Physics 127c: Statistical Mechanics

Superconductivity: Ginzburg-Landau Theory

Some of the key ideas for the Landau mean field description of phase transitions were developed in the context of superconductivity. It turns out that for conventional (low-T_c) superconductors, mean field theory is an accurate description because fluctuations are tiny except *very* close to the transition temperature. This is not the case for high T_c superconductors.

Free Energy Expansion

For a complex order parameter Ψ the Landau expansion of the free energy for small $|\Psi|$ would be

$$F = \int \left[\alpha(T) |\Psi|^2 + \frac{1}{2}\beta(T) |\Psi|^4 + \gamma(T) |\nabla\Psi|^2 \right] d^3x \tag{1}$$

For a charged superfluid we must add the coupling to the vector potential and also the magnetic energy, so that the full expression for a pair-superconductor is

$$F = \int \left[\alpha(T) |\Psi|^2 + \frac{1}{2}\beta(T) |\Psi|^4 + \gamma(T) \left| \left(\nabla + \frac{2ie}{\hbar c}\mathbf{A} \right) \Psi \right|^2 + \frac{B^2}{8\pi} \right] d^3x. \tag{2}$$

Near the transition temperature T_c we can write $\alpha(T) \to a(T - T_c)$, $\beta \to b$, and take a, b, γ to be independent of T. The free energy F must be minimized with respect to variations of Ψ and $\mathbf{A}$.

Minimizing with respect to $\mathbf{A}$ gives

$$\frac{\delta F}{\delta \mathbf{A}} = 0 = -\frac{2e\gamma}{\hbar c} i \left[\Psi^* \left(\nabla + \frac{2ie}{\hbar c}\mathbf{A} \right) \Psi - \Psi \left(\nabla - \frac{2ie}{\hbar c}\mathbf{A} \right) \Psi^* \right] + \frac{1}{4\pi} \nabla \times (\nabla \times \mathbf{A}) \tag{3}$$

or

$$\nabla \times \mathbf{B} = (4\pi/c)\mathbf{j} \tag{4}$$

with

$$\mathbf{j} = -\frac{4e}{\hbar}\gamma |\Psi|^2 \left(\nabla\phi + \frac{2e}{\hbar c}\mathbf{A} \right). \tag{5}$$

These expressions give the London penetration depth

$$\lambda^{-2} = 32\pi \left(\frac{e^2}{\hbar^2 c^2} \right) \gamma |\Psi|^2. \tag{6}$$

By comparison with the last lecture, we see

$$\gamma |\Psi|^2 = n_s \frac{\hbar^2}{8m}. \tag{7}$$

Again only this product of parameters has real significance, but often the choice is made

$$|\Psi|^2 = \frac{1}{2} n_s, \qquad \gamma = \frac{\hbar^2}{4m}, \tag{8}$$

and then

$$\mathbf{j} = -\frac{e\hbar}{m} |\Psi|^2 \left(\nabla\phi + \frac{2e}{\hbar c}\mathbf{A} \right). \tag{9}$$

These results show that near T_c the superfluid density varies as $n_s \propto (1 - T/T_c)$.

Minimizing with respect to Ψ (or actually Ψ^*) gives

$$\frac{\delta F}{\delta \Psi^*} = 0 = \alpha(T)\Psi + \beta(T)|\Psi|^2\Psi - \gamma(T)\left(\nabla + \frac{2ie}{\hbar c}\mathbf{A}\right)^2\Psi. \tag{10}$$

Using the convention (8) this can be written in a form that makes an analogy to the Schrodinger equation for particles of mass $2m$ apparent

$$\frac{1}{4m}\left(-i\hbar\nabla + \frac{2e}{c}\mathbf{A}\right)^2\Psi + \alpha(T)\Psi + \beta(T)|\Psi|^2\Psi = 0, \tag{11}$$

although this is a convenience for solving the equation, rather than anything deep, and purists don't like it because n_s is a stiffness constant and it is more natural to normalize the order parameter in terms of the strength of long range correlations, such as we did in the last lecture.

Near the transition temperature Eq. (10) becomes

$$-a(T_c - T)\Psi + b|\Psi|^2\Psi - \gamma\left(\nabla + \frac{2ie}{\hbar c}\mathbf{A}\right)^2\Psi = 0 \tag{12}$$

