# Physics 127c: Statistical Mechanics

## Kosterlitz-Thouless Transition

The transition to superfluidity in thin films is an example of the Kosterlitz-Thouless transition, an exotic new type of phase transition driven by the unbinding of vortex pairs. Many of the predictions of the theory have been verified in experiments on thin films of $He^4$ on a surface. This type of transition occurs in the universality class of two dimensional XY models, where the broken symmetry variable is an angle: the phase in superfluidity, the orientation of the spins in the xy plane for the magnet, etc.

The full treatment of the Kosterlitz-Thouless transition is a rather advanced topic, but the calculation illustrates many of the techniques introduced in the first two terms, and the result is interesting!

The novel behavior of the transition arises from the long wavelength logarithmic divergence of the phase or orientation fluctuations. It is therefore sufficient to take as the free energy

$$\bar{F} = \frac{F}{k_B T} = \frac{1}{2} K \int (\nabla\theta)^2 d^2x \tag{1}$$

where for the superfluid $\theta = \Phi$ and $K = \bar{\rho}_s / k_B T = (\hbar/m)^2 \rho_s / k_B T$. From now on we will use the reduced free energy $\bar{F}$.

### Special Features

**Phase fluctuations diverge** (This calculation mirrors the one on the *2d Heisenberg model* in Homework 4 of Ph127b.)

Introducing the usual expansion in Fourier modes

$$\theta(\mathbf{x}) = \sum_{\mathbf{q}} \theta_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{x}}, \tag{2}$$

the free energy becomes

$$\bar{F} = \frac{1}{2} K \Omega \sum_{\mathbf{q}} q^2 \left|\theta_{\mathbf{q}}\right|^2, \tag{3}$$

with $\Omega$ the area of the system. Equipartition gives

$$\left\langle \left|\theta_{\mathbf{q}}\right|^2 \right\rangle = \frac{1}{K\Omega q^2}, \tag{4}$$

and then the mean square fluctuation is

$$\sum_{\mathbf{q}} \left\langle \left|\theta_{\mathbf{q}}\right|^2 \right\rangle = \frac{\Omega}{(2\pi)^2} \int d^2q \frac{1}{K\Omega q^2} = \frac{1}{2\pi K} \int_{R^{-1}}^{\Lambda} \frac{dq}{q} \tag{5}$$

where $\Lambda$ is a large $q$ (small distance) cutoff, and the system size $R$ sets the small $q$ cutoff. The integral diverges logarithmically for large systems, $R \to \infty$.

Note that the free energy was only expanded up to quadratic order in deviations of $\theta(\mathbf{x})$ or $\theta_{\mathbf{q}}$. By analogy with the calculation for magnets, this is called the "spin wave approximation".

**Phase correlations decay with a power law** We now calculate the decay of correlations of the phase or angle coming from these small $q$ fluctuations, again starting from Eq. (1). We want to calculate the correlation function

$$G(x) = \left\langle e^{i(\theta(\mathbf{x})-\theta(\mathbf{0}))} \right\rangle = \langle \cos(\theta(\mathbf{x}) - \theta(\mathbf{0})) \rangle \tag{6}$$

since this gives the $\langle \psi(\mathbf{x})\psi^*(\mathbf{0}) \rangle$ correlation function for the superfluid, or the $\langle \mathbf{m}(\mathbf{x}) \cdot \mathbf{m}(\mathbf{0}) \rangle$ correlation function for the magnet (the average of the corresponding sin is zero since positive and negative $\theta(\mathbf{x}) - \theta(\mathbf{0})$ are equally likely). Expanding in the Fourier modes

$$\theta(\mathbf{x}) - \theta(\mathbf{0}) = \sum_{\mathbf{q}} \theta_{\mathbf{q}}(e^{i\mathbf{q}\cdot\mathbf{x}} - 1). \tag{7}$$

