Collective and Stochastic Effects in Arrays of Submicron Oscillators

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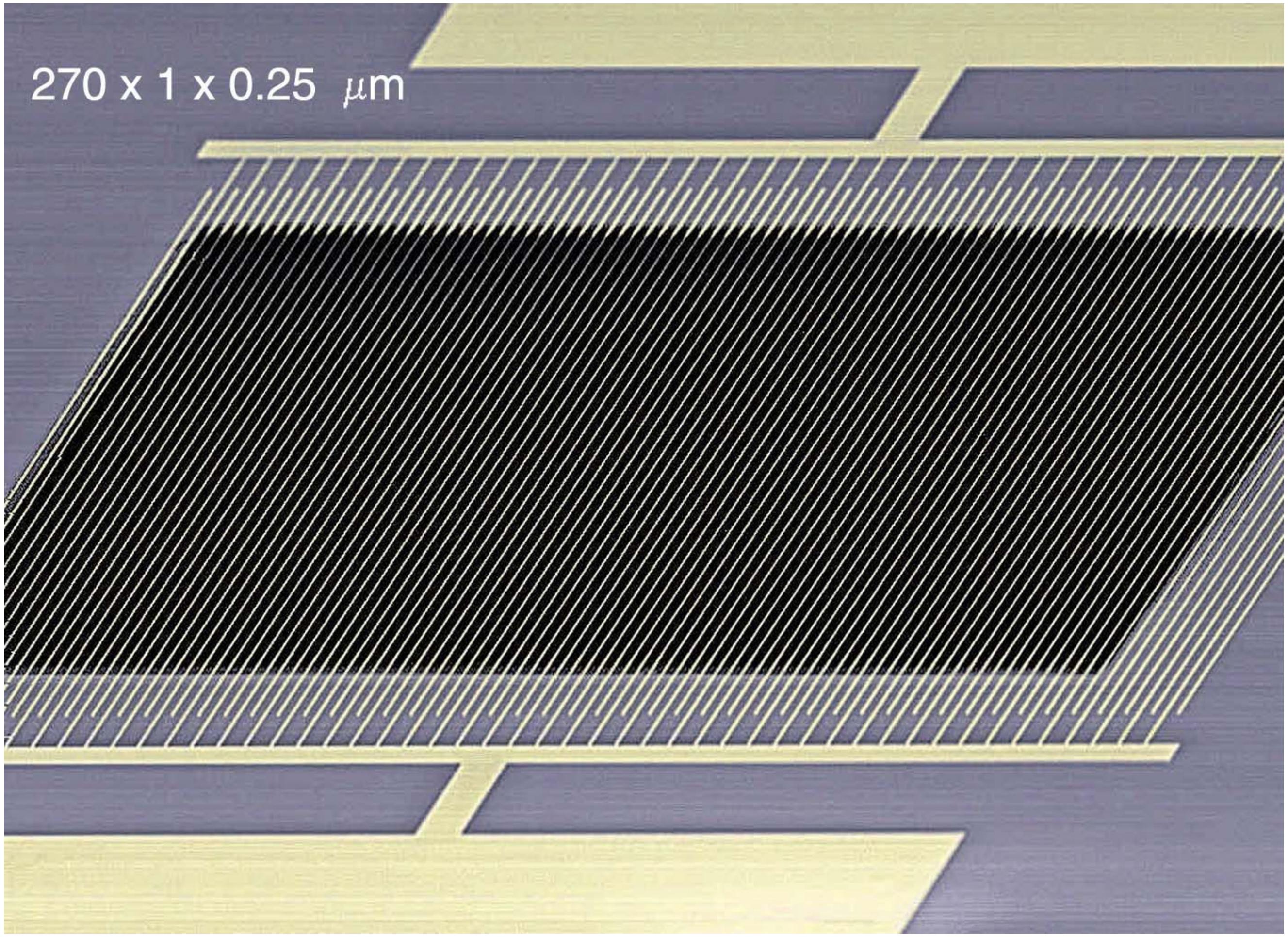
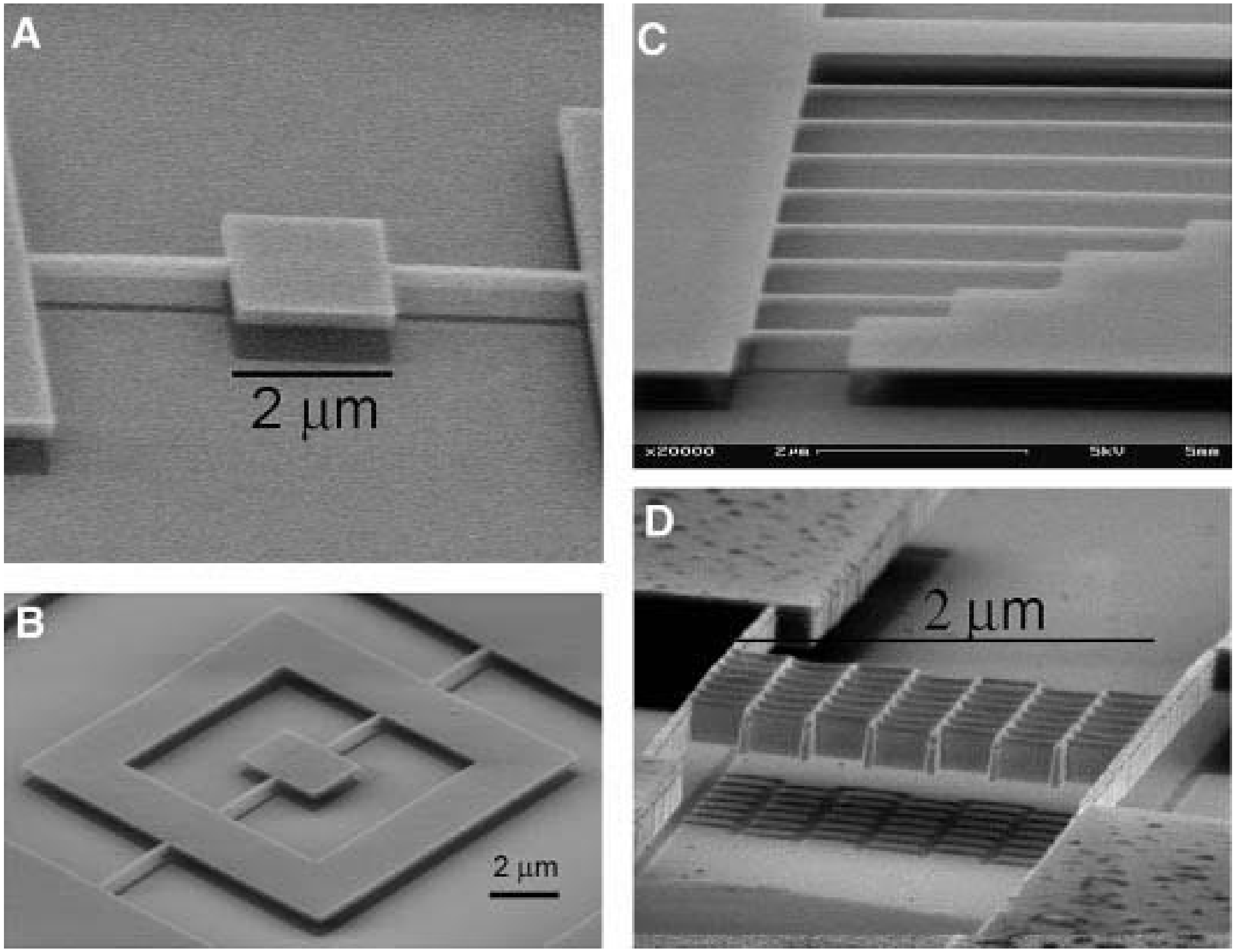
Support: NSF, Nato and EU, BSF, HRL

Outline

- Motivation: MEMS and NEMS
- Familiar examples of dynamics in a MEMS context
- Pattern formation in parametrically driven arrays
- Synchronization of arrays of oscillators
- Noise driven transitions between nonequilibrium states
- Conclusions



Array of μ-scale oscillators [From Buks and Roukes (2002)]



Single crystal silicon [From Craighead, Science (2000)]

MicroElectroMechanical Systems and NEMS

Arrays of tiny mechanical oscillators:

- driven, dissipative $\Rightarrow$ nonequilibrium
- nonlinear
- collective
- noisy
- (potentially) quantum

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New laboratory for nonlinear dynamics and pattern formation

- Apply knowledge from nonlinear dynamics, pattern formation etc. to technologically important questions
- Investigate pattern formation and nonlinear dynamics in new regimes
- Study new aspects of old questions

Modelling

$$0 = \ddot{x}_n + x_n$$

Modelling

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$$+ \delta_n x_n \qquad \text{with } \delta_n \text{ taken from distribution } g(\delta_n)$$

Modelling

$$
\begin{aligned}
0 = \ddot{x}_n + x_n \\
+ \delta_n x_n \\
+ \sum_m D_{nm}(x_{n+m} - 2x_n + x_{n-m})
\end{aligned}
$$

Modelling

$$
\begin{aligned}
0 = \ddot{x}_n + x_n \\
+ \delta_n x_n \\
+ \sum_m D_{nm}(x_{n+m} - 2x_n + x_{n-m}) \\
- \bar{\gamma}(\dot{x}_{n+1} - 2\dot{x}_n + \dot{x}_{n-1})
\end{aligned}
$$

Modelling

$$
\begin{aligned}
0 = \ddot{x}_n + x_n \\
+ \delta_n x_n \\
+ \sum_m D_{nm} (x_{n+m} - 2x_n + x_{n-m}) \\
- \bar{\gamma} (\dot{x}_{n+1} - 2\dot{x}_n + \dot{x}_{n-1}) \\
+ x_n^3
\end{aligned}
$$

Modelling

$$
\begin{aligned}
0 = \ddot{x}_n + x_n \\
+ \delta_n x_n \\
+ \sum_m D_{nm}(x_{n+m} - 2x_n + x_{n-m}) \\
- \bar{\gamma}(\dot{x}_{n+1} - 2\dot{x}_n + \dot{x}_{n-1}) \\
+ x_n^3 \\
+ \eta\left[(x_{n+1} - x_n)^2(\dot{x}_{n+1} - \dot{x}_n) - (x_n - x_{n-1})^2(\dot{x}_n - \dot{x}_{n-1})\right]
\end{aligned}
$$

Modelling

$$
\begin{aligned}
0 = \ddot{x}_n &+ x_n \\
&+ \delta_n x_n \\
&+ \sum_m D_{nm}(x_{n+m} - 2x_n + x_{n-m}) \\
&- \bar{\gamma}(\dot{x}_{n+1} - 2\dot{x}_n + \dot{x}_{n-1}) \\
&+ x_n^3 \\
&+ \eta\left[(x_{n+1} - x_n)^2(\dot{x}_{n+1} - \dot{x}_n) - (x_n - x_{n-1})^2(\dot{x}_n - \dot{x}_{n-1})\right] \\
&- \gamma \dot{x}_n(1 - x_n^2)
\end{aligned}
$$

Modelling

$$
\begin{aligned}
0 = \ddot{x}_n + x_n \\
+ \delta_n x_n \\
+ \sum_m D_{nm}(x_{n+m} - 2x_n + x_{n-m}) \\
- \bar{\gamma}(\dot{x}_{n+1} - 2\dot{x}_n + \dot{x}_{n-1}) \\
+ x_n^3 \\
+ \eta\left[(x_{n+1} - x_n)^2(\dot{x}_{n+1} - \dot{x}_n) - (x_n - x_{n-1})^2(\dot{x}_n - \dot{x}_{n-1})\right] \\
+ g_P \cos[(2 + \delta\omega_P)t]\, x_n
\end{aligned}
$$

Modelling

$$
\begin{aligned}
0 = \ddot{x}_n + x_n \\
+ \delta_n x_n \\
+ \sum_m D_{nm}(x_{n+m} - 2x_n + x_{n-m}) \\
- \bar{\gamma}(\dot{x}_{n+1} - 2\dot{x}_n + \dot{x}_{n-1}) \\
+ x_n^3 \\
+ \eta\left[(x_{n+1} - x_n)^2(\dot{x}_{n+1} - \dot{x}_n) - (x_n - x_{n-1})^2(\dot{x}_n - \dot{x}_{n-1})\right] \\
+ g_P \cos[(2 + \delta\omega_P)t]\, x_n \\
+ g_D \cos[(1 + \delta\omega_D)t]
\end{aligned}
$$

Modelling

$$
\begin{aligned}
0 = \ddot{x}_n + x_n \\
+ \delta_n x_n \\
+ \sum_m D_{nm}(x_{n+m} - 2x_n + x_{n-m}) \\
- \bar{\gamma}(\dot{x}_{n+1} - 2\dot{x}_n + \dot{x}_{n-1}) \\
+ x_n^3 \\
+ \eta\left[(x_{n+1} - x_n)^2(\dot{x}_{n+1} - \dot{x}_n) - (x_n - x_{n-1})^2(\dot{x}_n - \dot{x}_{n-1})\right] \\
+ g_P \cos[(2 + \delta\omega_P)t]\, x_n \\
+ g_D \cos[(1 + \delta\omega_D)t] \\
+ \text{Noise}
\end{aligned}
$$

Theoretical Approach

- Oscillators at frequency unity + small corrections
- Assume dispersion, coupling, damping, driving, noise, and nonlinear terms are small.
- Introduce small parameter ε with ε^p characterizing the size of these various terms.
- Then with the "slow" time scale $T = \varepsilon t$

$$x_n(t) = \left[A_n(T)e^{it} + c.c.\right] + \epsilon x_n^{(1)}(t) + \ldots$$

derive equations for $dA_n/dT = \cdots$.

Nonlinearity in MEMS

A simple nonlinear oscillator: the Duffing equation

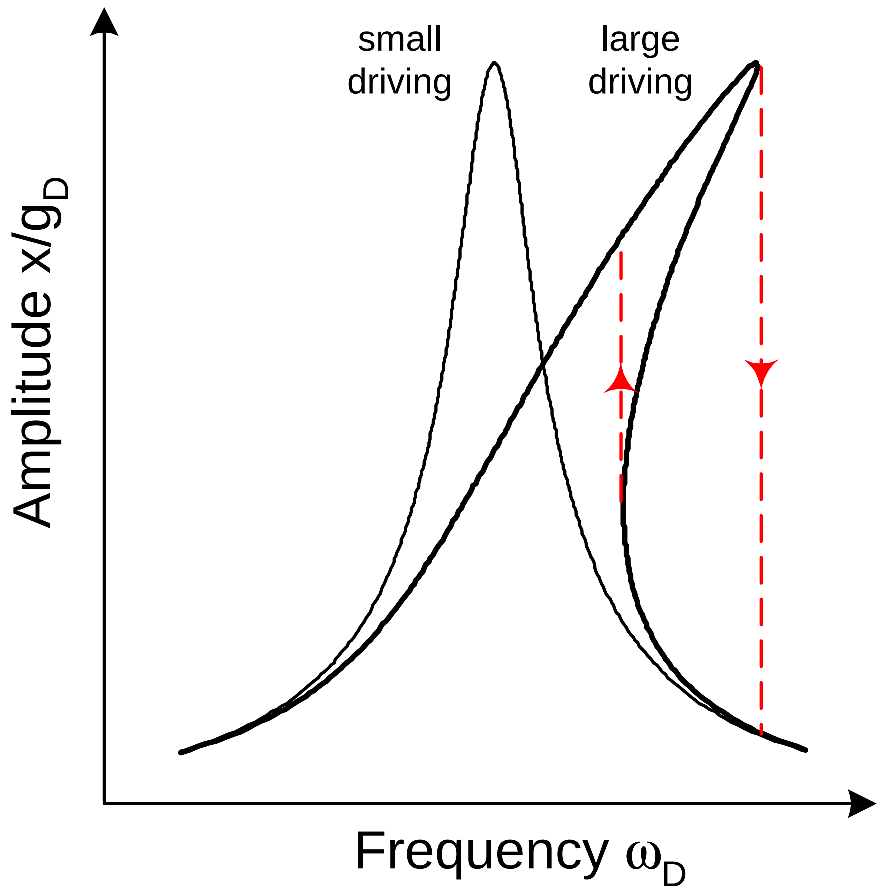
$$\ddot{x} + \gamma \dot{x} + x + x^3 = g_D \cos(\omega_D t)$$

Parameters:

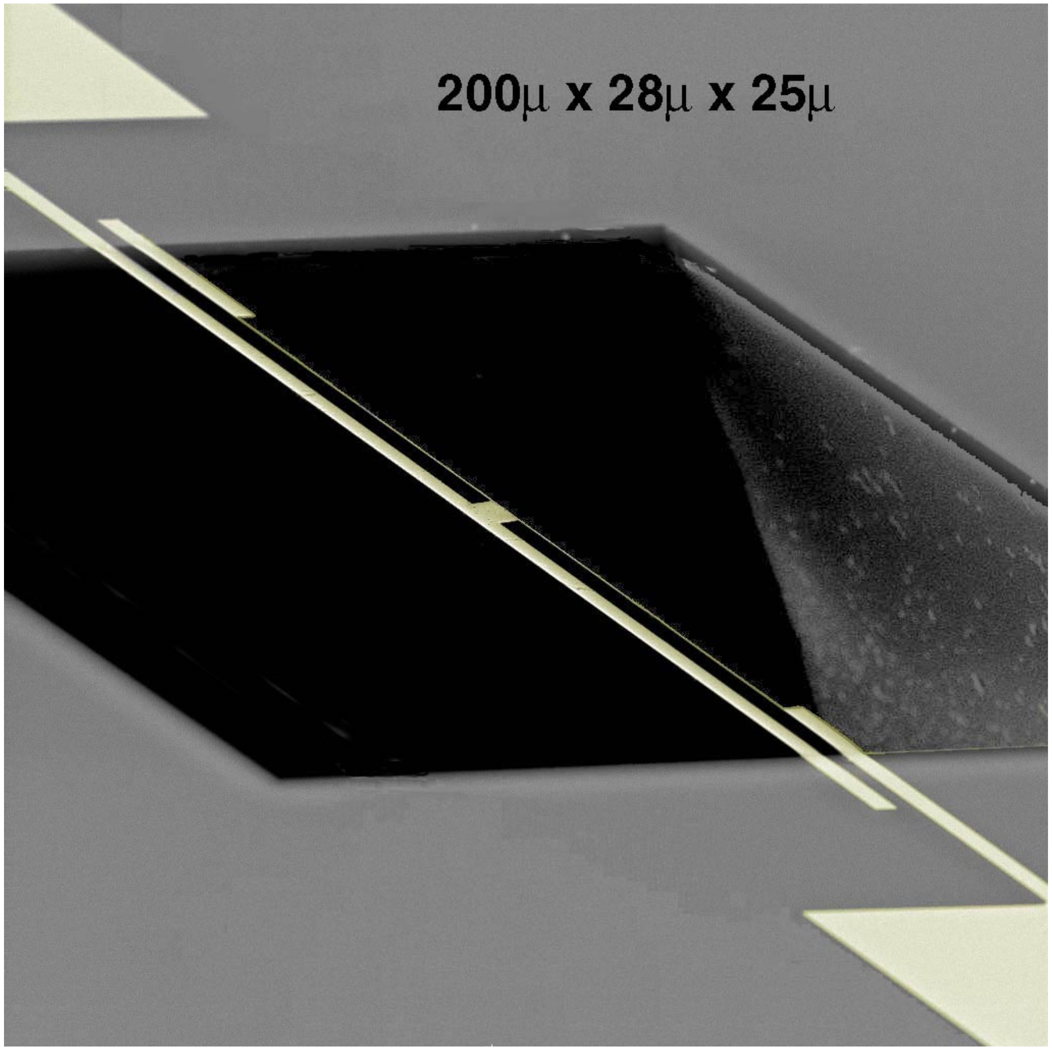
γ	damping
g_D	drive strength
ω_D	drive frequency

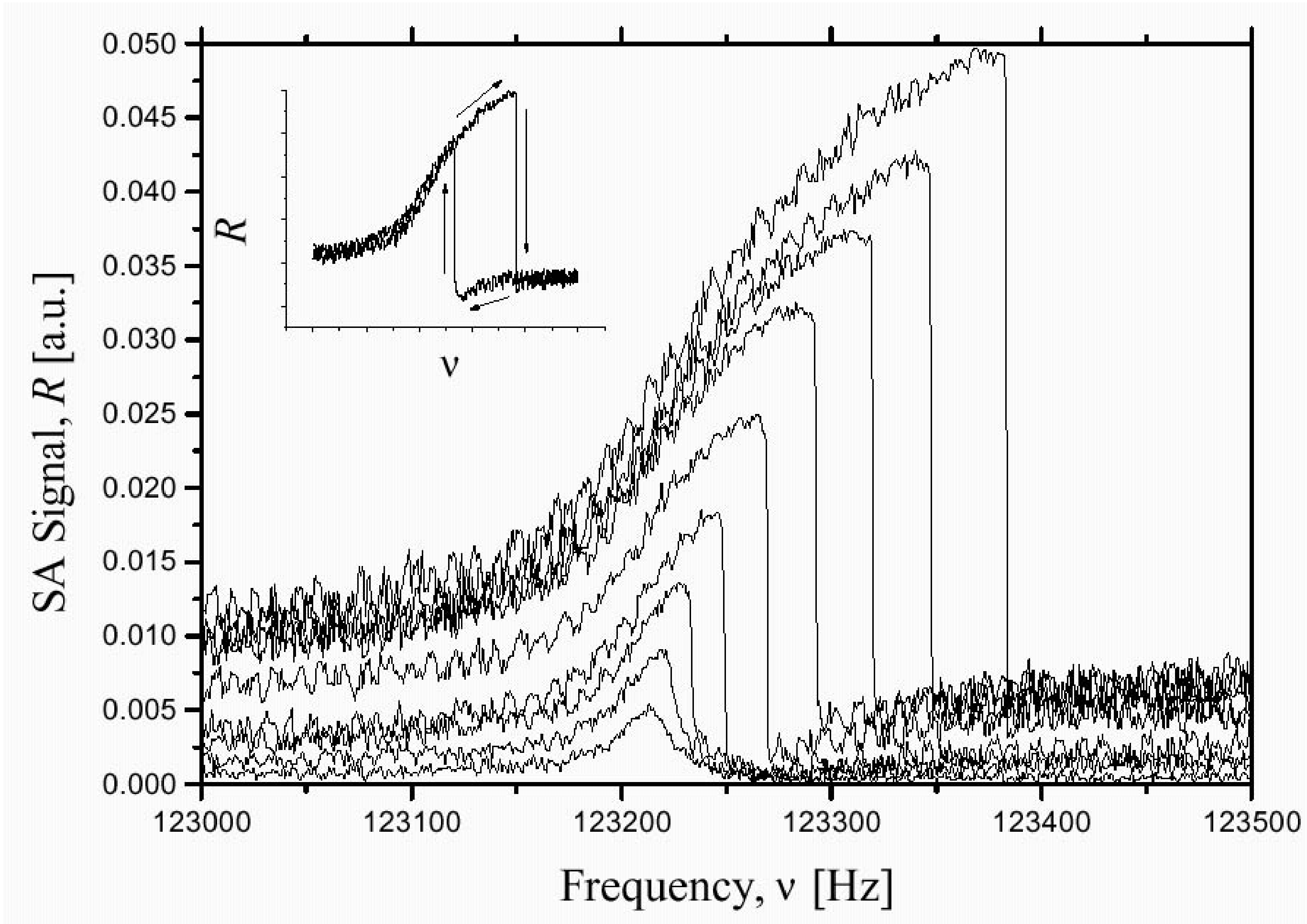
For chosen sign of x^3 term the spring gets *stiffer* with increasing displacement.

Frequency pulling



Experiment [Buks and Roukes, 2001]





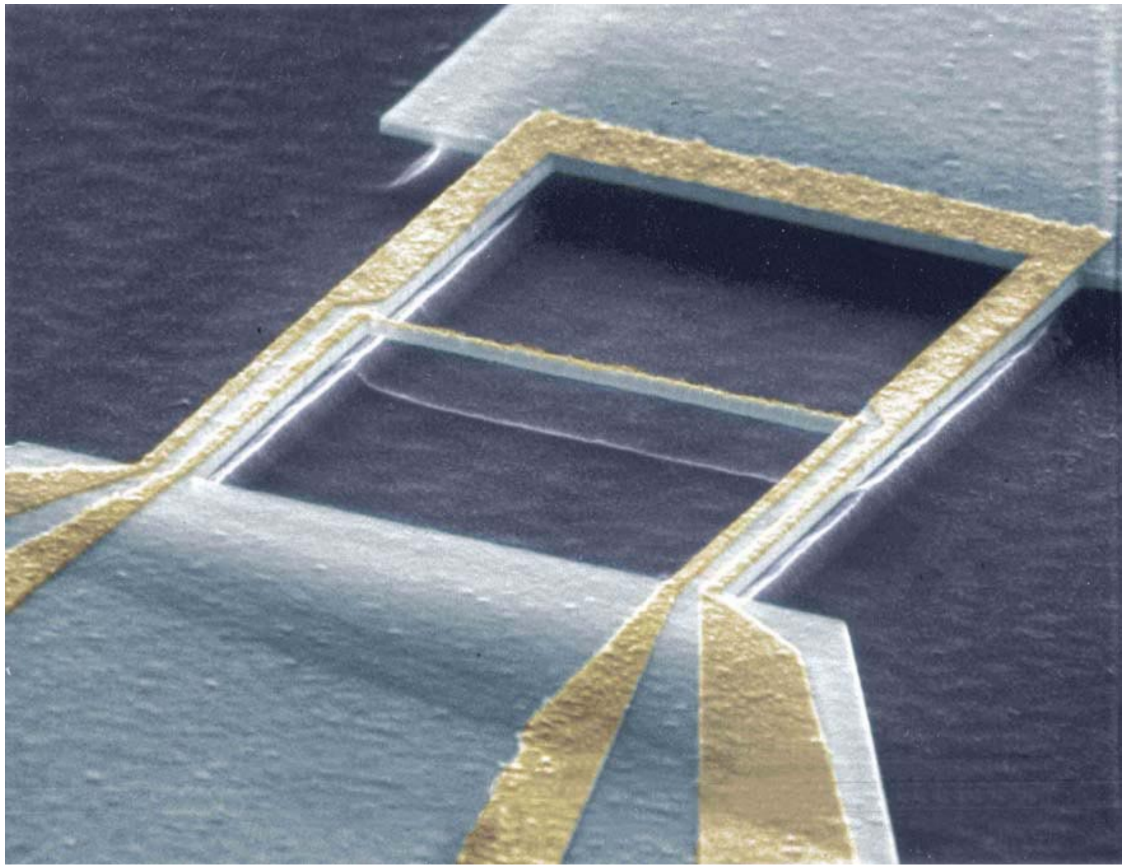
[Buks and Roukes, 2001]

Parametric Drive in MEMS

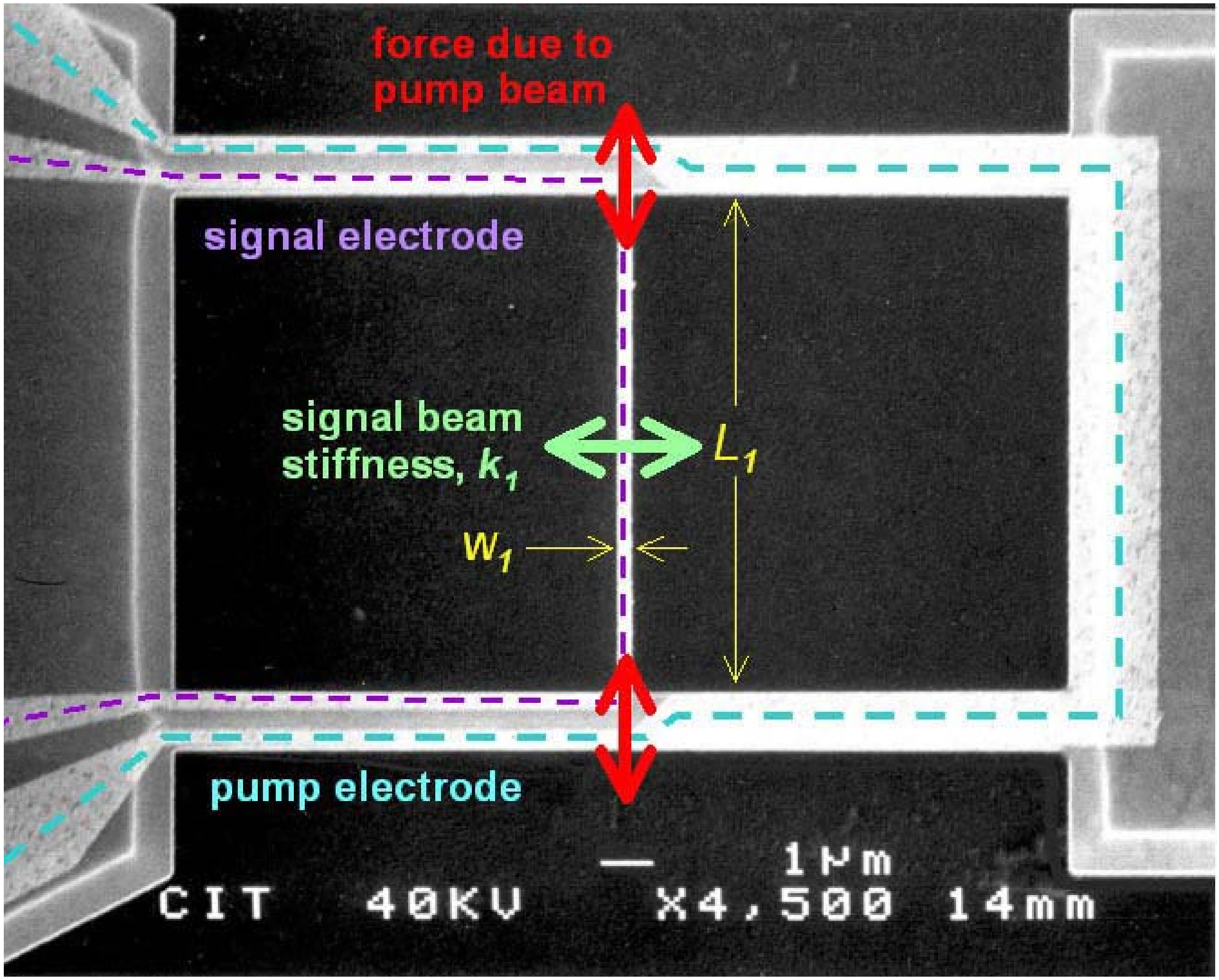
$$\ddot{x} + \gamma \dot{x} + (1 + g_P \cos \omega_P t)x + x^3 = 0$$

- oscillation of *parameter* of equation—here the spring constant
- $x = 0$ remains a solution in the absence of noise
- parametric drive decreases effective dissipation (for one quadrature of oscillations)
  - ◇ amplification for small drive amplitudes
  - ◇ instability for large enough drive amplitudes
- strongest response for $\omega_p = 2$

