{\left[\omega_D - \left(1 + \frac{3}{2}|x|^2\right)\right]^2 + (\gamma/2)^2}$$

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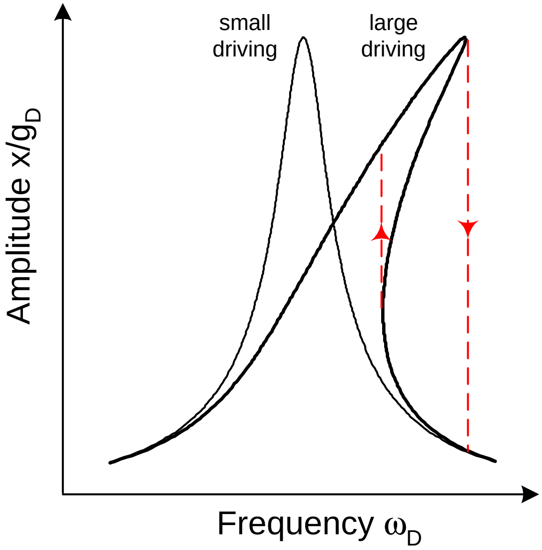
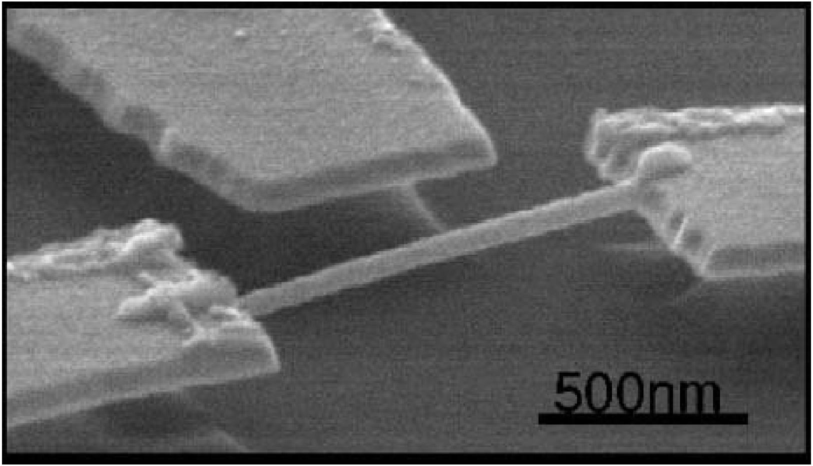
Nonlinearity: Frequency Pulling

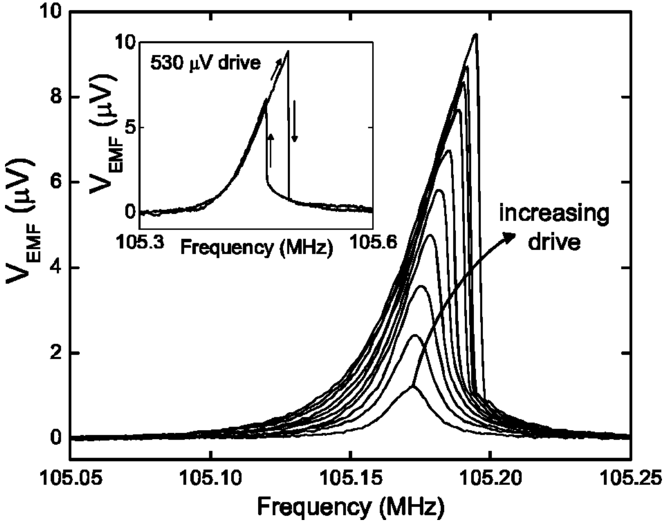


Experiment



Platinum Wire [Husain et al., Appl. Phys. Lett. **83**, 1240 (2003)]

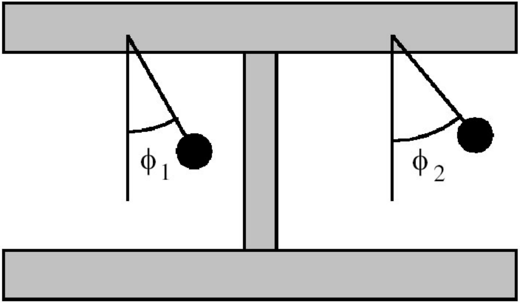
Results



530 μV drive
V_EMF (μV)
10
5
0
105.3
Frequency (MHz)
105.6
increasing drive
10
8
6
4
2
0
105.05
105.10
105.15
105.20
105.25
Frequency (MHz)

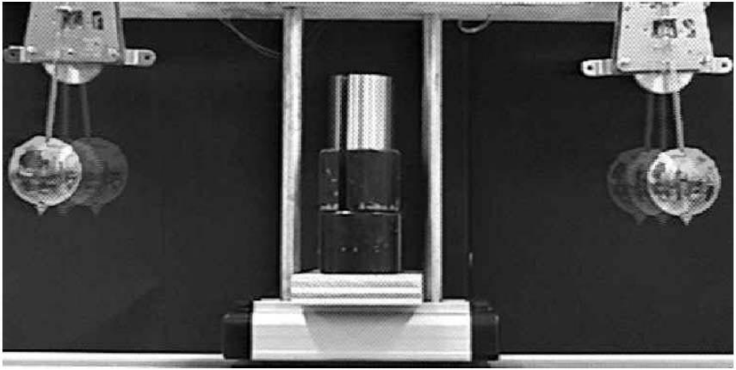
Outline

Huygen's Clocks 1665



Pendulum clocks hanging on the same wall kept perfect time and oscillated in antiphase.