The average is given by integrating over the Gaussian distribution of each $\theta_{\mathbf{q}}$ given by the Boltzmann factor from the free energy Eq. (3)

$$G(x) = \frac{\int \cdots \int (\prod_{\mathbf{q}} d\theta_{\mathbf{q}}) \exp\left(-\sum_{\mathbf{q}} \frac{1}{2} K\Omega q^2 \left|\theta_{\mathbf{q}}\right|^2 - i\theta_{\mathbf{q}}(e^{i\mathbf{q}\cdot\mathbf{x}} - 1)\right)}{\int \cdots \int (\prod_{\mathbf{q}} d\theta_{\mathbf{q}}) \exp\left(-\sum_{\mathbf{q}} \frac{1}{2} K\Omega q^2 \left|\theta_{\mathbf{q}}\right|^2\right)}, \tag{8}$$

where the first term in the sum in the exponential in the numerator is from the Boltzmann factor, the second is from Eq. (7), and the products of exponentials has been written as a single sum.

The integrals are most reliably done by writing $\theta_{\mathbf{q}}$ in real and imaginary parts

$$\theta_{\mathbf{q}} = R_{\mathbf{q}} + i I_{\mathbf{q}}, \tag{9}$$

where the fact that $\theta(\mathbf{x})$ is real means $\theta_{-\mathbf{q}} = \theta^*_{\mathbf{q}}$ so that $R_{\mathbf{q}}$ is even and $I_{\mathbf{q}}$ odd in $\mathbf{q}$. Then we can write the sum in the numerator

$$\sum_{\mathbf{q}} \ldots = \sum_{\mathbf{q}} \frac{1}{2} K\Omega q^2 (R^2_{\mathbf{q}} + I^2_{\mathbf{q}}) - i R_{\mathbf{q}}(\cos \mathbf{q}\cdot\mathbf{x} - 1) + i I_{\mathbf{q}} \sin \mathbf{q}\cdot\mathbf{x} \tag{10}$$

where the other terms such as $R_{\mathbf{q}} \sin \mathbf{q}\cdot\mathbf{x}$ vanish on summing over $\mathbf{q}$ and $-\mathbf{q}$. Now complete the squares

$$\sum_{\mathbf{q}} \ldots = \sum_{\mathbf{q}} \frac{1}{2} K\Omega q^2 \left[ \left( R_{\mathbf{q}} - \frac{i(\cos \mathbf{q}\cdot\mathbf{x} - 1)}{K\Omega q^2} \right)^2 + \left( I_{\mathbf{q}} + \frac{i \sin \mathbf{q}\cdot\mathbf{x}}{K\Omega q^2} \right)^2 \right] + \sum_{\mathbf{q}} \frac{(1 - \cos \mathbf{q}\cdot\mathbf{x})}{K\Omega q^2}, \tag{11}$$

where to get the last term use $(\cos \mathbf{q}\cdot\mathbf{x} - 1)^2 + (\sin \mathbf{q}\cdot\mathbf{x})^2 = 2(1 - \cos \mathbf{q}\cdot\mathbf{x})$. The integrals over $R_{\mathbf{q}}$, $I_{\mathbf{q}}$ in the numerator of Eq. (8) cancel the integrals in the denominator (the shift of the center of the Gaussians does not change the integral) so that

$$G(x) = e^{-g(x)} \tag{12}$$

with

$$g(x) = \frac{1}{K\Omega} \sum_{\mathbf{q}} \frac{(1 - \cos \mathbf{q}\cdot\mathbf{x})}{q^2}. \tag{13}$$

Using as usual

$$\sum_{\mathbf{q}} \rightarrow \frac{\Omega}{(2\pi)^2} \int d^2 q, \tag{14}$$

and introducing $\phi$ the angle of $\mathbf{q}$ in the plane measured from the direction of $\mathbf{x}$ gives

$$g(x) = \frac{1}{2\pi K} \int_0^{\Lambda} \frac{dq}{q} \frac{1}{2\pi} \int_0^{2\pi} d\phi [1 - \cos(q\,|x| \cos \phi)] \tag{15}$$