MEMS Elastic Parametric Drive

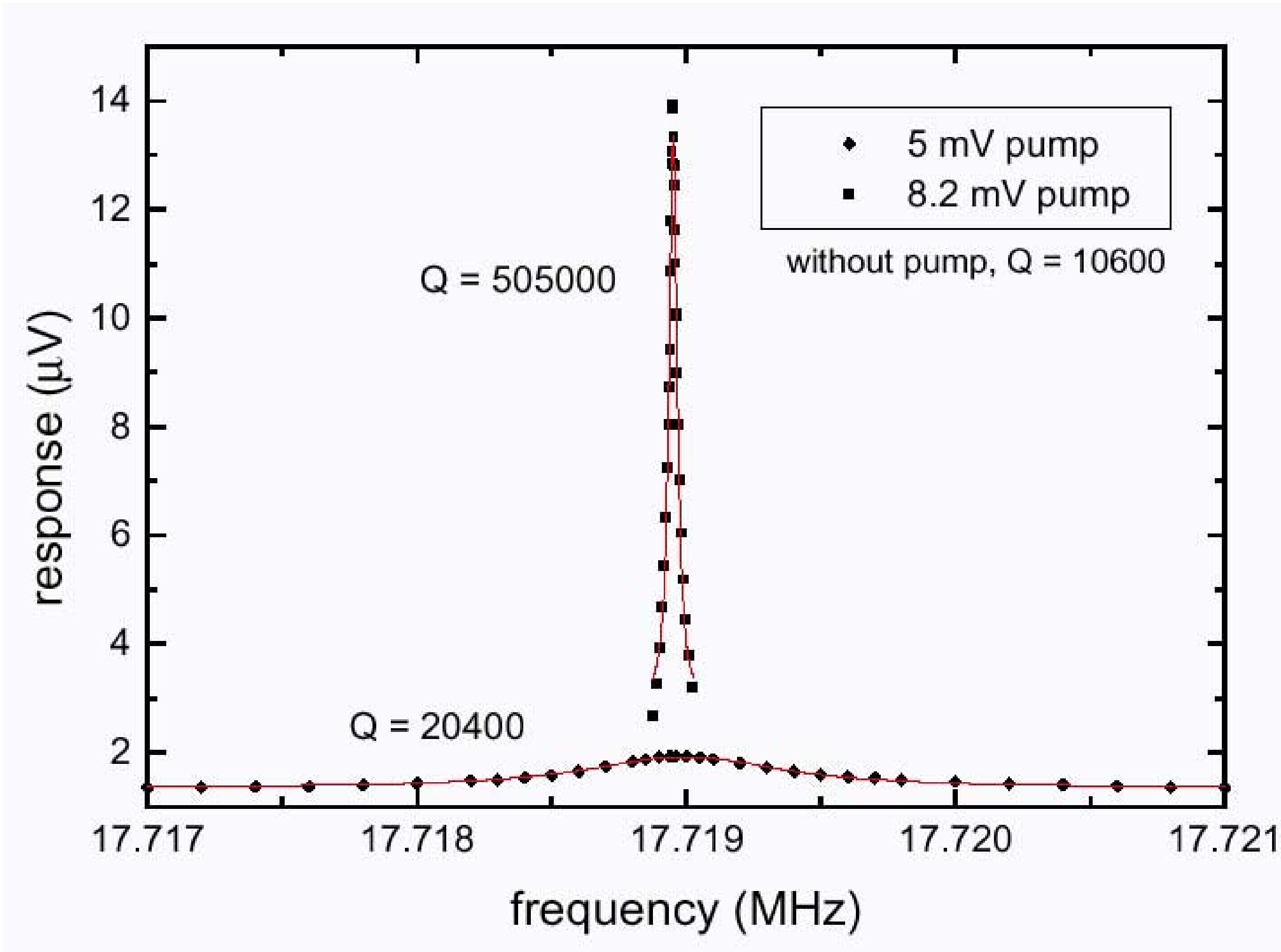


[Harrington and Roukes]



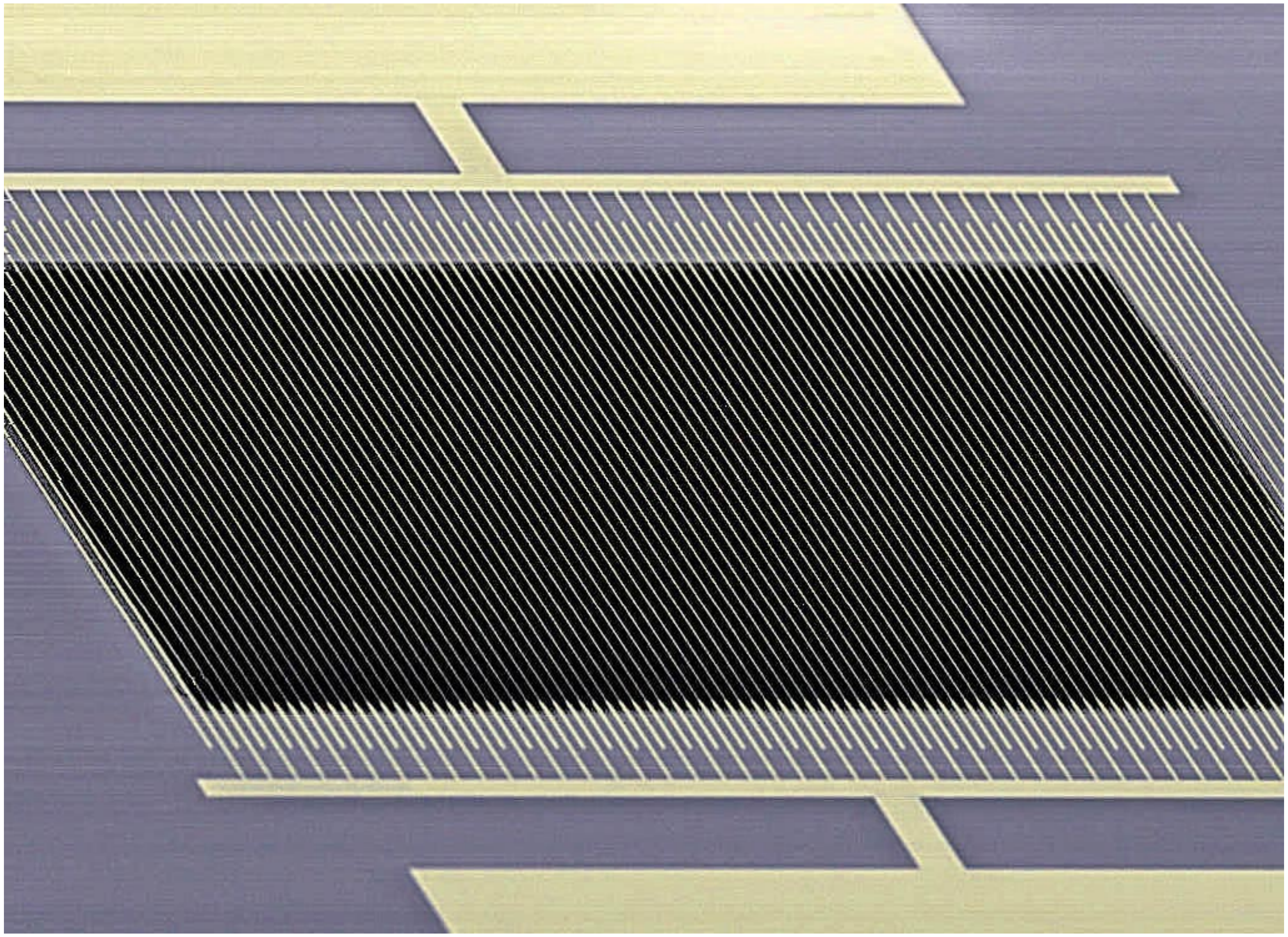
[Harrington and Roukes]

Amplification



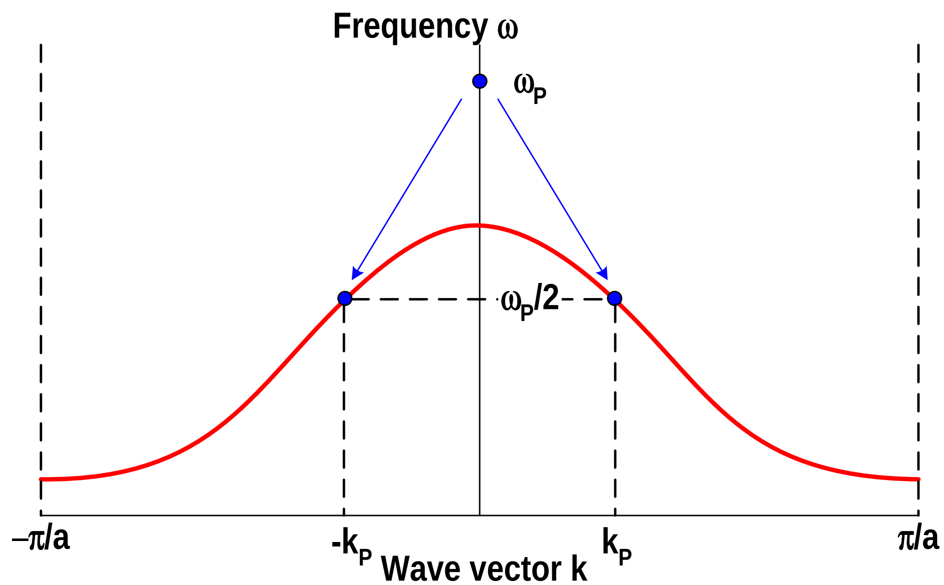
[Harrington and Roukes]

Parametric Instability in Arrays of Oscillators



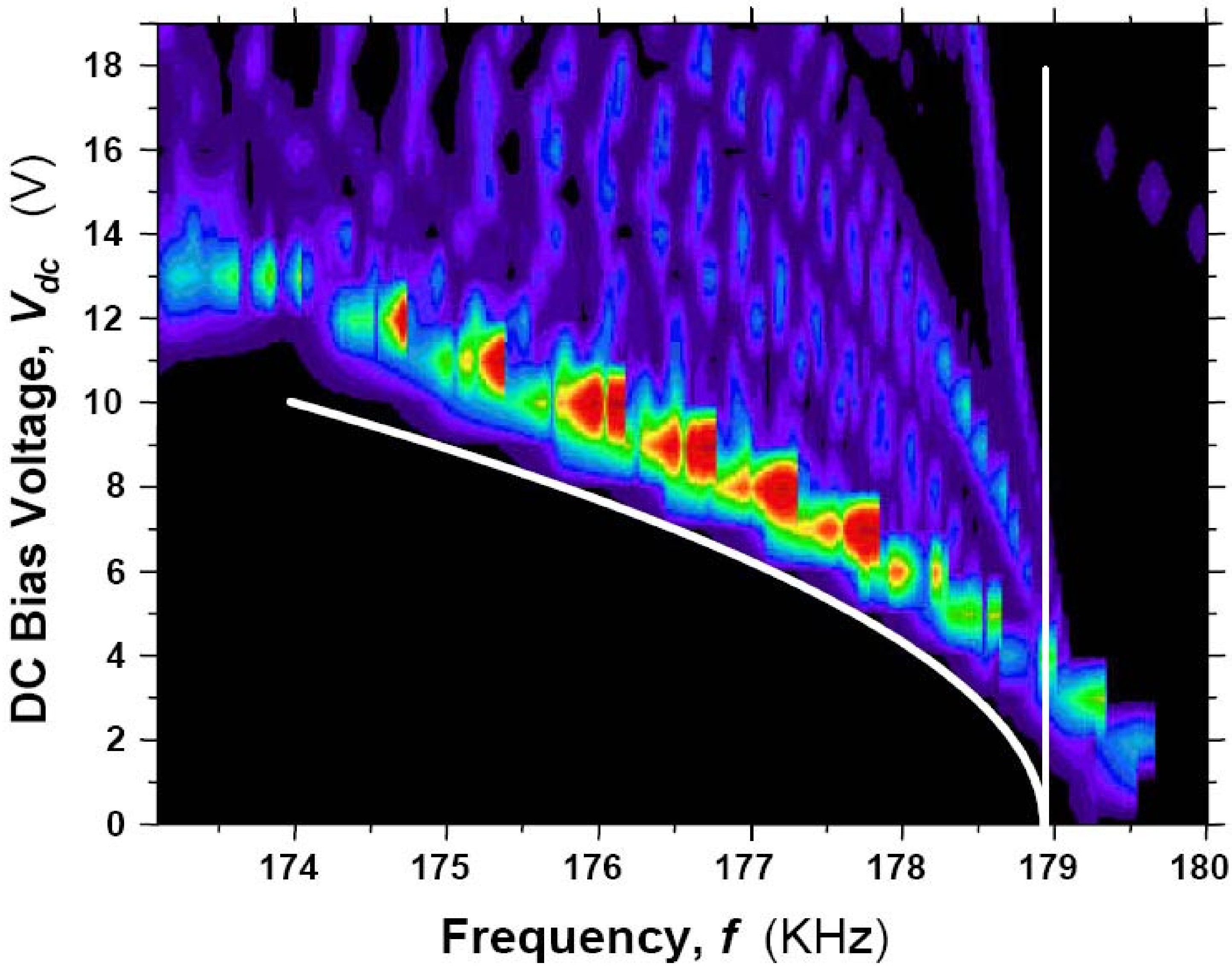
[Buks and Roukes, 2001]

Simple Intuition

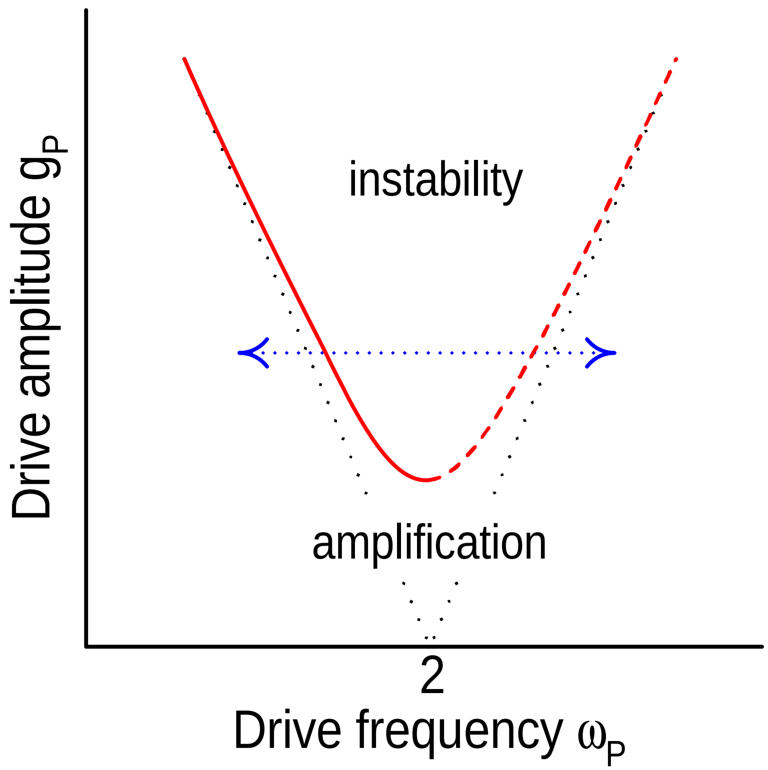


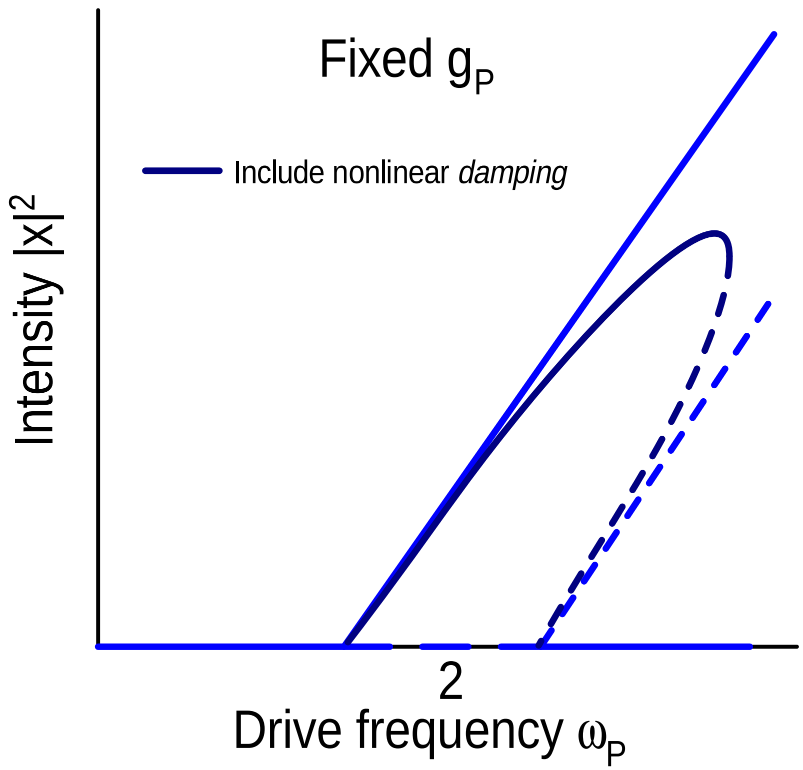
Above the parametric instability nonlinearity is essential to understand the oscillations.