Experimental Reconstruction



Bennett, Schatz, Rockwood, and Wiesenfeld (Proc. Roy. Soc. Lond. 2002)

Synchronization since Huygens

- Synchronization: collective effects in nonlinear oscillators
- Wide applicability
  - Steven Strogatz — *Sync: The Emerging Science of Spontaneous Order*
  - Arkady Pikovsky, Michael Rosenblum, Jürgen Kurths — *Synchronization: A Universal Concept in Nonlinear Sciences*
- Examples
  - Biology: fireflies, brain
  - Technology
    - laser arrays (increased power)
    - phase coherent detectors (increased sensitivity)
    - MEMS arrays (counteract fabrication differences)
  - …

Paradigm I

Synchronization occurs through dissipation acting on the phase differences

- Huygen's clocks (cf. Bennett, Schatz, Rockwood, and Wiesenfeld)
- Winfree-Kuramoto phase equation

$$\dot{\theta}_n = \omega_n - \sum_m K_{nm} \sin(\theta_n - \theta_{n+m})$$

with ω_n taken from distribution $g(\omega)$.

- Aronson, Ermentrout and Kopell analysis of two coupled oscillators
- Matthews, Mirollo and Strogatz magnitude-phase model

Results for the Phase Model

Mean field model with all-to-all coupling (Kuramoto, 1975)

$$\dot{\theta}_n = \omega_n - K\,N^{-1}\sum_m \sin(\theta_n - \theta_{n+m})$$

with ω_n taken from a symmetric distribution $g(\omega)$ (e.g. gaussian) of width w

- Order parameter

$$\Psi = N^{-1}\sum_n r_n\, e^{i\theta_n} = R\, e^{i\Theta}, \qquad r_n = 1$$

  Synchronization occurs if $R \neq 0$

- Phase difference $\bar{\theta}_n = \theta_n - \Theta$ satisfies

$$\dot{\bar{\theta}}_n = \omega_n - F(\bar{\theta}_n) \qquad \text{with} \qquad F(\theta) = K\,R\sin\theta$$

  (order parameter frequency $\dot{\Theta} = 0$ for symmetric distribution $g(\omega)$)

- Self-consistency condition

$$R = N^{-1}\sum_n \langle\cos\bar{\theta}_n\rangle, \qquad \sum_n \langle\sin\bar{\theta}_n\rangle = 0$$

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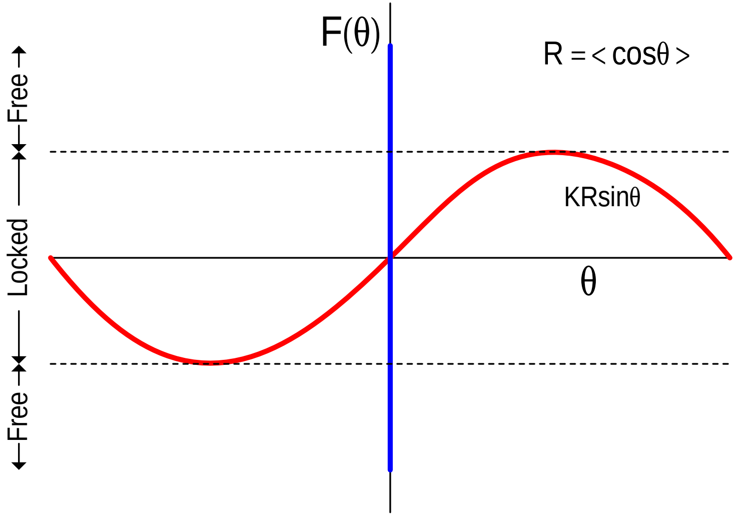