$$= \frac{1}{2\pi K} \int_0^{\Lambda} dq \frac{1 - J_0(q\,|x|)}{q}, \tag{16}$$

with $\Lambda$ a large $q$ cutoff. Without the Bessel function in the numerator the integral would diverge logarithmically from the small $q$ range. Since $J_0(q\,|x|) \to 0$ for $q$ large, and $J_0(q\,|x|) \to 0$ for $q \lesssim 1/\,|x|$ we have for large $x$

$$g(x) = \frac{1}{2\pi K} \ln(c\,|x|) \tag{17}$$

with $c$ some constant. This gives for the correlation function

$$G(x) \propto |x|^{-\eta} \qquad \text{with} \qquad \eta = \frac{1}{2\pi K} = \frac{k_B T}{2\pi \bar{\rho}_s}. \tag{18}$$

Thus the order parameter correlation function decays as a *power law* with an exponent $\eta$ that depends on temperature. The correlations decay more rapidly as $T$ increases, as you might expect.

Equation (18) predicts power law correlations for *all* temperatures, whereas we would expect exponential decay at large enough temperatures. We should actually have some confidence in the result. The calculation was tractable because of the quadratic nature of the effective Hamiltonian. But this arises *not* from an approximation that the deviations of the angles $\theta(\mathbf{x})$ from some uniform state are small (which we would doubt), but from the assumption of small *gradients*, i.e. that the *spatial variation* of $\theta(\mathbf{x})$ is slow, and the difference between "neighboring"angles is small. The power law correlations at long distances comes from the small $q$ part of the behavior, which is where this approximation should be good! What we have left out is the possibility of vortex excitations, for which Eq. (1) does not apply everywhere. Equation (18) turn out to be accurate, until vortex excitations proliferate. How this develops is the next topic.

**Vortex proliferation** We have seen that the energy of a single vortex depends logarithmically on the system size

$$\frac{E_v}{k_B T} = \pi K \ln\left(\frac{\alpha R}{a}\right), \tag{19}$$

and so we might expect no thermal excitation of a vortex. However the entropy, proportional to the log of the number of ways we can put down a single vortex, also depends on the log of the system size. An estimate would be

$$S_v = k_B \ln\left(\frac{C R^2}{a^2}\right), \tag{20}$$

with $C$ some numerical constant. The (reduced) free energy $\bar{F}_v = (E_v - T S_v)/k_B T$ diverges with increasing system size as

$$\bar{F}_v = (\pi K - 2)\ln R + \cdots, \tag{21}$$

where the $\cdots$ denotes unimportant constant terms. As first pointed out by Kosterlitz and Thouless, this diverging free energy *switches sign* at a critical temperature $T_{KT}$ given by

$$K_{KT}^{-1} = \frac{k_B T_{KT}}{\bar{\rho}_s} = \frac{\pi}{2}. \tag{22}$$

Above this temperature, at least in the approximation of isolated vortices, $\bar{F}_v$ is *negative*, and vortices should proliferate. Notice from Eq. (18) that $\eta = 1/4$ at $T = T_{KT}$ so that the correlations decay as $|x|^{-1/4}$ here.

### Complete picture

Perhaps surprisingly, given the simplicity of the assumptions, the preceding results turn out to be *exactly correct*, with the one modification that $K$ (or $\bar{\rho}_s$) is itself temperature dependent. There is a phase transition at $T_{KT}$. Above this temperature correlations decay exponentially. Below this temperature there is no long range

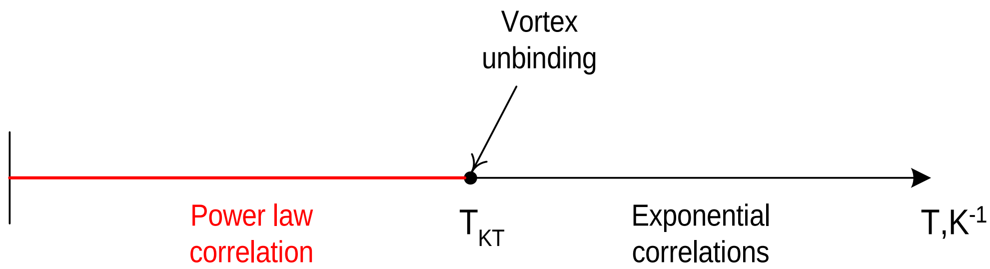


Figure 1: Schematic of the Kosterlitz-Thouless transition.