- Mode Competition
- Pattern formation

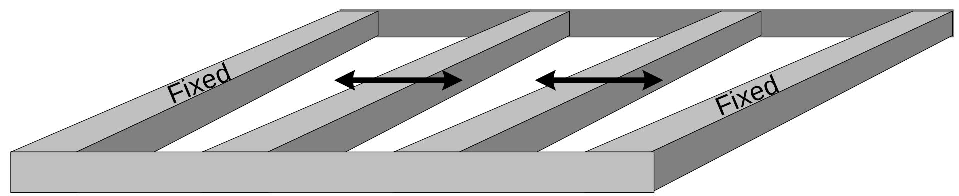

DC Bias Voltage, V_{dc} (V)
18
16
14
12
10
8
6
4
2
0
174
175
176
177
178
179
180
Frequency, f (KHz)

One Beam Theory





Two Beam Theory

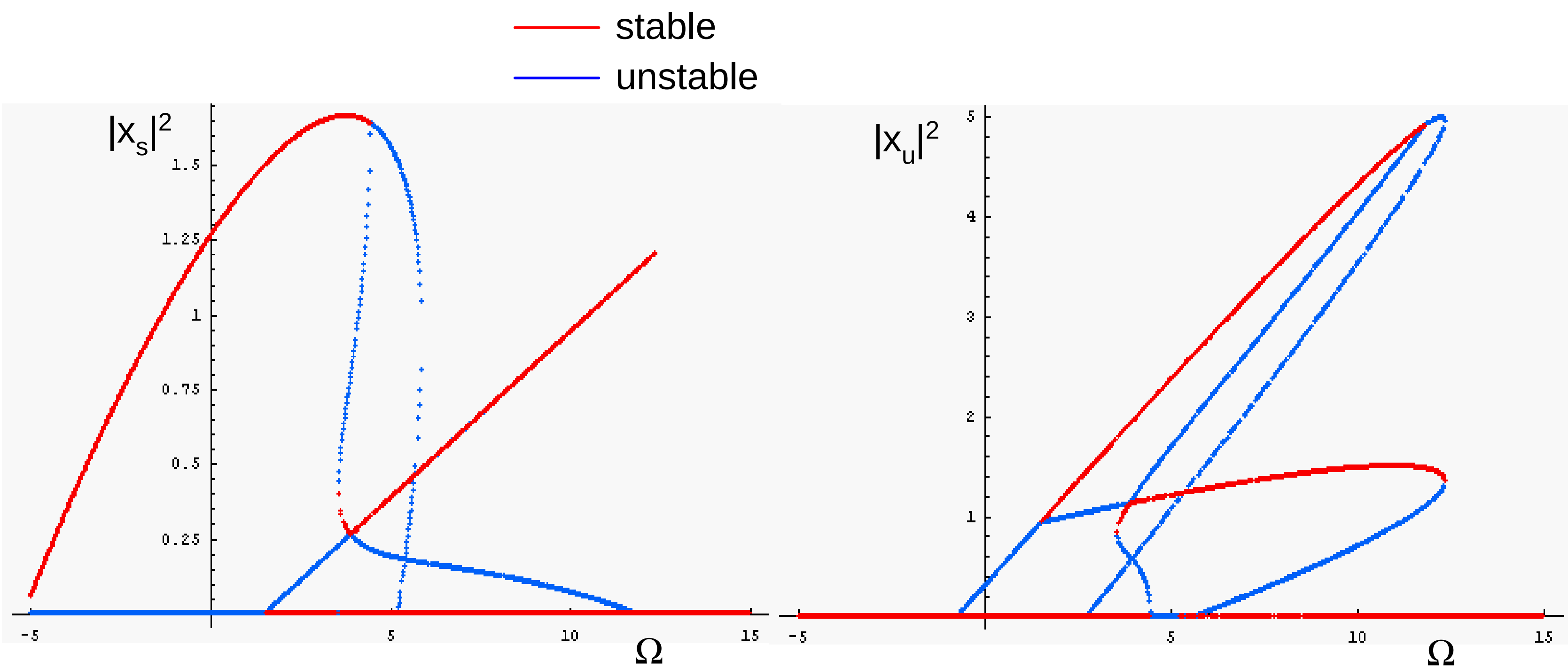


$$0 = \ddot{x}_n + x_n + x_n^3$$
$$+ \Delta^2 (1 + g_P \cos[(2 + \varepsilon \Omega_P) t])(x_{n+1} - 2x_n + x_{n-1})$$
$$- \gamma (\dot{x}_{n+1} - 2\dot{x}_n + \dot{x}_{n-1})$$
$$+ \eta \left[(x_{n+1} - x_n)^2 (\dot{x}_{n+1} - \dot{x}_n) - (x_n - x_{n-1})^2 (\dot{x}_n - \dot{x}_{n-1}) \right]$$

Local Duffing (elasticity) + Electrostatic Coupling (dc and modulated) + Dissipation (currents) + Nonlinear Damping (also currents)

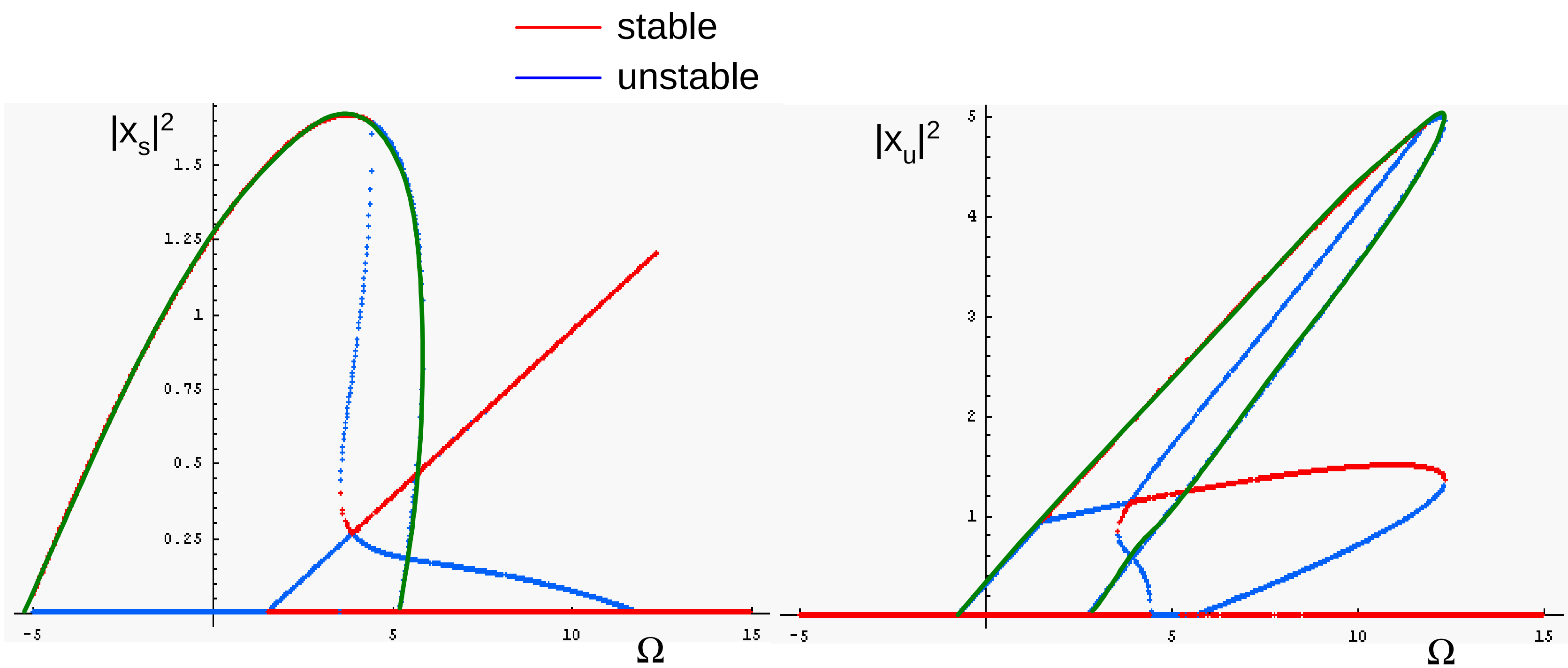
[Lifshitz and MCC Phys. Rev. B67, 134302 (2003)]

Periodic Solutions



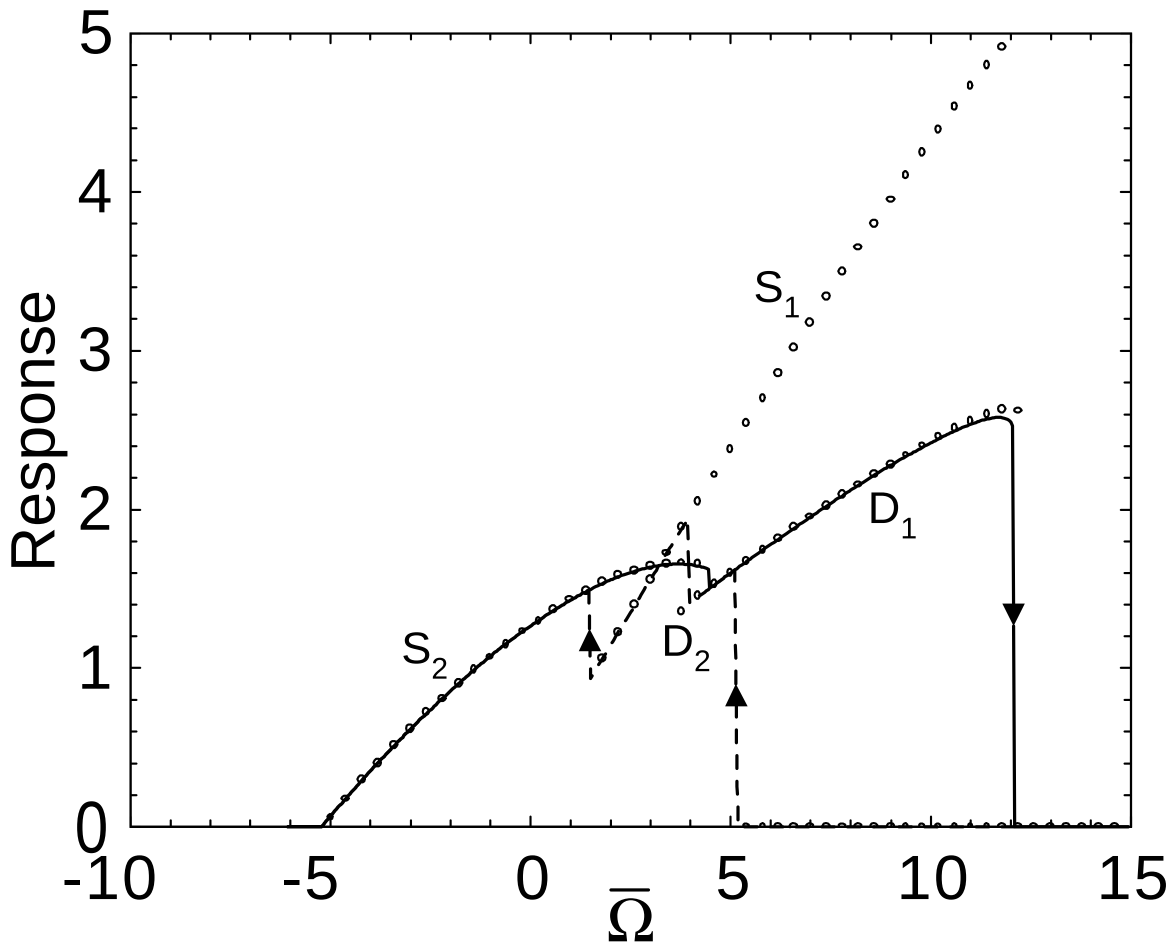
Intensity of symmetric mode $|x_s|^2$ and antisymmetric mode $|x_u|^2$ as frequency is scanned.

Periodic Solutions

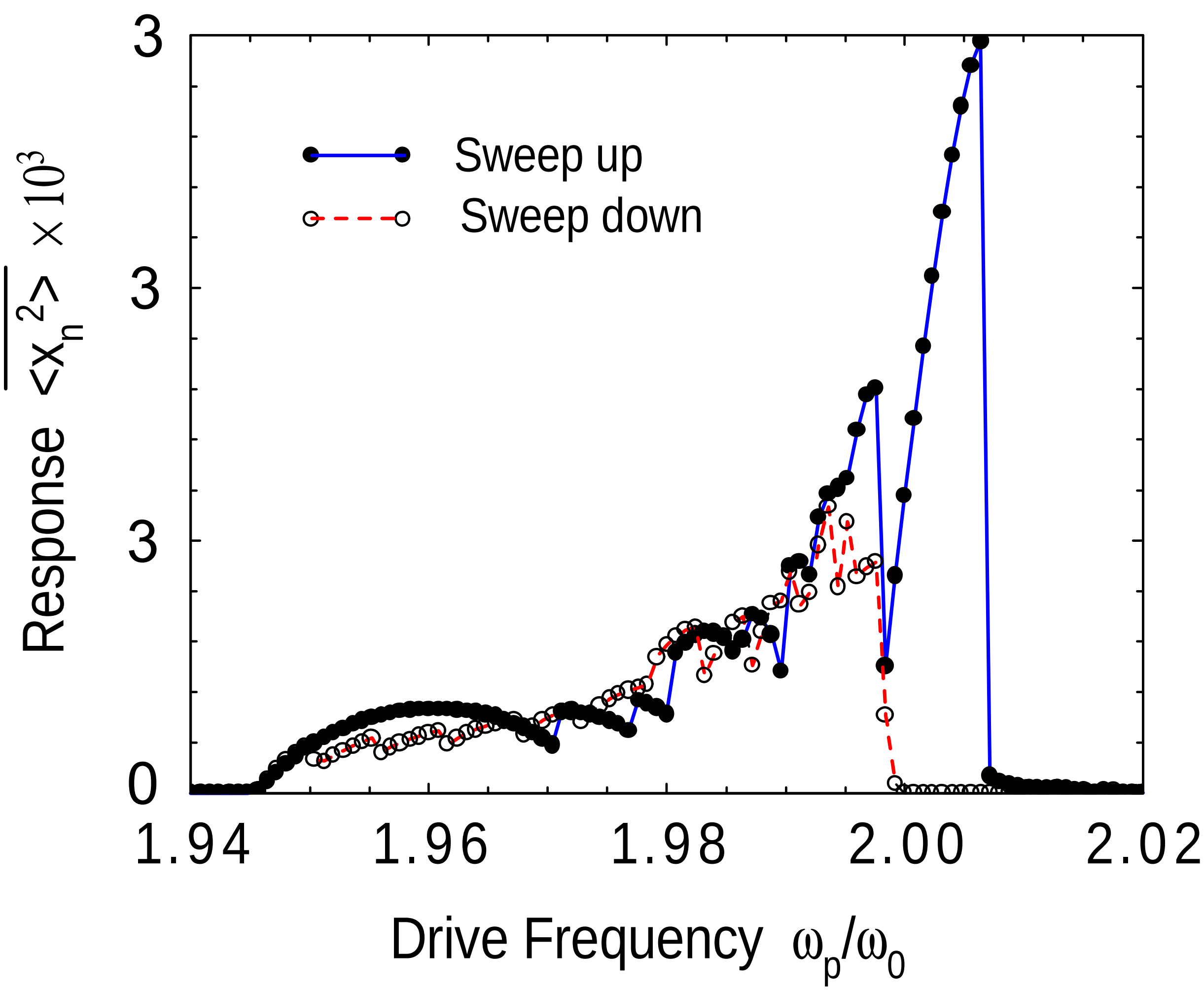


The green lines correspond to a single excited mode, the remainder to coupled modes.