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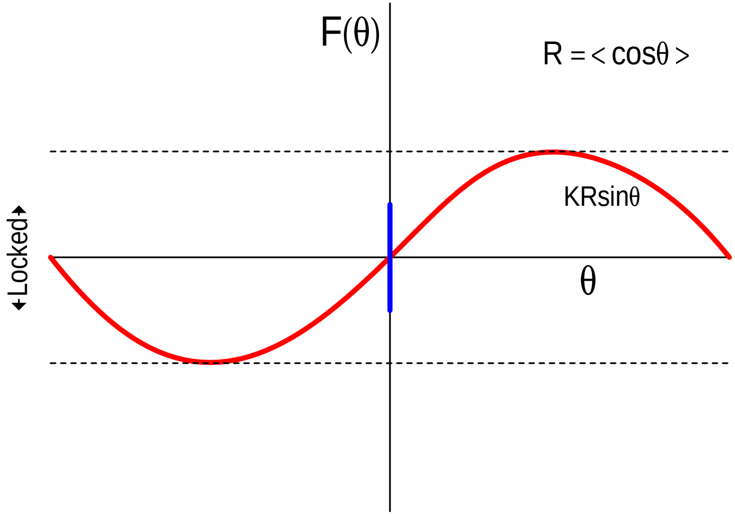
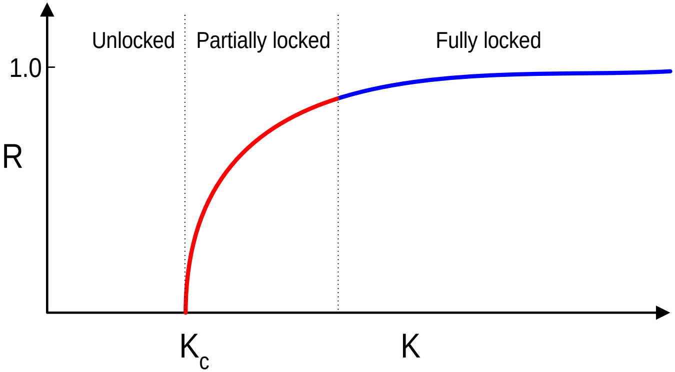


Synchronization Transition



- Onset of synchronization for $K > K_c = 2(\pi g(0))^{-1}$
- Also: divergence of susceptibility at $K = K_c$

Paradigm II

MEMS equation

$$0 = \ddot{x}_n + (1+\omega_n)x_n - \nu(1-x_n^2)\dot{x}_n + ax_n^3 + \sum_m D_{nm}(x_m - x_n)$$

Synchronization in MEMS $\Rightarrow$ alternative mechanism

Synchronization occurs by nonlinear frequency pulling and reactive coupling

Outline

The Model

Investigate all-to-all coupling using complex amplitude formulation

$$\dot{A}_n = i(\omega_n - \alpha |A_n|^2)A_n + (1 - |A_n|^2)A_n + \frac{i\beta}{N}\sum_{m=1}^{N}(A_m - A_n)$$

(cf. *Synchronization* by Pikovsky, Rosenblum, and Kurths)

Write as equations for magnitude and phase $A_n = r_n e^{i\theta_n}$

$$\dot{\bar{\theta}}_n = \bar{\omega}_n + \alpha(1 - r_n^2) + \frac{\beta R}{r_n}\cos\bar{\theta}_n$$

$$\dot{r}_n = (1 - r_n^2)r_n + \beta R \sin\bar{\theta}_n$$

with $\bar{\theta}_n = \theta_n - \Theta$, $\bar{\omega}_n = \omega_n - \alpha - \beta - \dot{\Theta}$

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with $\bar{\theta}_n = \theta_n - \Theta$, $\bar{\omega}_n = \omega_n - \alpha - \beta - \dot{\Theta}$

Easy Limit

Narrow frequency distributions and large α.

- Magnitude relaxes rapidly to the value given by setting $\dot{r} = 0$

$$(1 - r^2)r = -\beta R \sin \bar{\theta}$$

If $r \simeq 1$ (OK for large α)

$$1 - r^2 \simeq -\beta R \sin \bar{\theta}$$

- Phase equation becomes Kuramoto equation with $K = \alpha\beta$

$$\dot{\bar{\theta}}_n = \bar{\omega}_n + \alpha(1 - r_n^2) + \frac{\beta R}{r_n} \cos \bar{\theta}_n \qquad \rightarrow \qquad \dot{\bar{\theta}} \simeq \bar{\omega} - \alpha\beta R \sin \bar{\theta}$$

(ignore βR compared with $\alpha\beta R$)

- Synchronization occurs ($R \neq 0$) for $\alpha\beta > 2(\pi g(0))^{-1}$.

Easy Limit

Narrow frequency distributions and large α.