order (consistent with the Mermin-Wagner theorem for any continuous order parameter in two dimensions). However there are power-law correlations with the exponent $\eta(T)$ given by Eq. (18) where the temperature dependent $\bar{\rho}_s(T)$ (as would be measured in an experiment) is to be used. Below $T_{KT}$ the long range order is eliminated by the accumulation of small phase fluctuations ("spin wave theory"). The transition occurs by the proliferation of free vortices, the topological defect of the broken symmetry. Since at any temperature there are thermally excited vortex pairs, with small separations at low temperatures, probably a better way to think of the transition is as a *vortex pair unbinding*. At $T_{KT}$ the superfluid density jumps discontinuously between a nonzero value and zero. The ratio $\Delta\bar{\rho}_s(T_{KT})/k_B T_{KT}$ is *universal* and takes on the value $2/\pi$. Power law correlations are associated with a critical point; the power law correlations for all $T < T_{KT}$ can be understood in terms of a *critical line*. The correlations behave as $|x|^{-1/4}$ at $T_{KT}$, again universal behavior at the transition temperature.

The simple calculations break down in ignoring the effect of the vortices on the $\theta_{\mathbf{q}}$ modes. In fact the vortices act to renormalize the stiffness $K$ of these modes. This is what makes $\bar{\rho}_s \to 0$ for $T > T_{KT}$, and is why Eq. (18) is not correct at all temperatures. This can be understood using a RNG treatment, as outlined in the next section.

## RNG Treatment

We first want to see how the vortices change the effective long-distance stiffness constant. We already know that vortex pairs reduce the mass flow $\rho_s \mathbf{v}_s$ (cf. the discussion of critical velocities in Lecture 5, where the vortex pair in the tube reduces the total flow). We can calculate the effect more completely from the change in free energy of an imposed superfluid velocity $\mathbf{v}$. (In this chapter I will define $\mathbf{v} = \nabla\Phi$ without a factor of $\hbar/m$.) The full, renormalized $\bar{\rho}_s$ is then

$$\bar{\rho}_s^R = \lim_{v\to 0} \frac{F(v) - F(0)}{\frac{1}{2}\Omega v^2}, \tag{23}$$

or in terms of $K = \bar{\rho}_s/k_B T$

$$K^R = \lim_{v\to 0} \frac{\bar{F}(v) - \bar{F}(0)}{\frac{1}{2}\Omega v^2}. \tag{24}$$

The free energy is

$$\bar{F} = -\ln \mathrm{Tr}\left[e^{-\bar{H}_{eff}(v)}\right] \tag{25}$$

with the effective Hamiltonian (divided by $k_B T$)

$$\bar{H}_{eff} = \frac{1}{2} K \int d^2x (\mathbf{v} + \mathbf{v}_v)^2. \tag{26}$$

Here Tr denotes the integral over all configurations of the vortices, and $\mathbf{v}_v(\mathbf{x})$ is the superfluid velocity field due to the vortices. (The small angle fluctuations are supposed already included in the "bare" $K$.) Expanding out

$$\bar{H}_{eff} = \frac{1}{2}K\int d^2x\, v^2 + \frac{1}{2}K\int d^2x\, v_v^2 + K\int d^2x\, \mathbf{v}_v\cdot\mathbf{v}. \tag{27}$$