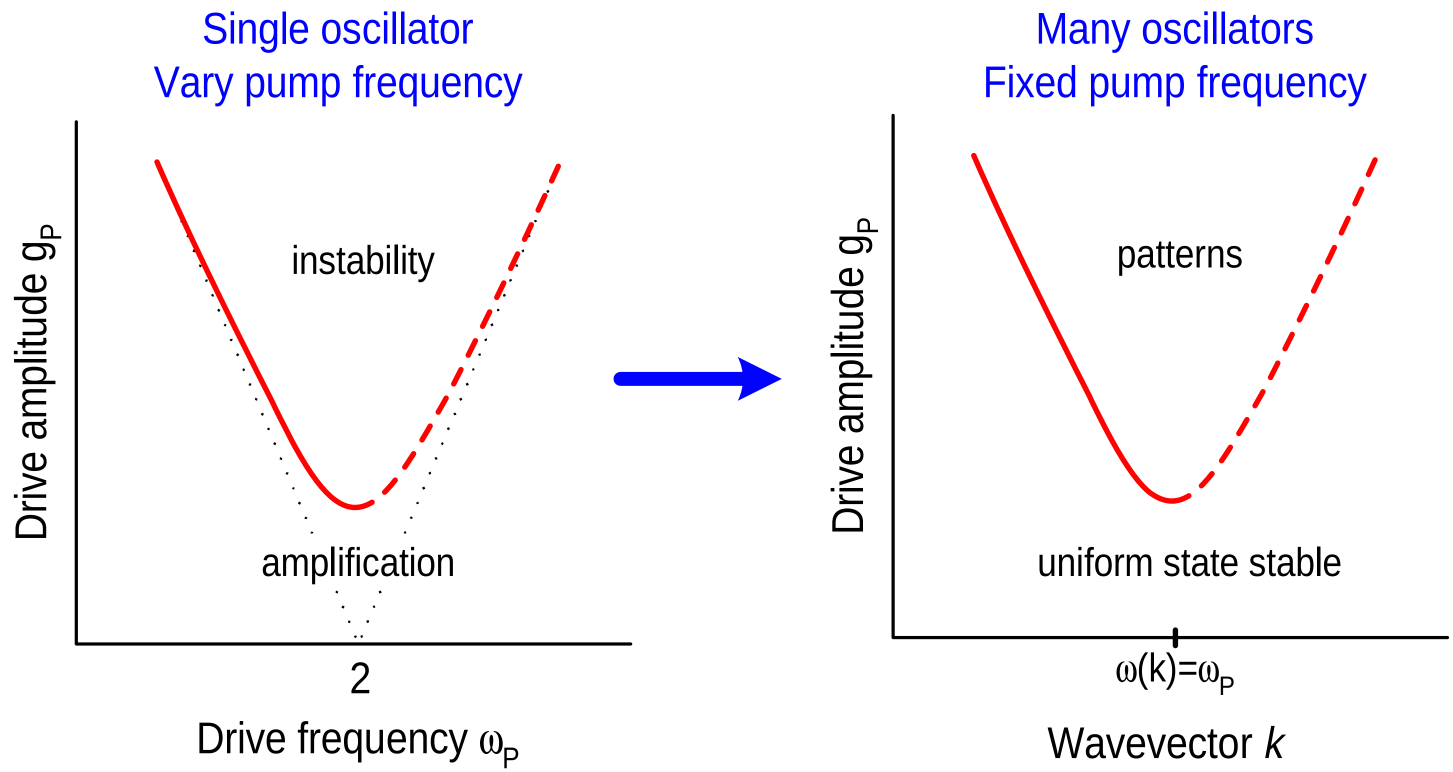
Hysteresis for Two Beams



Simulations of 67 Beams



Many Beams

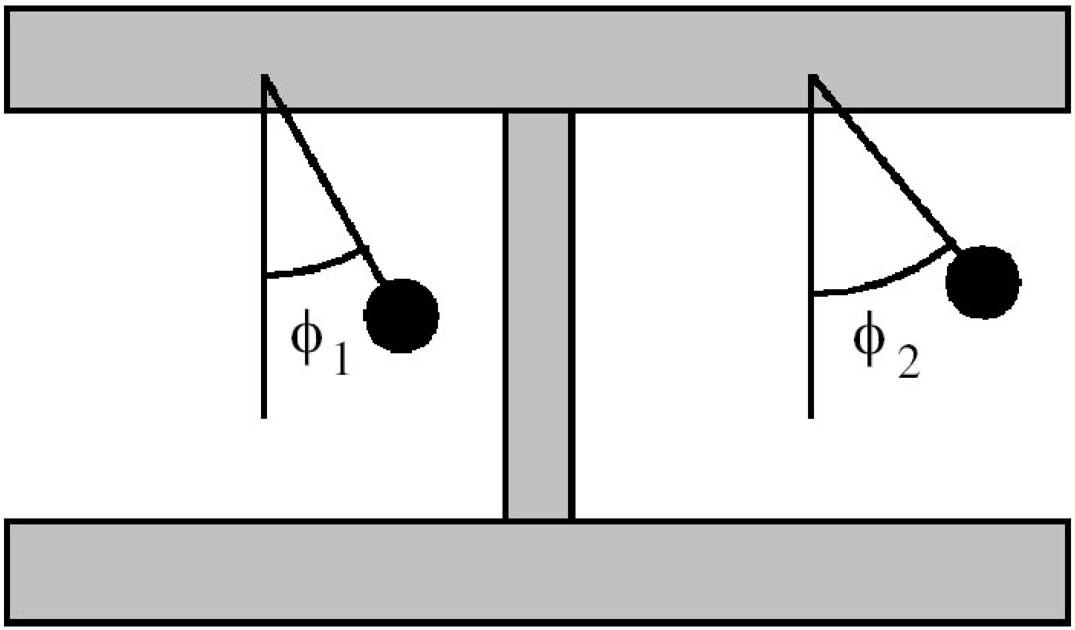


Continuum approximation: new amplitude equation [Bromberg, MCC and Lifshitz (preprint, 2005)]

$$\frac{\partial A}{\partial T} = A + \frac{\partial^2 A}{\partial X^2} + i\frac{2}{3}\left(4|A|^2\frac{\partial A}{\partial X} + A^2\frac{\partial A^*}{\partial X}\right) - 2|A|^2 A - |A|^4 A$$

Synchronization

Huygen's Clocks (1665)



From: Bennett, Schatz, Rockwood, and Wiesenfeld (Proc. Roy. Soc. Lond. 2002)

Paradigm I: Synchronization occurs through dissipation acting on the phase differences

- Huygen's clocks (cf. Bennett, Schatz, Rockwood, and Wiesenfeld)
- Winfree-Kuramoto phase equation

$$\dot{\theta}_n = \omega_n - \sum_m K_{nm} \sin(\theta_n - \theta_{n+m})$$

with ω_n taken from distribution $g(\omega)$. Continuum limit (short range coupling)

$$\dot{\theta} = \omega(x) + K\nabla^2\theta + O(\nabla(\nabla\theta)^3)$$

—phase diffusion, not propagation (eg. no $(\nabla\theta)^2$ term)

- Aronson, Ermentrout and Kopell analysis of two coupled oscillators
- Matthews, Mirollo and Strogatz magnitude-phase model

Synchronization in MEMS $\Rightarrow$ alternative mechanism

Paradigm II: Synchronization occurs by nonlinear frequency pulling and reactive coupling

MEMS equation

$$0 = \ddot{x}_n + (1+\omega_n)x_n - \nu(1-x_n^2)\dot{x}_n - a x_n^3 + \sum_m D_{nm}(x_{n+m} - 2x_n + x_{n-m})$$

leads to

$$\dot{A}_n = i(\omega_n - \alpha\,|A_n|^2)A_n + (1-|A_n|^2)A_n + i\sum_m \beta_{mn}(A_m - A_n)$$

with $a \Rightarrow \alpha,\ D \Rightarrow \beta.$

(cf. *Synchronization* by Pikovsky, Rosenblum, and Kurths)

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(cf. *Synchronization* by Pikovsky, Rosenblum, and Kurths)

Analyze mean field version (all-to-all coupling): $\beta_{mn} \to \beta/N$

Definitions of Synchronization

1. Order parameter

$$\Psi = N^{-1} \sum_n A_n = N^{-1} \sum_n r_n \, e^{i\theta_n} = R \, e^{i\Theta}$$

   Synchronization occurs if $R \neq 0$

2. Full locking: $\dot{\theta}_n = \Omega$ for all the oscillators

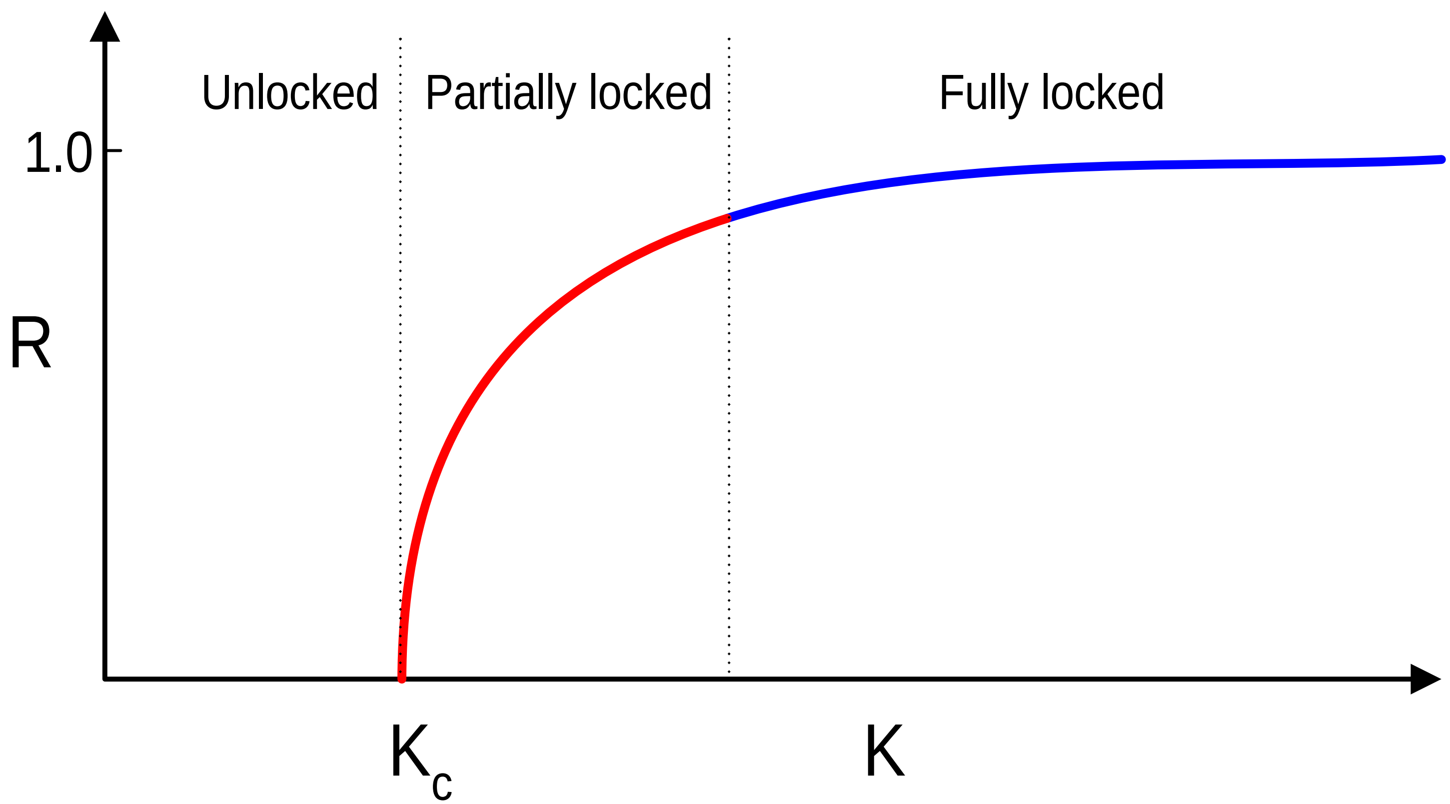
3. Partial frequency locking

$$\bar{\omega}_n = \lim_{T \to \infty} \frac{\theta_n(T) - \theta_n(0)}{T}$$

   and then $\bar{\omega}_n = \Omega$ for some $O(N)$ subset of oscillators

4. …

Results for the mean field phase model (Kuramoto 1975)



Calculations [MCC, Zumdieck, Lifshitz, and Rogers (2004)]

- Linear instability of unsynchronized $R = 0$ state (for Lorentzian, triangular, top-hat $g(\omega)$)
- Linear instability of fully locked state
- Simulations of amplitude-phase model for up to 10000 oscillators with all-to-all coupling

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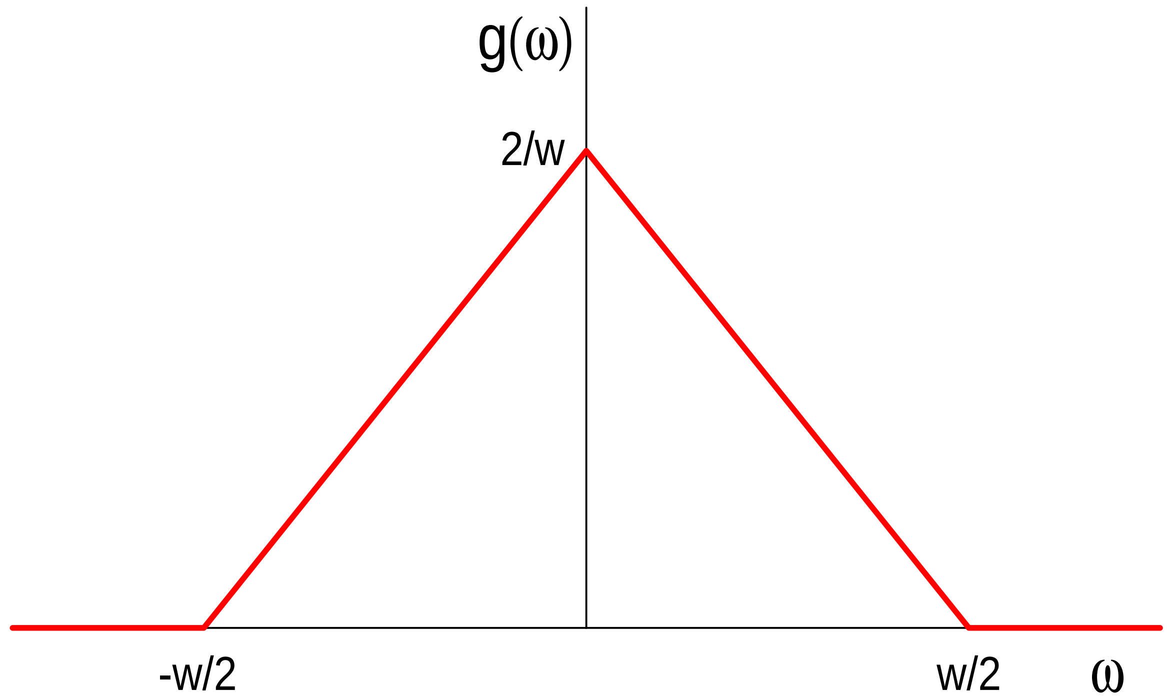
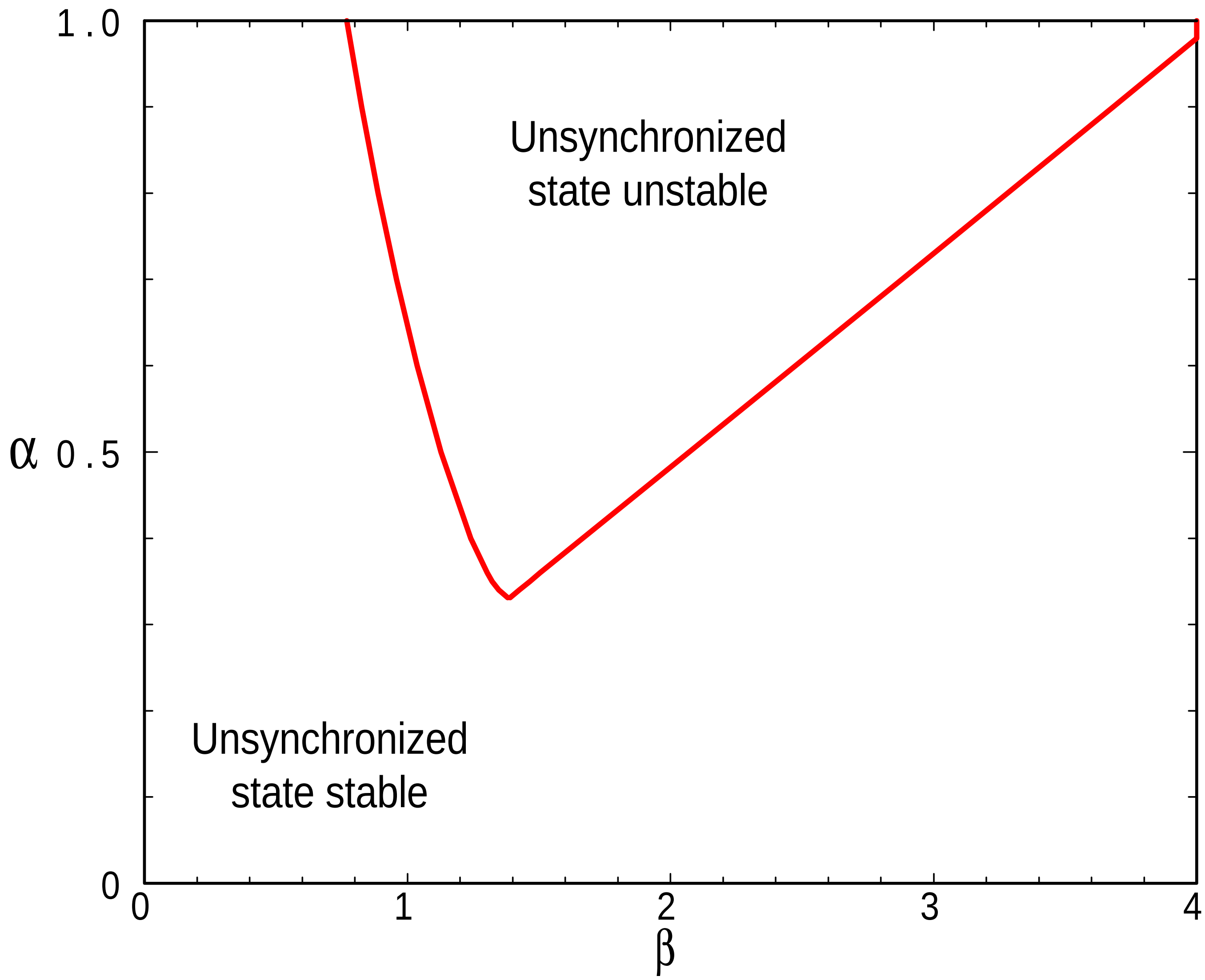
Results