Correlation Length

The solution to Eq. (12) for the uniform state is

$$|\Psi|^2 = a(T_c - T)/b \tag{13}$$

giving the usual square root growth of the order parameter for $T < T_c$ found in mean field theories. The corresponding free energy density for $T < T_c$ is

$$f = \frac{F}{V} = -\frac{a^2(T_c - T)^2}{2b}, \tag{14}$$

giving the jump in the specific heat at T_c by two differentiations with respect to temperature. From our calculation of the zero temperature energy we would guess

$$F(T \simeq T_c) \sim -N\frac{(k_B T_c)^2}{\varepsilon_F}\left(1 - \frac{T}{T_c}\right)^2 \tag{15}$$

so that

$$\frac{a^2}{b} \sim \frac{n k_B^2}{\varepsilon_F}. \tag{16}$$

For spatial variations of the order parameter Eq. (12) yields the length scale ξ given by

$$\xi^2 = \frac{\gamma}{\alpha} = \frac{\gamma}{a}(T_c - T)^{-1}, \tag{17}$$

or $\xi = \xi_0(1 - T/T_c)^{-1/2}$ with the temperature independent length scale $\xi_0 = \gamma/aT_c$. On the other hand from Eq. (7) we estimate (supposing $n_s(T \to 0) \sim n$)

$$\gamma\frac{a}{b}T_c \sim n\frac{\hbar^2}{2m}, \tag{18}$$

so that

$$\xi_0^2 \sim \frac{\hbar^2}{2m\varepsilon_F}\left(\frac{\varepsilon_F}{k_B T_c}\right)^2. \tag{19}$$

The correlation length is a therefore a factor of $\varepsilon_F/k_B T_c$ larger than $\left(\hbar^2/2m\varepsilon_F\right)^{1/2}$ which is of order the interparticle spacing. This can be traced to the fact that the stiffness constant is determined by the total density of electrons, whereas the energy coefficients a, b are given by the fraction of particles corresponding to the energy band of width about $k_B T_c$ around the Fermi surface that is affected by the pairing.

The Ginzburg criterion for the temperature T_G near T_c when fluctuations become important can be estimated as $f\xi^3 \sim k_B T_c$. With the above results this is estimated as $1 - T_G/T_c = t_G$ given by

$$\frac{n\,(k_B T_c)^2}{\varepsilon_F} t_G^{1/2}\left[\frac{\hbar^2}{2m\varepsilon_F}\left(\frac{\varepsilon_F}{k_B T_c}\right)^2\right]^{3/2} \sim k_B T_c \tag{20}$$

i.e. $t_G \sim (k_B T_c/\varepsilon_F)^4$. This is *very* small for conventional (not high-T_c) superconductors, so that fluctuation corrections and the critical region near T_c are usually immeasurable.

Behavior in a Magnetic Field

Dimensionless Equations

In a constant imposed magnetic field $\mathbf{H}$ (i.e. the field due to external current sources) the appropriate free energy to minimize is

$$G_H = F - \mathbf{B}(\mathbf{x}) \cdot \mathbf{H}/4\pi. \tag{21}$$

In the normal state $\mathbf{B}(\mathbf{x}) = \mathbf{H}$, whereas in the bulk superconducting state $\mathbf{B} = 0$. Thus the simplest idea would be that superconductivity is killed at a *thermodynamic critical field* H_c given by

$$\frac{H_c^2}{8\pi} = \frac{\alpha^2}{2\beta}. \tag{22}$$

For fields larger than this the system becomes normal. This would be disappointing, since for typical superconductors this critical field would be only a few hundred gauss even at low temperatures. Luckily, the behavior in a field is more complicated than this, and in some materials superconductivity persists up to much higher fields, a vital result for the technological applications that are common today. Abrikosov was awarded the Nobel Prize in Physics in 2003 for his understanding of this physics.

To proceed it is useful to simplify the Ginzburg-Landau equation by introducing dimensionless units: Measure lengths in units of the London penetration depth λ, magnetic fields in units of $\sqrt{2}H_c$, the order parameter in units of $|\alpha|/\beta$, and the energy density in units of $H_c^2/4\pi$, i.e. define

$$\mathbf{x}' = \mathbf{x}/\lambda, \tag{23}$$

$$\mathbf{A}' = \mathbf{A}/\sqrt{2}H_c\lambda, \tag{24}$$

$$\mathbf{B}' = \mathbf{B}/\sqrt{2}H_c, \tag{25}$$

$$f' = 4\pi f/H_c^2, \tag{26}$$

$$\Psi' = \Psi/(|\alpha|/\beta), \tag{27}$$

etc. Then (dropping the primes) the free energy density is

$$f = -|\Psi|^2 + \frac{1}{2}|\Psi|^4 + \left|\left(-i\kappa^{-1}\nabla + \mathbf{A}\right)\Psi\right|^2 + B^2 \tag{28}$$

with

$$\kappa = \frac{\lambda}{\xi} = \frac{1}{4}\left(\frac{\hbar c}{e\gamma}\right)\left(\frac{\beta}{2\pi}\right)^{1/2}. \tag{29}$$

Although λ and ξ both depend strongly on temperature, this mainly cancels out in the ratio, and κ is roughly temperature independent. It is the key parameter in determining the nature of the behavior in a magnetic field. Since λ and ξ derive from quite different physics, κ varies from small to large values in different materials.