The first term is an additive constant, what we would have without vortices; the second plays the role of the unperturbed Hamiltonian $\bar{H}_0$ for the vortices (no external $\mathbf{v}$), and the third is the small perturbation. Now expanding the exponential to second order in $\mathbf{v}$

$$-\ln \mathrm{Tr}\left[e^{-\bar{H}_{eff}(v)}\right] \simeq \frac{1}{2}K\Omega\, v^2 - \ln \mathrm{Tr}\left[e^{-\bar{H}_0}\left\{1 - K\int d^2x\,\mathbf{v}_v\cdot\mathbf{v} + \frac{1}{2}K^2\left(\int d^2x\,\mathbf{v}_v\cdot\mathbf{v}\right)^2 + \cdots\right\}\right] \tag{28}$$

$$\simeq \bar{F}_0 + \frac{1}{2}K\Omega v^2 - \ln\left[1 + \left\langle \frac{1}{2}K^2\left(\int d^2x\,\mathbf{v}_v\cdot\mathbf{v}\right)^2\right\rangle_0\right] \tag{29}$$

$$\simeq \bar{F}_0 + \frac{1}{2}K\Omega v^2 - \frac{1}{2}K^2\left\langle\left(\int d^2x\,\mathbf{v}_v\cdot\mathbf{v}\right)^2\right\rangle_0, \tag{30}$$

where the average $\langle\ \rangle_0$ is with respect to $\bar{H}_0$, and $\langle \mathbf{v}_v\rangle_0 = 0$ has been used. Using isotropy and homogeneity we have

$$\left\langle\left(\int d^2x\,\mathbf{v}_v\cdot\mathbf{v}\right)^2\right\rangle_0 = \sum_{ij} v_i v_j \int d^2x \int d^2x' \left\langle v_{v,i}(\mathbf{x}) v_{v,j}(\mathbf{x}')\right\rangle_0 \tag{31}$$

$$= \frac{1}{2}\Omega v^2 \int d^2x\, \langle \mathbf{v}_v(\mathbf{x})\cdot\mathbf{v}_v(\mathbf{0})\rangle_0\,. \tag{32}$$

Thus from Eq. (24)

$$K^R = K - \tfrac{1}{2}K^2\int d^2x\,\langle \mathbf{v}_v(\mathbf{x})\cdot\mathbf{v}_v(\mathbf{0})\rangle_0\,. \tag{33}$$

This relates the superfluid density to the velocity-velocity correlation function, a result reminiscent linear response theory.

The velocity $\mathbf{v}_v$ is due to the vortices. Now we introduce a configuration of vortices represented by the vorticity density $n_v(\mathbf{x})$

$$\nabla\times\mathbf{v}_v = 2\pi n_v(\mathbf{x})\hat{\mathbf{z}} = 2\pi\hat{\mathbf{z}}\sum_\alpha k_\alpha\delta(\mathbf{x}-\mathbf{X}_\alpha), \tag{34}$$

with the $\alpha$th vortex having sign $k_\alpha = \pm 1$ and position $\mathbf{X}_\alpha$ (the higher charged vortices have a larger energy, and can be neglected). We want to relate the $\mathbf{v}_v$ correlation function to the vortex density correlation function, which we will then evaluate from the statistical mechanics of the interacting vortices. The final result is

$$\int d^2x\,\langle \mathbf{v}_v(\mathbf{x})\cdot\mathbf{v}_v(\mathbf{0})\rangle_0 = \pi^2\int d^2x\, x^2\,\langle n_v(\mathbf{x})n_v(\mathbf{0})\rangle_0\,. \tag{35}$$

It seems to me I should be able to do this by integrating Eq. (34) directly, but the standard approach goes through Fourier space.

Introducing the Fourier representation and using $\left\langle \mathbf{v}_v(\mathbf{q})\cdot\mathbf{v}_v(\mathbf{q}')\right\rangle_0 \propto \delta_{\mathbf{q},-\mathbf{q}'}$

$$\int d^2x\,\langle \mathbf{v}_v(\mathbf{x})\cdot\mathbf{v}_v(\mathbf{0})\rangle_0 = \int d^2x \sum_{\mathbf{q},\mathbf{q}'} e^{i\mathbf{q}\cdot\mathbf{x}}\left\langle \mathbf{v}_v(\mathbf{q})\cdot\mathbf{v}_v(\mathbf{q}')\right\rangle_0 \tag{36a}$$

$$= \int d^2x \sum_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{x}}\,\langle \mathbf{v}_v(\mathbf{q})\cdot\mathbf{v}_v(-\mathbf{q})\rangle_0 \tag{36b}$$

$$= \Omega\lim_{q\to 0}\,\langle \mathbf{v}_v(\mathbf{q})\cdot\mathbf{v}_v(-\mathbf{q})\rangle_0\,. \tag{36c}$$