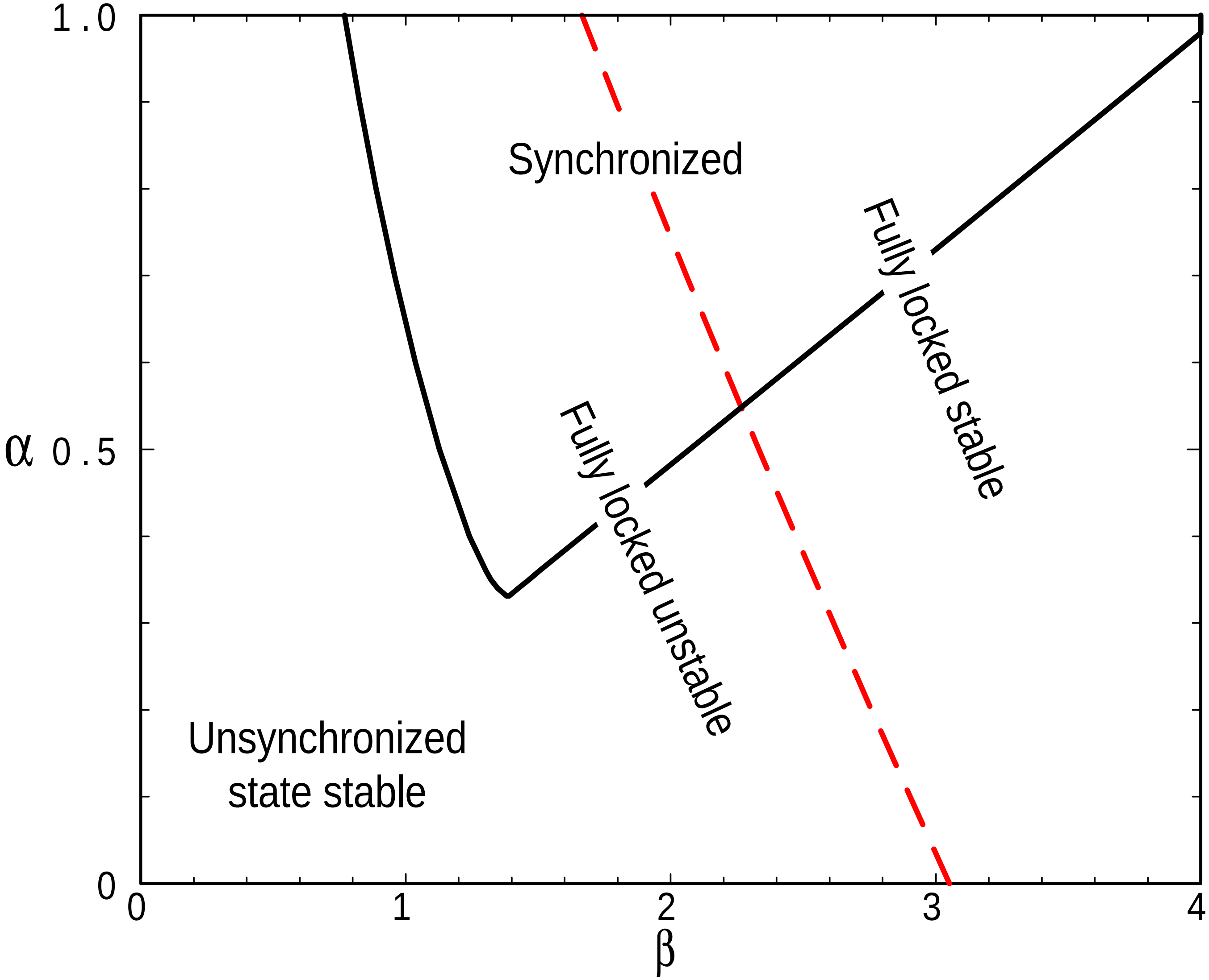
- Order parameter frequency $\Omega = \dot{\Theta}$ not trivially given by $g(\omega)$
- For fixed $\alpha > \alpha_{\rm min}$ there are two values of β giving linear instability
- Linear instability of fully locked state may be through stationary or Hopf bifurcation
- No “amplitude death” as in Matthews et al.
- Complicated phase diagram with regions of coexisting states

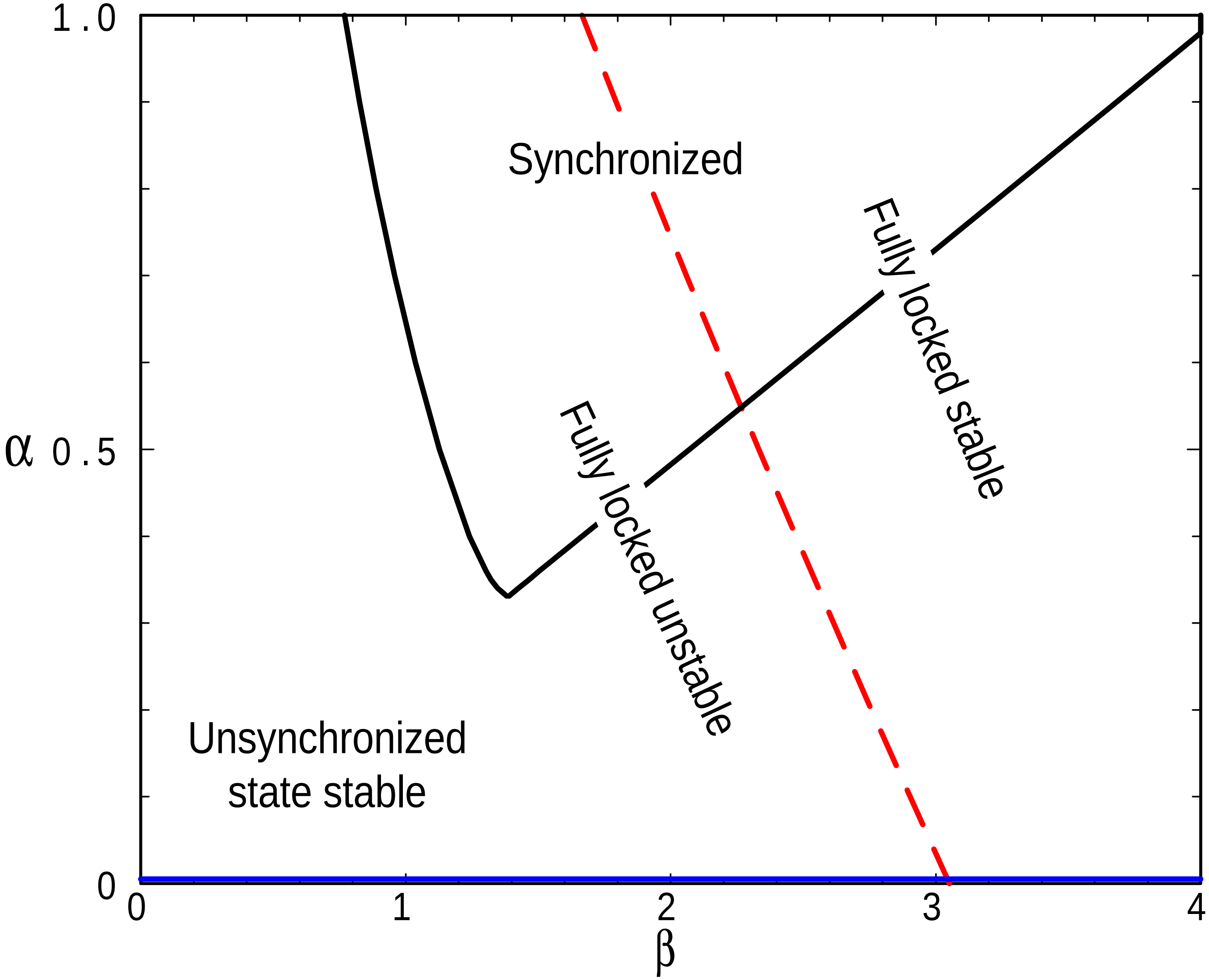
Results for a triangular distribution

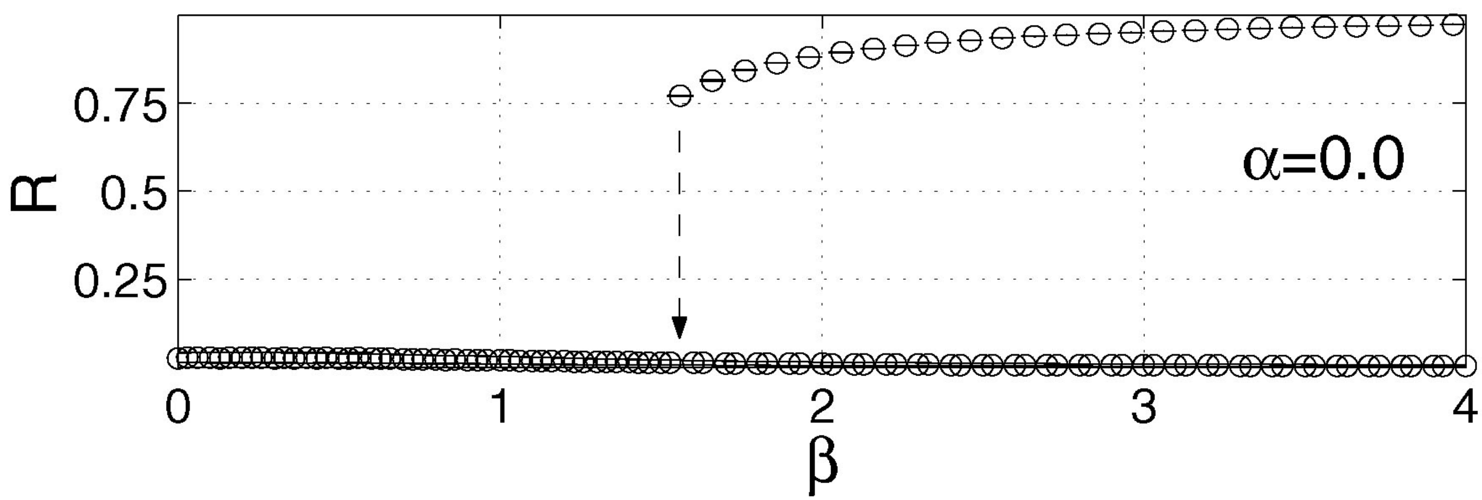


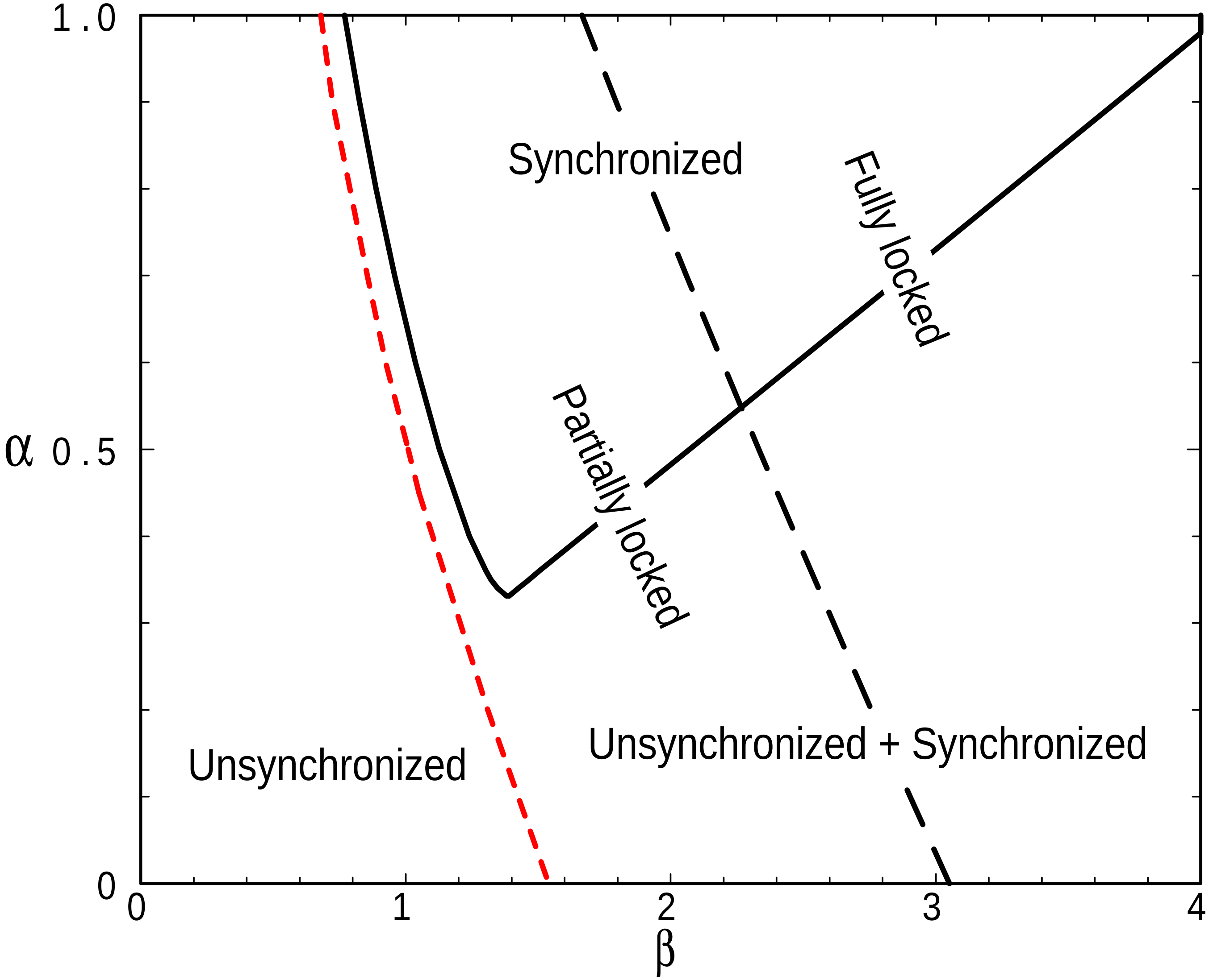
Show results for $w = 2\ldots$

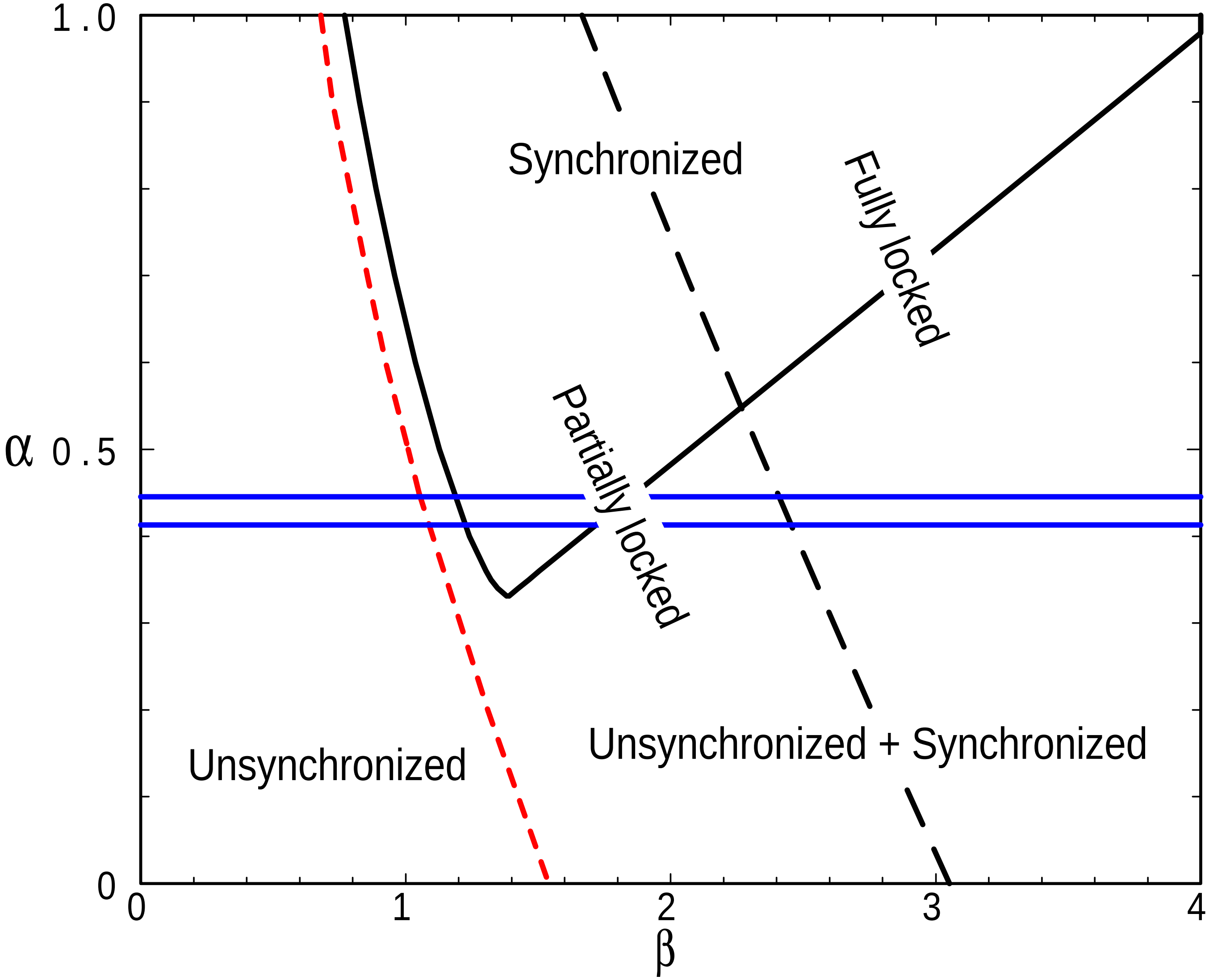

1.0
α
0.5
0
0
1
2
3
4
β
Unsynchronized
state unstable
Unsynchronized
state stable