- Magnitude relaxes rapidly to the value given by setting $\dot{r} = 0$

$$(1 - r^2)r = -\beta R \sin \bar{\theta}$$

If $r \simeq 1$ (OK for large α)

$$1 - r^2 \simeq -\beta R \sin \bar{\theta}$$

- Phase equation becomes Kuramoto equation with $K = \alpha\beta$

$$\dot{\bar{\theta}}_n = \bar{\omega}_n + \alpha(1 - r_n^2) + \frac{\beta R}{r_n} \cos \bar{\theta}_n \qquad \rightarrow \qquad \dot{\bar{\theta}} \simeq \bar{\omega} - \alpha\beta R \sin \bar{\theta}$$

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Summary of Results

[MCC, Zumdieck, Lifshitz, and Rogers (2004, 2006)]

$$\dot{A}_n = i(\omega_n - \alpha\,|A_n|^2)A_n + (1 - |A_n|^2)A_n + i\,\frac{\beta}{N}\sum_m (A_m - A_n)$$

with ω_n from distribution $g(\omega)$

- Analytics
  - Linear instability of unsynchronized $R = 0$ state for Lorentzian, triangular, top-hat $g(\omega)$ (cf. Matthews et al.)
  - Instability of fully locked state
- Numerical simulations of amplitude-phase model for up to 10000 oscillators with all-to-all coupling

Outline

Full locking

Amplitude-phase equations

$$\dot{\bar{\theta}} = \bar{\omega} + \alpha(1 - r^2) + \frac{\beta R}{r}\cos\bar{\theta}$$
$$\dot{r} = (1 - r^2)r + \beta R \sin\bar{\theta}$$

If all the oscillators are locked

$$\dot{\bar{\theta}} = \dot{r} = 0$$

- solve cubic equation for $r(\bar{\theta})$
- solve phase equation

$$\bar{\omega} = \frac{\beta R}{r(\bar{\theta})}(\alpha \sin\bar{\theta} - \cos\bar{\theta}) = F(\bar{\theta})$$

- test stability of single oscillator solution $(r(\bar{\omega}), \bar{\theta}(\bar{\omega}))$

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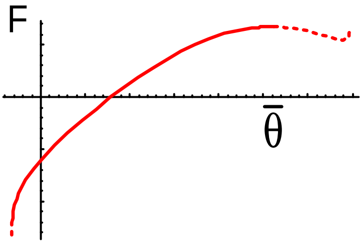
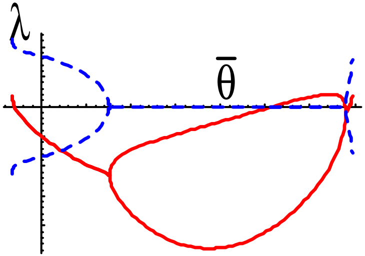
- test stability of single oscillator solution $(r(\bar{\omega}), \bar{\theta}(\bar{\omega}))$