(Normally we would just evaluate the result at $\mathbf{q}=0$, but here we need to keep the limit.) In terms of $n_v(\mathbf{q})$, the Fourier transform of $n_v(\mathbf{x})$ we can write

$$\lim_{\mathbf{q}\to\mathbf{0}} \langle \mathbf{v}_v(\mathbf{q}) \cdot \mathbf{v}_v(-\mathbf{q}) \rangle_0 = 4\pi^2 \lim_{q\to 0} \frac{\langle n_v(\mathbf{q}) n_v(-\mathbf{q}) \rangle_0}{q^2}. \tag{37}$$

Now we want to calculate $\Omega \langle n_v(\mathbf{q}) n_v(-\mathbf{q}) \rangle_0$ for small $q$:

$$\Omega \langle n_v(\mathbf{q}) n_v(-\mathbf{q}) \rangle_0 = \Omega^{-1} \int d^2x \int d^2x' \left\langle n_v(\mathbf{x}) n_v(\mathbf{x}') \right\rangle_0 e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}')} \tag{38}$$

$$\simeq \Omega^{-1} \int d^2x \int d^2x' \left\langle n_v(\mathbf{x}) n_v(\mathbf{x}') \right\rangle_0 [1 - i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}') - \frac{1}{2} q_i q_j (\mathbf{x}-\mathbf{x}')_i (\mathbf{x}-\mathbf{x}')_j + \cdots]. \tag{39}$$

The first term is zero since the total vortex charge $\int d^2x \, \langle n_v(\mathbf{x}) \rangle$ is zero. The second is odd in $\mathbf{x}-\mathbf{x}'$ and integrates to zero. This leaves the last term, which gives (doing the angular average, giving a factor of $1/2$)

$$\Omega \langle n_v(\mathbf{q}) n_v(-\mathbf{q}) \rangle_0 \simeq -\frac{1}{4} q^2 \int d^2x \, x^2 \, \langle n_v(\mathbf{x}) n_v(\mathbf{0}) \rangle_0 \,. \tag{40}$$

Hence with Eq. (36) we get Eq. (35).

Thus from Eqs. (33) and (35) we have our final formal result for the renormalization of the superfluid density by vortices

$$K^R = K + \frac{\pi^2}{2} K^2 \int d^2x \, x^2 \, \langle n_v(\mathbf{x}) n_v(\mathbf{0}) \rangle_0 \,. \tag{41}$$

We are left with calculating the correlation function of the vortex density. The vortices interact with a logarithmic dependence on separation, which in the present notation is

$$\bar{E}_{\text{pair}} = 2\pi K \ln\left(\frac{|x|}{a}\right), \tag{42}$$

(where I have absorbed constants inside the logarithm into the small scale cutoff $a$). The problem actually reduces to a two dimensional gas of charges. Equation (41) is analogous to the dielectric constant for the charged gas: the polarization of intervening charge pairs changes the interaction of well separated charges. This is a nontrivial problem! We can make progress assuming a dilute gas. This is reasonable if the core energy of the vortex is large compared to $k_B T$, or in other words if the *fugacity* $y = \exp(-E_c/k_B T)$, is small. In this limit we can evaluate the correlation function in terms of the Boltzmann factor for $E_{\text{pair}}$

$$\langle n_v(\mathbf{x}) n_v(\mathbf{0}) \rangle_0 = -2a^{-4} y^2 \exp\left[-2\pi K \ln\left(\frac{|x|}{a}\right)\right], \tag{43}$$

where the factor of $a^{-4}$ is from two factors of the average density which we estimate as $\langle n_v \rangle = a^{-2} y$, and the minus sign is because the charges must be opposite to get a low energy configuration. The factor of 2 is for the two configurations $+$ at $\mathbf{x}$ and $-$ at $\mathbf{0}$ and the reverse, but actually any numerical prefactor can be absorbed into a slightly redefined fugacity, since $E_c$ is not precisely known. The important part is the dependence on $x$, which is the exponential of the interaction potential.