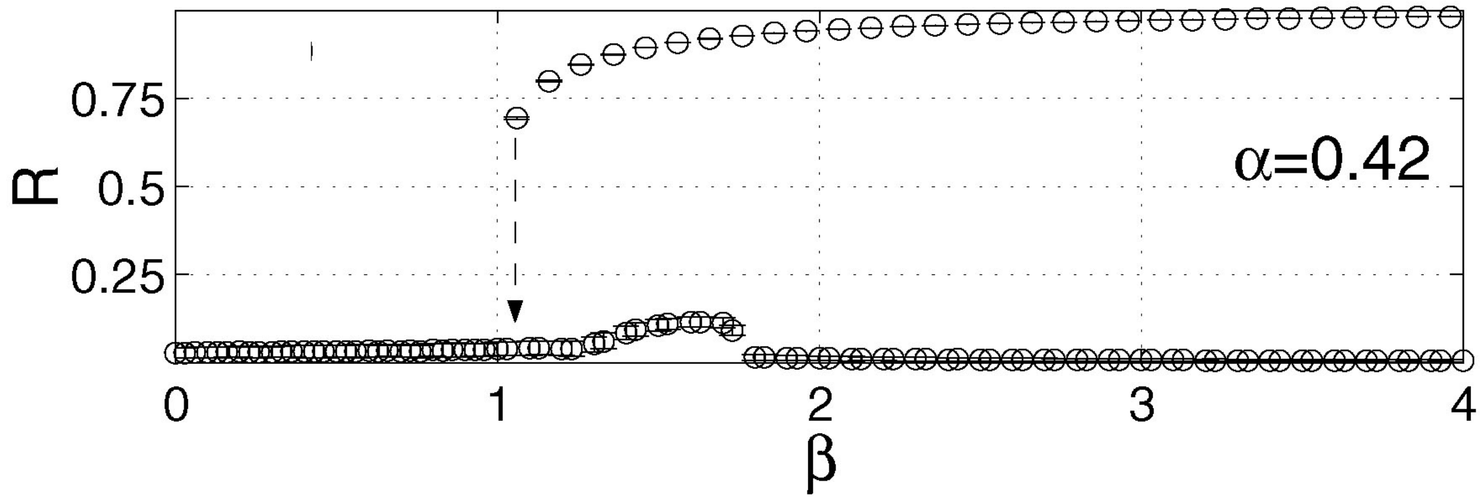

1.0
0.5
0
α
0
1
2
3
4
β
Synchronized
Fully locked stable
Fully locked unstable
Unsynchronized
state stable

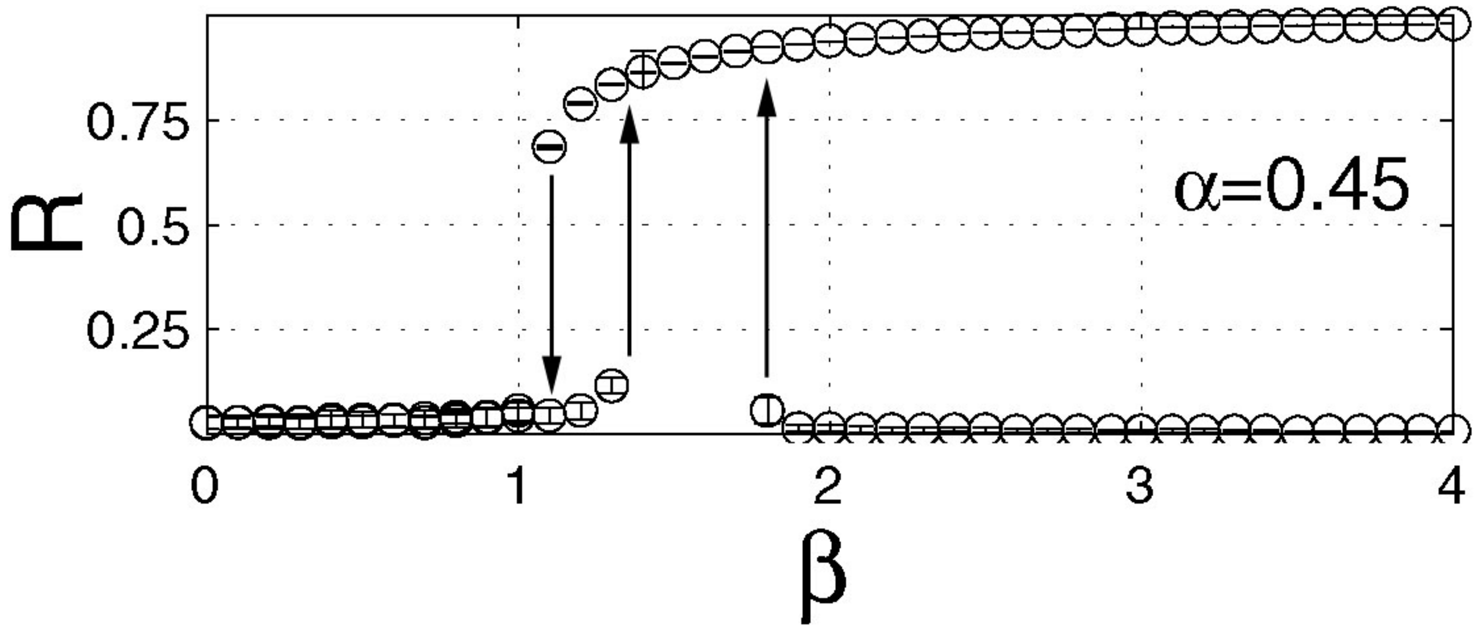

1.0
α 0.5
0
0
1
2
3
4
β
Synchronized
Fully locked stable
Fully locked unstable
Unsynchronized
state stable

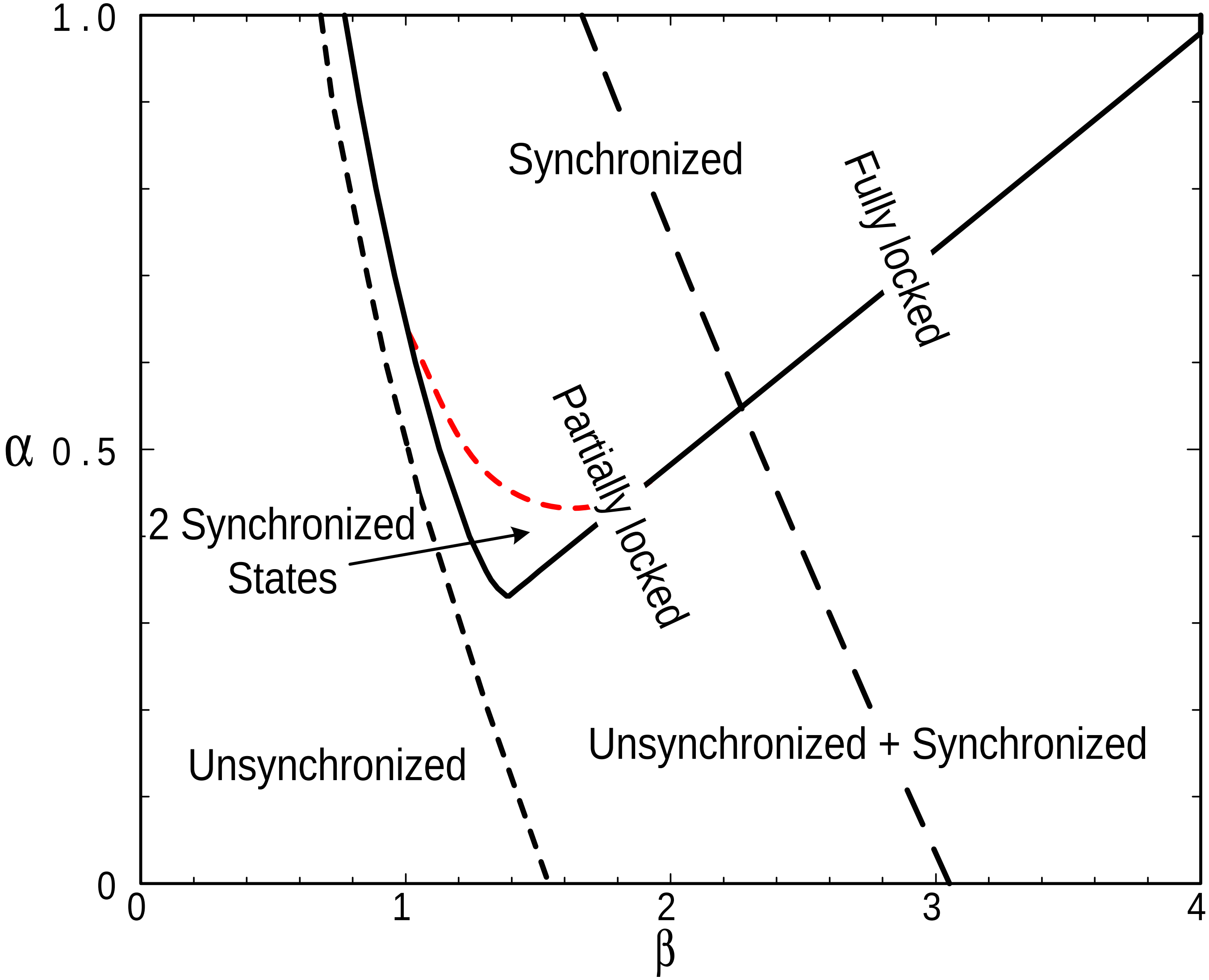

0.75
0.5
0.25
R
α=0.0
0
1
2
3
4
β

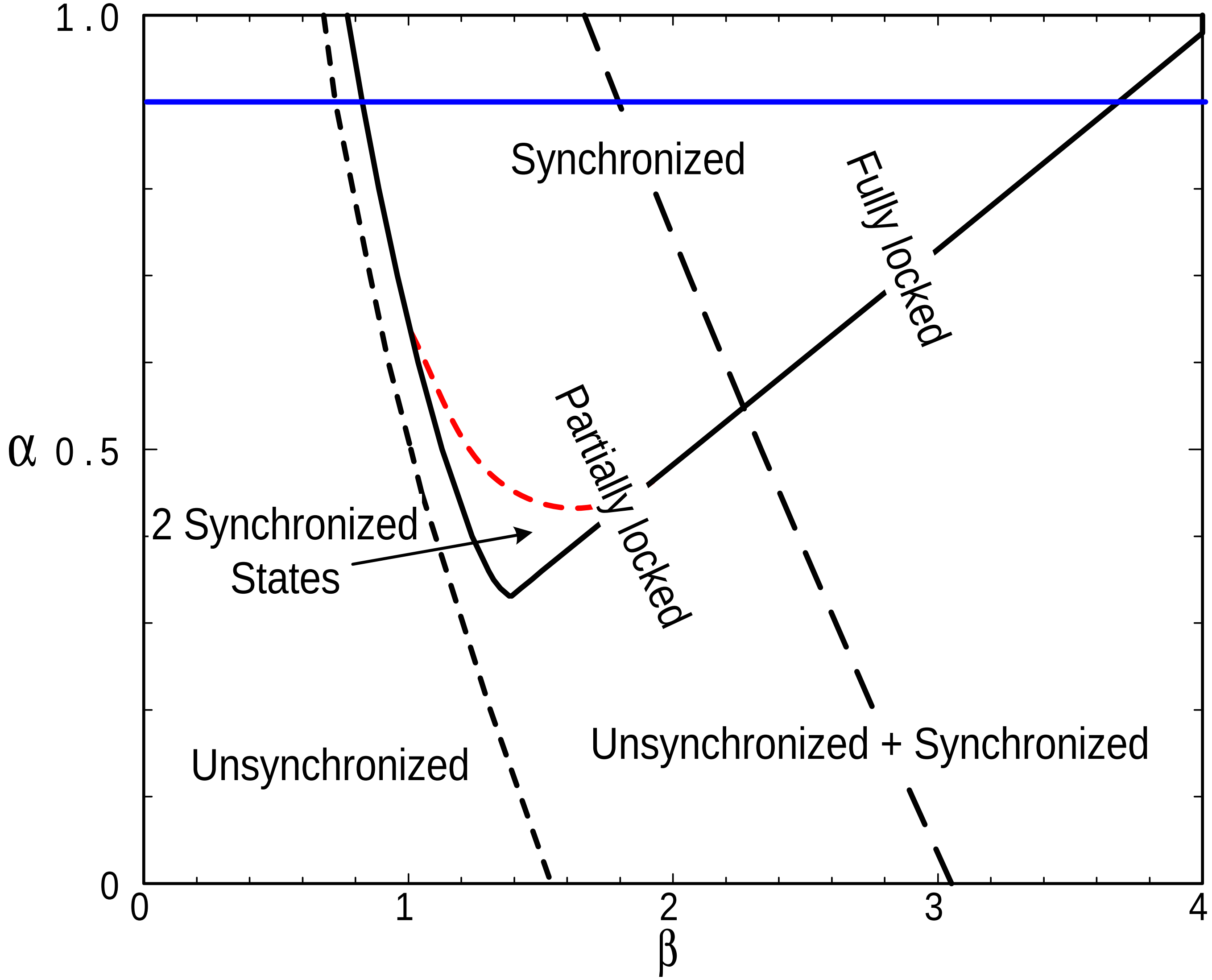

1.0
α 0.5
0
0
1
2
3
4
β
Synchronized
Fully locked
Partially locked
Unsynchronized
Unsynchronized + Synchronized

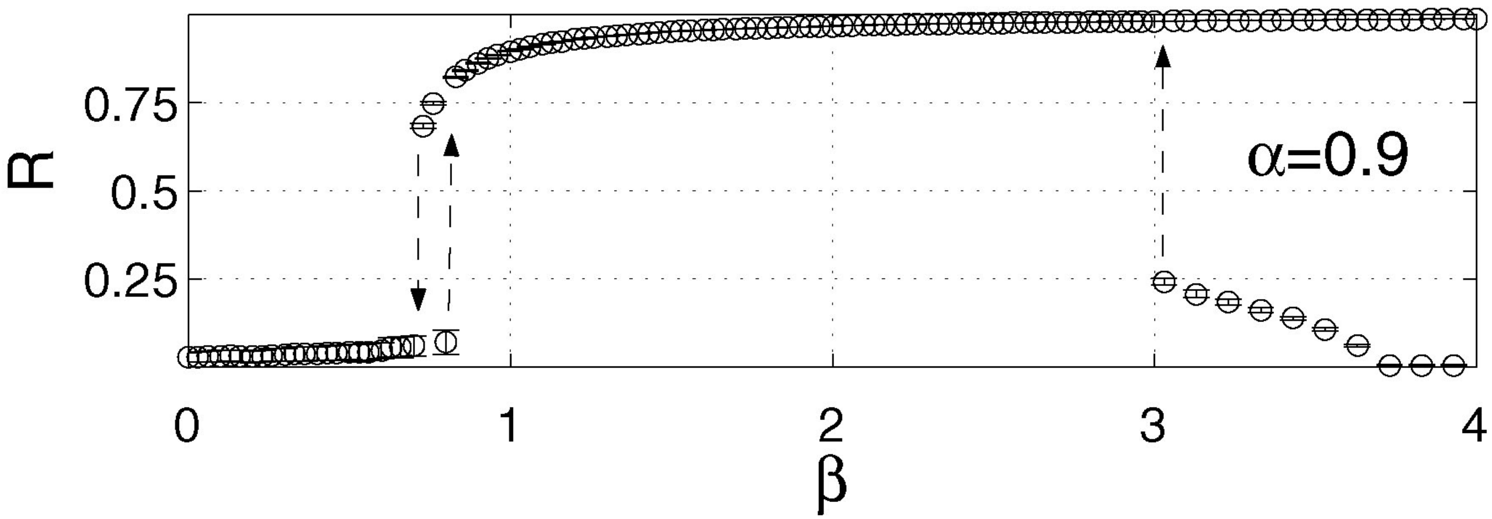

1.0
α 0.5
0
0
1
2
3
4
β
Synchronized
Fully locked
Partially locked
Unsynchronized
Unsynchronized + Synchronized