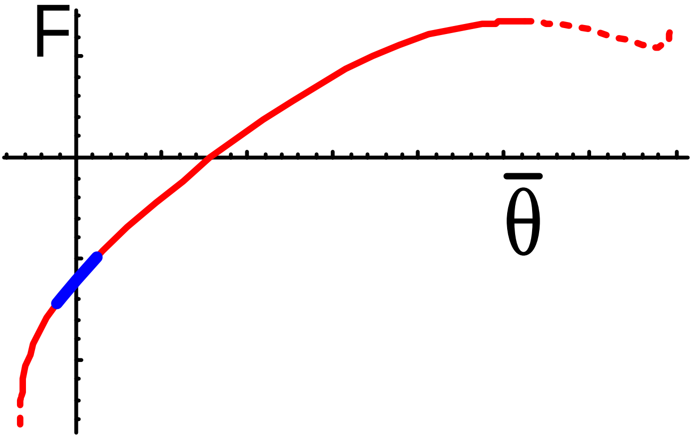
Example

$\alpha = 1.0, \beta R = 1.2$

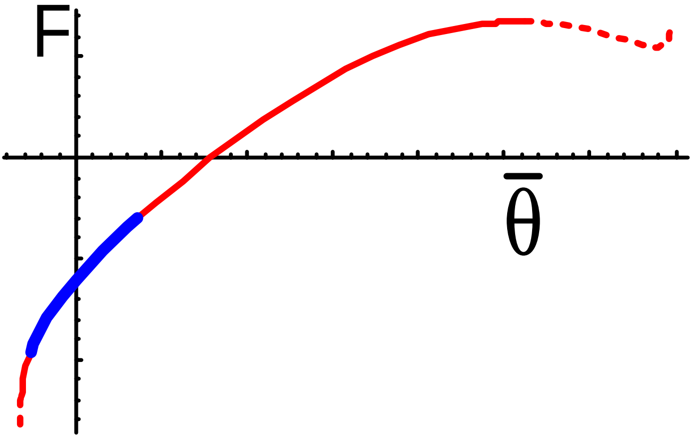




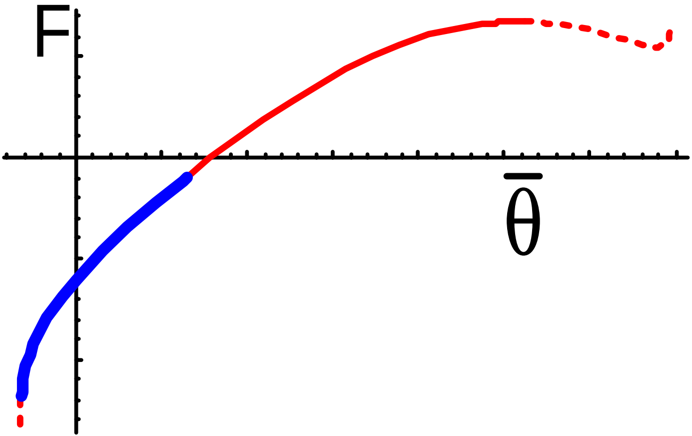
Solution for narrow distribution



Solution for wider distribution



Critical distribution width



Outline

Numerics

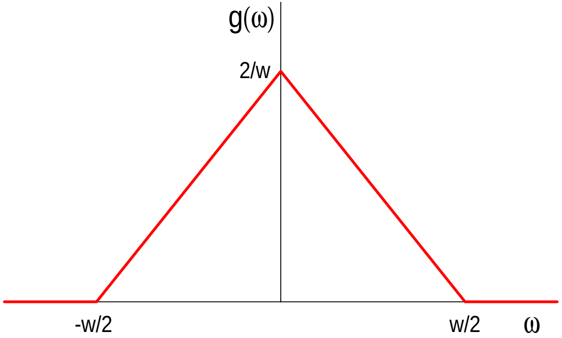
- Up to N=100,000 oscillators used (interested in phenomena that survive $N \to \infty$ limit)
- Lorentzian (with cutoff), Gaussian, and top-hat (uniform) distributions
- Usually choose $g(0) = 1$
- Scan behavior as function of α and β

Graphical Example

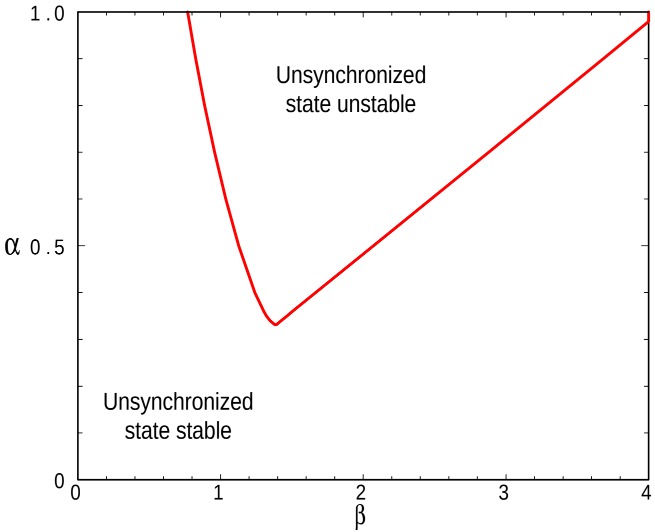
Complex A plane:

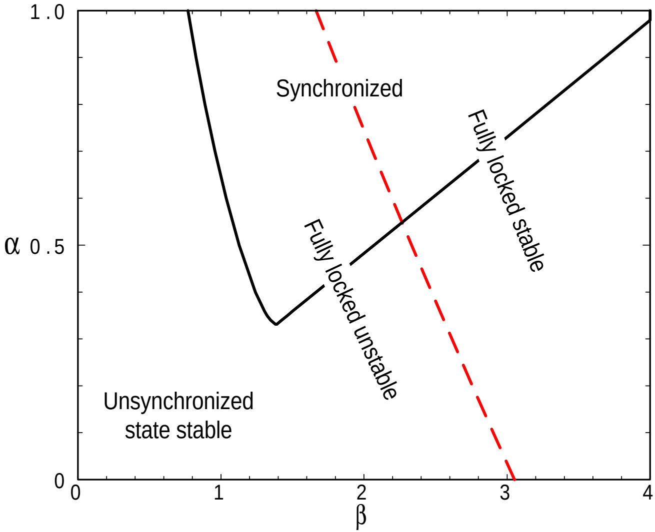
Outline

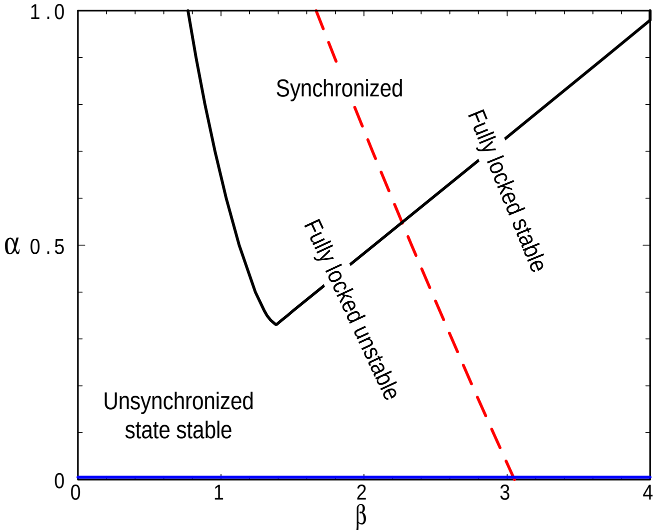
Results for a Triangular Distribution

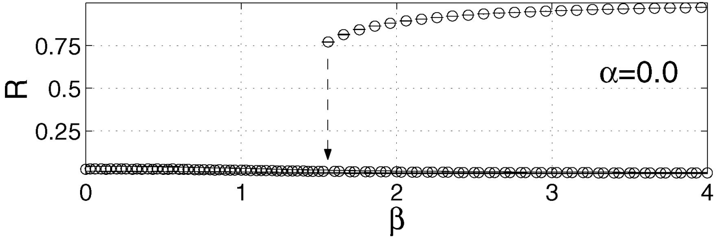


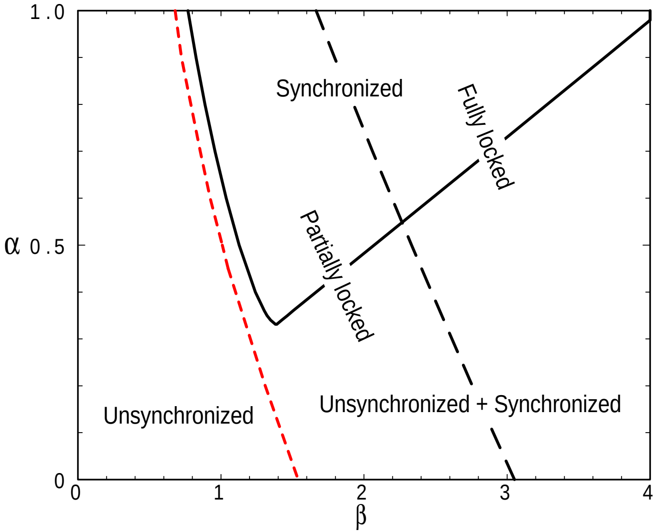
Show results for $g(0) = 1$ ($w = 2$)

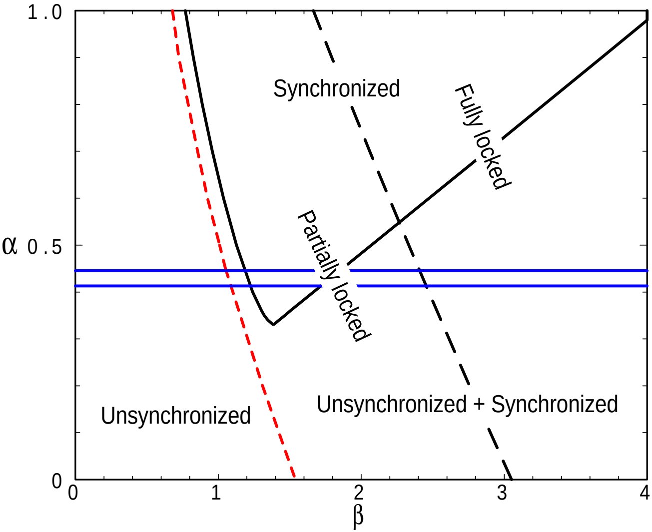

1.0
α 0.5
0
0
1
2
3
4
β
Unsynchronized
state unstable
Unsynchronized
state stable

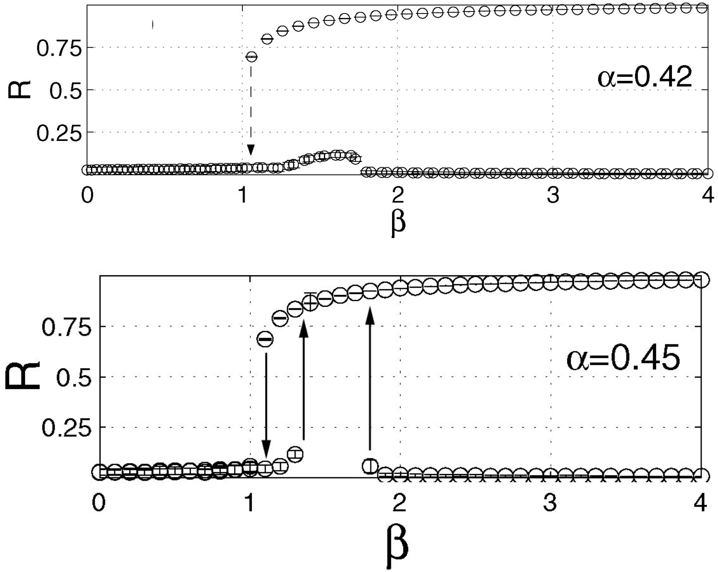

1.0
α 0.5
0
0
1
2
3
4
β
Synchronized
Fully locked stable
Fully locked unstable
Unsynchronized
state stable