It is convenient to rewrite Eq. (41) in terms of $K^{-1} = k_B T / \bar{\rho}_s$. Because we are assuming a dilute gas, so that the second term is small, this finally gives

$$\left(K^{-1}\right)^R \simeq K^{-1} + 2\pi^3 y^2 \int_a^\infty \frac{dr}{a} \left|\frac{r}{a}\right|^{3-2\pi K} . \tag{44}$$

This expression imagines starting with "microscopic" $y$ and $K$ and calculating the "macroscopic" $K = K^R$ taking into account the reduction in the total momentum and energy due to the polarization (realignment and stretching) of the vortex pairs.

Inspecting the integral reproduces the elementary Kosterlitz-Thouless result: for $K^{-1} > \pi/2$ the integral diverges. We can understand this more completely using a renormalization group treatment allowing the coupling constants $K$ and $y$ to depend on scale factor $l$, and evaluating the integral over all scales piece by piece. Suppose we integrate out the small separations between $a$ and $a\prime = a(1+\delta l)$

$$\left(K^{-1}\right)^R \simeq K^{-1} + 2\pi^3 y^2 \int_a^{a+a\delta l} \frac{dr}{a} \left|\frac{r}{a}\right|^{3-2\pi K} + 2\pi^3 y^2 \int_{a'}^{\infty} \frac{dr}{a} \left|\frac{r}{a}\right|^{3-2\pi K}. \tag{45}$$

The first integral gives an additive correction that can be absorbed into a new $K$:

$$\left(K^{-1}\right)' = K^{-1} + 2\pi^3 y^2 \delta l. \tag{46}$$

The second integral can be put in the same form as before, now in terms of the cutoff $a'$, by defining a new $y$:

$$\left(y'\right)^2 = y^2 (1+\delta l)^{4-2\pi K}. \tag{47}$$

These transformations leave the expression for $K_R$ unchanged, with $K \to K'$, $y \to y'$, $a \to a'$, and the process can be iterated.

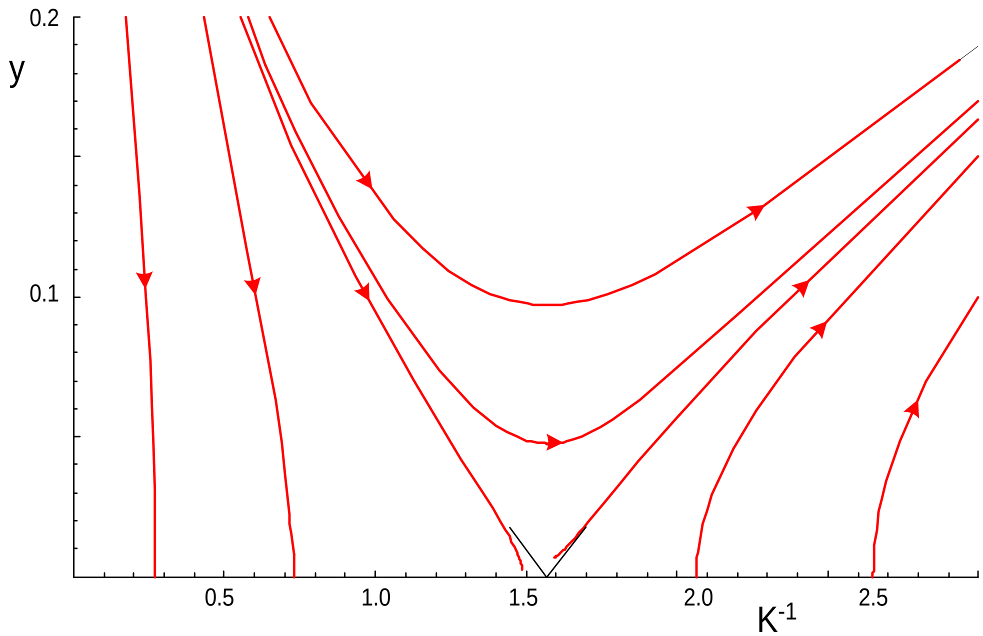


Figure 2: RNG flows for $K^{-1}$ and $y$ generated by numerical solutions of Eqs. (46) and (47).