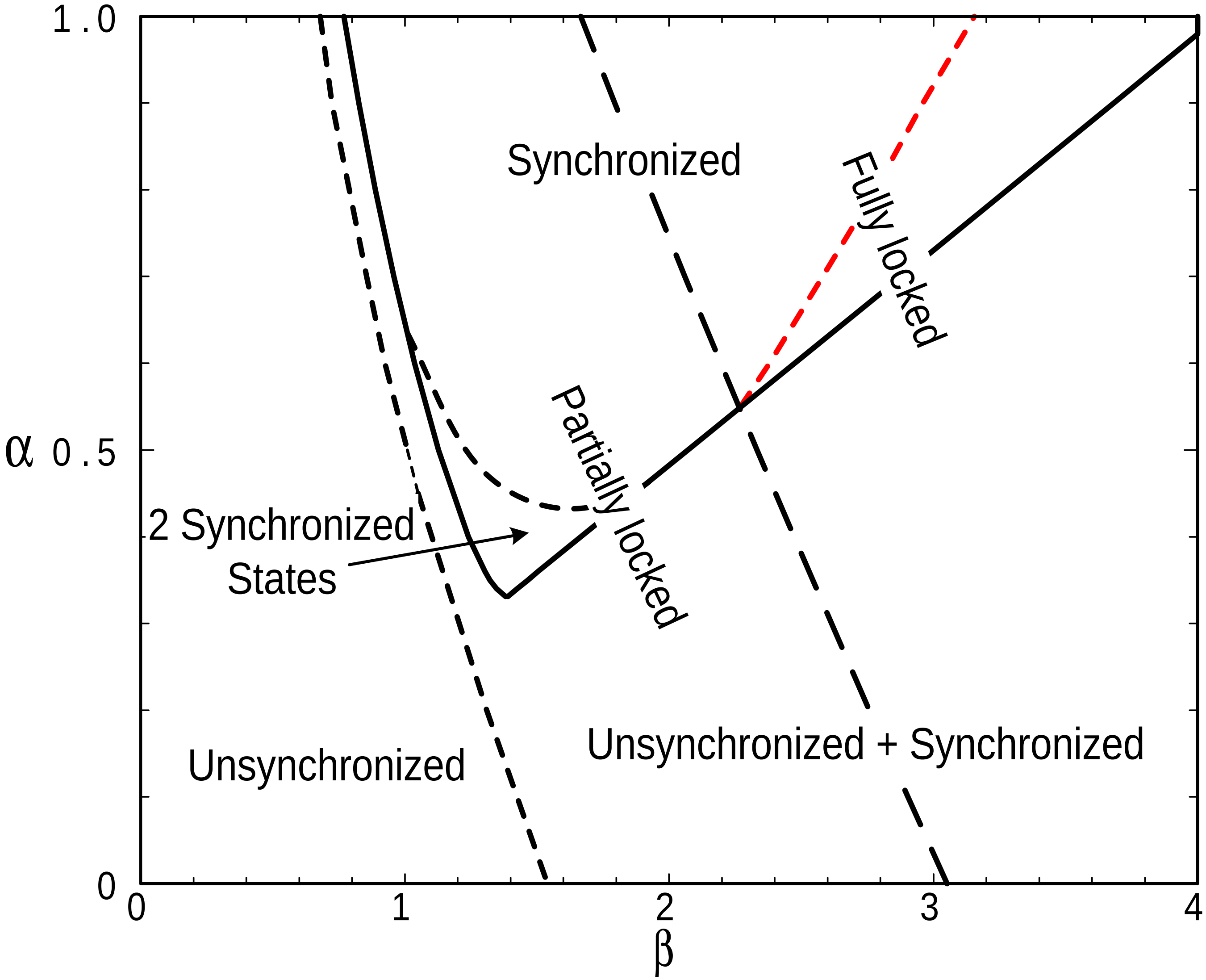

R
0.75
0.5
0.25
0
1
2
3
4
β
α=0.42

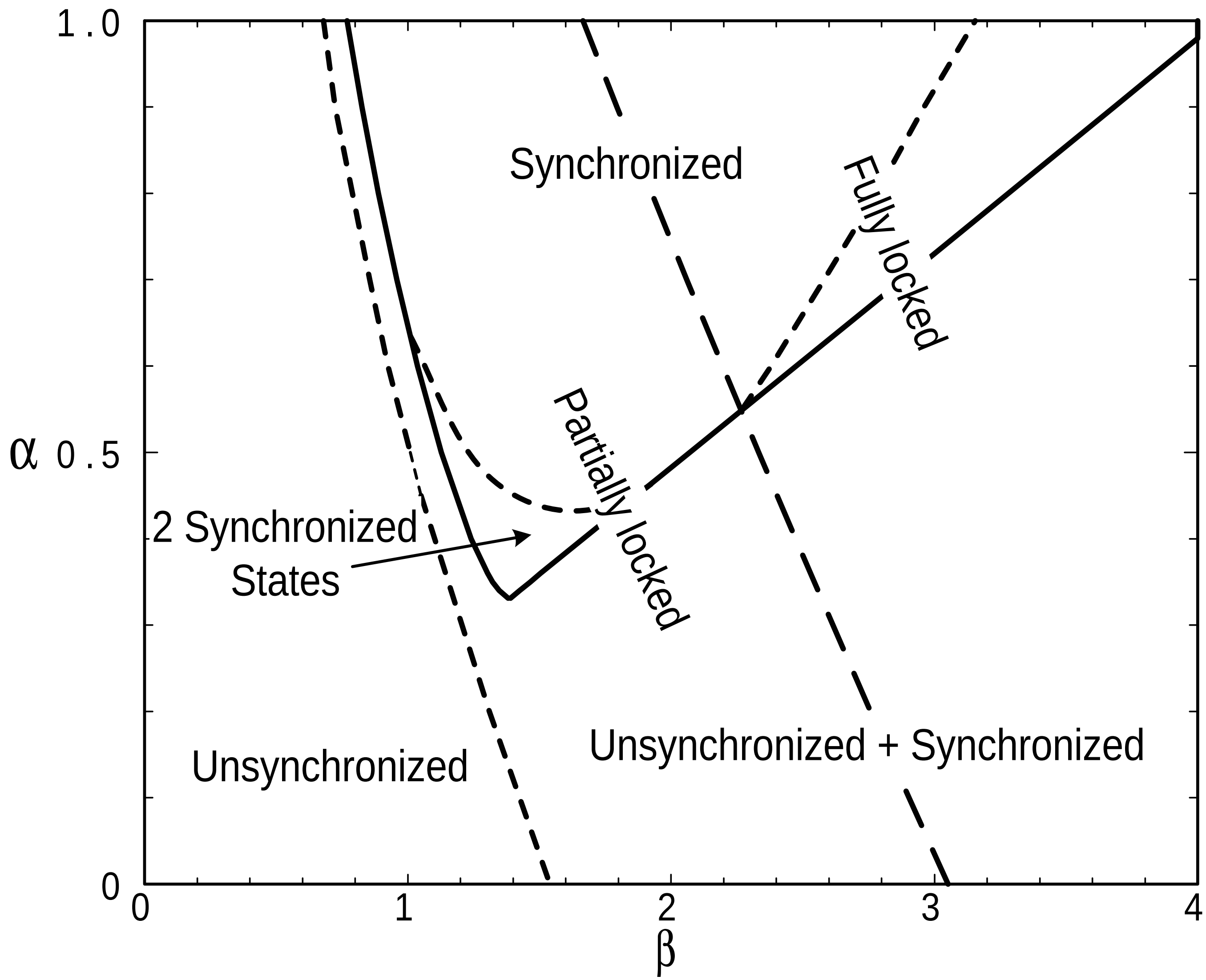

R
0.75
0.5
0.25
0
1
2
3
4
β
α=0.45


1.0
0.5
0
α
0
1
2
3
4
β
Synchronized
Fully locked
Partially locked
2 Synchronized
States
Unsynchronized
Unsynchronized + Synchronized

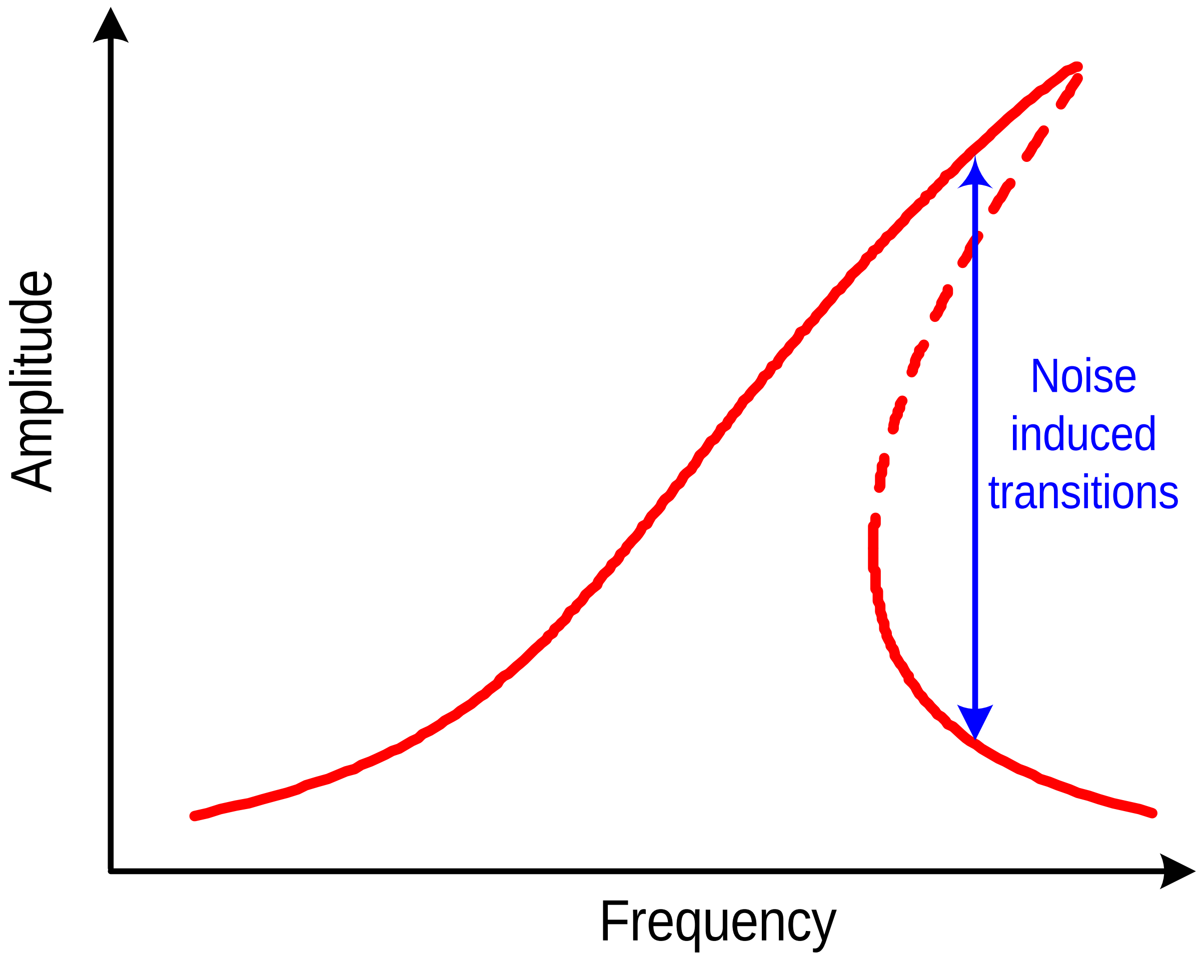

1.0
0.5
0
α
0
1
2
3
4
β
Synchronized
Fully locked
Partially locked
2 Synchronized
States
Unsynchronized
Unsynchronized + Synchronized


R
0.75
0.5
0.25
0
1
2
3
4
β
α=0.9


1.0
α 0.5
0
0
1
2
3
4
β
Synchronized
Fully locked
Partially locked
2 Synchronized
States
Unsynchronized
Unsynchronized + Synchronized


1.0
0.5
0
α
0
1
2
3
4
β
Synchronized
Fully locked
Partially locked
2 Synchronized
States
Unsynchronized
Unsynchronized + Synchronized

Noise Induced Transitions between Nonequilibrium States

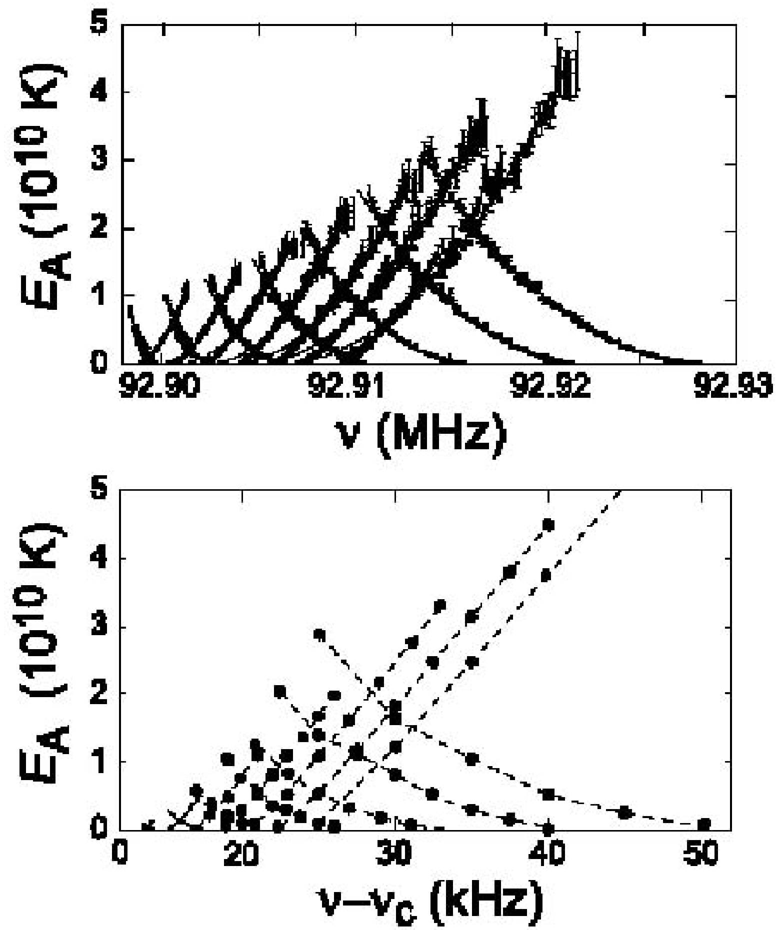


Current Status

- Lots of theory for single oscillators, few experimental tests
- No (?) results for many degree of systems
- Connection with non-equilibrium potential (Graham…). In weak noise limit $\eta \to 0$

$$P(\mathbf{x}) \sim \exp\left(-\Phi(\mathbf{x})/\eta\right)$$

  - ◇ Smoothness properties of $\Phi(\mathbf{x})$?
  - ◇ Prediction of deterministic dynamics from $\Phi(\mathbf{x})$?
- Recent experiments…



[From Aldridge and Cleland (cond-mat/0406528, 2004)]

Conclusions

Sub-micron oscillator arrays provide a new laboratory for nonlinear and nonequilibrium physics

- New features:
    - ◇ importance of noise and eventually quantum effects as dimensions shrink
    - ◇ discreteness
- Motivates new directions for theoretical investigation
- Physics of pattern formation, synchronization etc. may be useful in technological applications