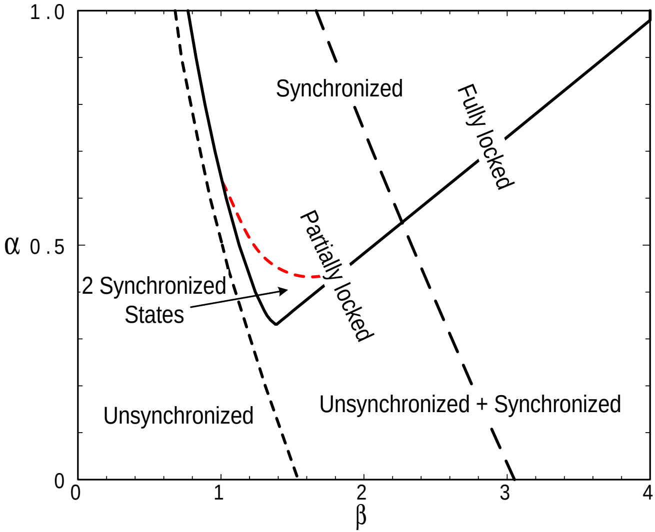

1.0
α 0.5
0
0
1
2
3
4
β
Synchronized
Fully locked stable
Fully locked unstable
Unsynchronized
state stable

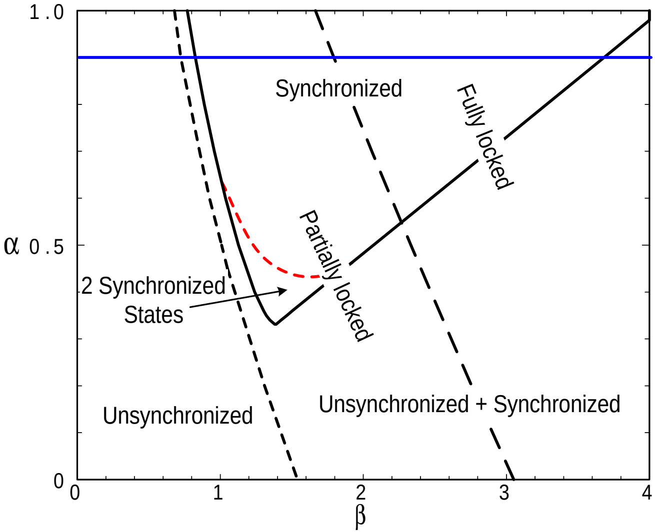

R
0.75
0.5
0.25
0
1
2
3
4
β
α=0.0

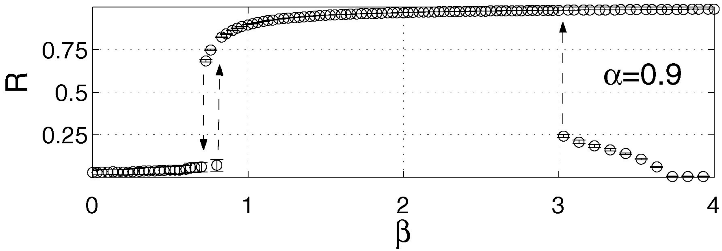

1.0
α 0.5
0
0
1
2
3
4
β
Synchronized
Fully locked
Partially locked
Unsynchronized
Unsynchronized + Synchronized

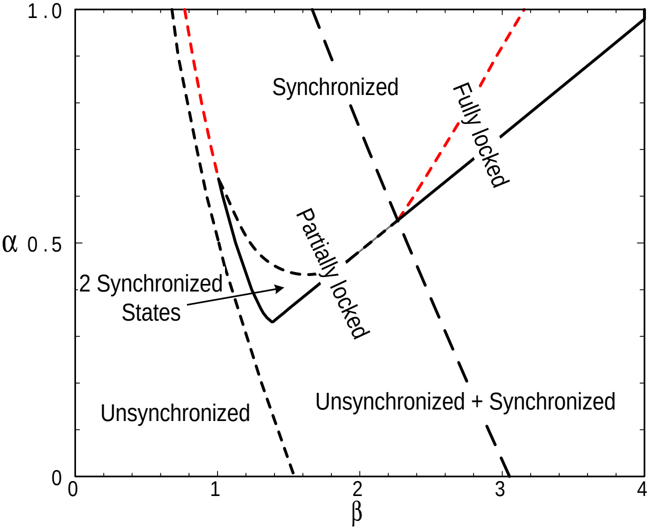

1.0
α 0.5
0
0
1
2
3
4
β
Synchronized
Fully locked
Partially locked
Unsynchronized
Unsynchronized + Synchronized