The equations (46) and (47) can be written as differential equations for the evolution of $K^{-1}(l)$ and $y(l)$

$$\frac{dK^{-1}}{dl} = 2\pi^3 y^2, \tag{48a}$$

$$\frac{dy}{dl} = (2 - \pi K) y. \tag{48b}$$

These are the RNG *flows* for the Kosterlitz-Thouless transition. Some numerically generated solutions for the flows are shown in Fig. 2. We have derived the results for small $y$.

The final step of the RNG is to rescale lengths $L \to L/(1+\delta l)$, so that the cutoff returns to its original value. Thus lengths such as the correlation length evolve as

$$\frac{dL}{dl} = -L \qquad \text{i.e.} \qquad L \propto e^{-l}. \tag{49}$$

The line $y = 0$ is a *fixed line*. For $K^{-1} < \pi/2$ the line is *stable*, for $K^{-1} > \pi/2$ the line is unstable. The superfluid state corresponds to initial values of $K$ and $y$ such that $y \to 0$ and $K^{-1} \to K_\infty^{-1} < \pi/2$ as $l \to \infty$. The value $K_\infty$ is the large distance reduced stiffness constant $\bar{\rho}_s/k_B T$ that would be measured in experiment. As we increase temperature, the initial $y$ increases, and $K$ decreases, until we reach a temperature $T_{KT}$ which gives initial values of $y$, $K$ such that the RNG flow terminates at $y = 0$, $K_\infty^{-1} = \pi/2$. For a slightly larger temperature the RNG flows pass near this point, but then flow away to large $y$ and $K^{-1}$. Presumably this corresponds to the disordered state, although we cannot follow the behavior to large $y$. For $T = T_{KT}$ the physical (large length scale) superfluid density is $\bar{\rho}_s = (2/\pi)k_B T_{KT}$; for slightly larger temperatures $\bar{\rho}_s = 0$. Thus the phase transition is signalled by a *discontinuous jump in the superfluid density*. The ratio of the jump in $\bar{\rho}_s$ at the transition to $k_B$ times the transition temperature is *universal*. Note that $\bar{\rho}_s$ is a stiffness constant, not a thermodynamic variable, so this is not a first order transition. In fact the entropy is continuous, and the specific heat show only a very weak singularity at $T_{KT}$. This is because the energy in the vortices is small—most resides in the elementary excitations or quadratic modes. However the vortices are vital to the properties such as the superfluid density.Near $y = 0$, $K^{-1} = \pi/2$ the flow trajectories are hyperbolae, and the scaling behavior of other quantities such as the correlation length near the transition can be derived from this. For example it is found that the correlation length diverges approaching $T_{KT}$ from above as $\xi \propto \exp(c/\sqrt{T - T_c})$, rather than the usual power law divergence.

## Further Reading

The original paper is *Ordering, Metastability, and Phase Transitions in 2-Dimensional Systmes*, by J. M. Kosterlitz and D. J. Thouless, J. Phys. **C1181**, (1973), available here. Kosterlitz introduced the RNG treatment in *Critical Properties Of 2-Dimensional XY-Model* J. Phys. **C7**, 1046 (1974) or here, but the standard reference on this is *Renormalization, Vortices, and Symmetry-Breaking Perturbations in 2-Dimensional Planar Model*, by J. V. Jose, L. P. Kadanoff, S. Kirkpatrick, and D. R. Nelson, Phys. Rev. **B16**, 1217 (1977) or online.

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