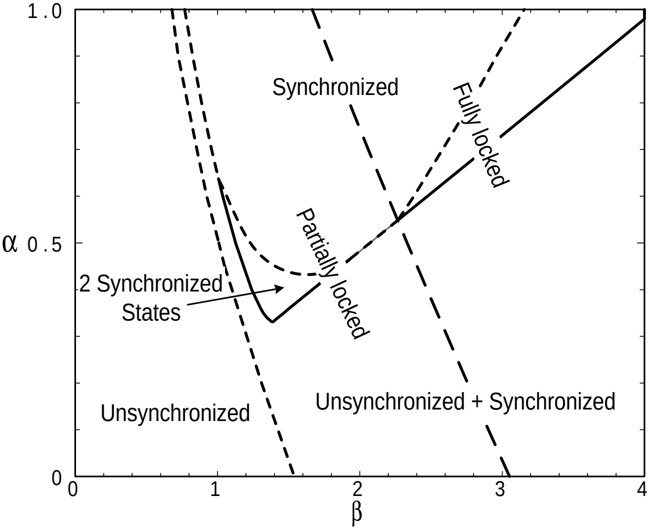

α=0.42
R
0.75
0.5
0.25
0
1
2
3
4
β
α=0.45
R
0.75
0.5
0.25
0
1
2
3
4
β


1.0
α 0.5
0
0
1
2
3
4
β
Synchronized
Fully locked
Partially locked
2 Synchronized
States
Unsynchronized
Unsynchronized + Synchronized


1.0
α 0.5
0
0
1
2
3
4
β
Synchronized
Fully locked
Partially locked
2 Synchronized
States
Unsynchronized
Unsynchronized + Synchronized

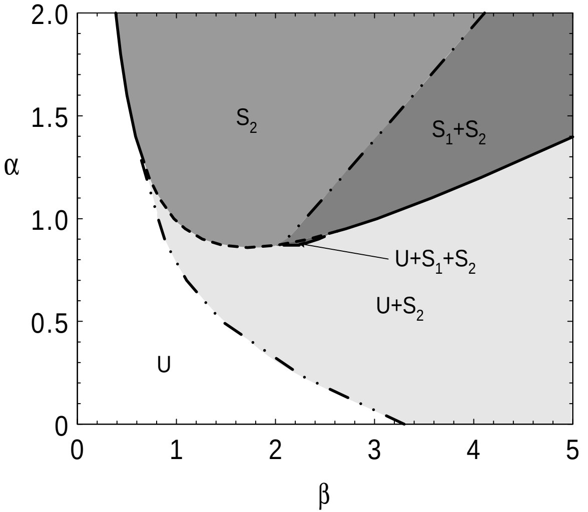

R
0.75
0.5
0.25
α=0.9
0
1
2
3
4
β


1.0
α 0.5
0
0
1
2
3
4
β
Synchronized
Fully locked
Partially locked
2 Synchronized
States
Unsynchronized
Unsynchronized + Synchronized


1.0
α 0.5
0
0
1
2
3
4
β
Synchronized
Fully locked
Partially locked
2 Synchronized
States
Unsynchronized
Unsynchronized + Synchronized

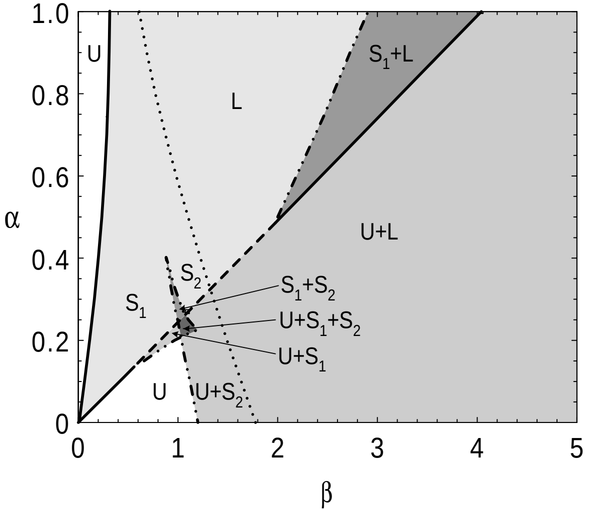
Lorentzian

$g(0) = 1$



Top-Hat

$g(0) = 1$



Outline

Conclusions

I've described one aspect of theoretically modelling micron and submicron scale oscillators:

- Synchronization due to nonlinear frequency pulling and reactive coupling [MCC, Zumdieck, Lifshitz and Rogers, Phys. Rev. Lett. **93**, 224101 (2004)]

Other areas of interest:

- Linear fluctuations in solution [Paul and MCC, Phys. Rev. Lett. **92**, 235501 (2004)]
- Nonlinear collective effects of parametrically driven high-Q arrays [Lifshitz and MCC, Phys. Rev. **B67**, 134302 (2003)]
- A QND scheme to measure discrete levels in a quantum MEMS oscillator [Santamore, Doherty, and MCC, Phys. Rev. **B70**, 144301 (2004)]
- Noise induced transitions between driven (nonequilibrium) states