In the dimensionless units the Ginzburg-Landau equation is

$$\left(-i\kappa^{-1}\nabla + \mathbf{A}\right)^2 \Psi - \Psi + |\Psi|^2 \Psi = 0, \tag{30}$$

and the current current equation is

$$\nabla \times \mathbf{B} = \frac{1}{2}\left[\Psi^* \left(-i\kappa^{-1}\nabla + \mathbf{A}\right)\Psi - \Psi\left(i\kappa^{-1}\nabla + \mathbf{A}\right)\Psi^*\right]. \tag{31}$$

The flux quantization condition in the scaled units is given by identifying the current from Eq. (31)

$$\mathbf{j} \propto |\Psi|^2 \left(\kappa^{-1}\nabla\phi + \mathbf{A}\right) \tag{32}$$

so that in the bulk of a superconductor where $\mathbf{j} = 0$

$$\int \mathbf{B}\cdot\mathbf{dS} = \oint \mathbf{A}\cdot\mathbf{dl} = \text{integer} \times 2\pi\kappa^{-1}. \tag{33}$$

Thus the flux quantum is $2\pi\kappa^{-1}$ in the scaled units.

Surface Energy

If we set the external field to the thermodynamic critical field H_c the normal and superfluid states have the same free energy and can be in contact, and the question of the surface energy between them can be addressed. We choose the direction normal to the surface to be $\mathbf{z}$ and $\mathbf{A} = A(z)\hat{\mathbf{x}}$, $\mathbf{B} = B(z)\hat{\mathbf{y}}$ the governing equations are

$$-\kappa^{-2}\frac{d^2\Psi}{dz^2} + \Psi - |\Psi|^2\Psi + A^2\Psi = 0, \tag{34a}$$

$$\frac{d^2 A}{dz^2} - |\Psi|^2 A = 0. \tag{34b}$$

where it is consistent to assume only z dependence and no phase variation. The surface free energy is the difference of G_H from the value with all superconductor or normal state. In the scaled units and at $H = H_c$ this is

$$\Sigma = \int_{-\infty}^{\infty} dz \left[-|\Psi|^2 + \frac{1}{2}|\Psi|^4 + \left|\left(-i\kappa^{-1}\frac{d}{dz} + \mathbf{A}\right)\Psi\right|^2 + \left(B - \frac{1}{\sqrt{2}}\right)^2\right]. \tag{35}$$

If Eq. (34a) is multiplied by Ψ^*, integrated over z, and then we integrate by parts, Σ can be simplified to

$$\Sigma = \int_{-\infty}^{\infty} dz \left[-\frac{1}{2}|\Psi|^4 + \left(B - \frac{1}{\sqrt{2}}\right)^2\right]. \tag{36}$$

Note that the integrand is zero in the superconductor ($\Psi = 1$, $B = 0$) and the normal state ($\Psi = 0$, $B = 1/\sqrt{2}$). Thus Σ gets contributions just from the interface region, as makes sense. Equations (34a-34b) must be solved numerically, but we can deduce the main results without a full solution. For κ large, $\xi \ll \lambda$, the the field penetrates a large distance into the superconductor compared to the thickness of the interface. Thus in the interface region the integrand in Eq. (36) is negative, and the surface energy is *negative*. On the other hand for small κ, $\lambda \gg \xi$, the field cannot penetrate even the small Ψ region before the superconducting state is established, the integrand is positive in the interface region, and so the surface tension is *negative*. The dividing value of κ is given by $\Sigma = 0$, which means $|\Psi(z)|^2 = \sqrt{2}(B(z) - 1/\sqrt{2})$. It can be shown that this satisfies Eq. (34a) for $\kappa = 1/\sqrt{2}$. This divides superconductors into two classes:

Type I: $\kappa < 1/\sqrt{2}$: positive surface tension

Type II: $\kappa > 1/\sqrt{2}$: negative surface tension

For type I superconductors the thermodynamic critical field H_c does indeed give the boundary between superconducting and normal states. For type II superconductors however the negative surface energy favors a *mixed state* of regions of normal and superconductor intermixed on length scales of order ξ or λ for fields $H_{c1} < H < H_c$ where the negative surface energy favors the invasion of normal regions with magnetic field into the superconductor, and for $H_c < H < H_{c2}$ where the negative surface energy further stabilizes the superconducting state.

Lower Critical Field

The lower critical field H_{c1} for a type II superconductor occurs when the energy for a single quantized flux line or vortex becomes negative. There is a positive contribution to the energy density from the phase gradient $\nabla\phi \sim r^{-1}$ at a distance r from the vortex core. From Eq. (28) this gives an energy density per unit length of line of order $\kappa^{-2}r^{-2}$ in the scaled units (taking $|\Psi| \sim 1$), over distances from ξ to λ the London penetration depth. The integral gives the energy cost

$$\Delta G_{+} \sim \kappa^{-2}\ln(\kappa). \tag{37}$$

In addition there is a negative contribution from the single quantum of flux in the magnetic field H_{c1}. This decrease in magnetic energy (flux $\times$ external field) per length of line is of order

$$\Delta G_{-} \sim \kappa^{-1}H_{c1}. \tag{38}$$

This the lower critical field is of order

$$H_{c1} \sim \kappa^{-1}\ln\kappa. \tag{39}$$

The calculation can be done essentially exactly for large κ when the local London equation can be used, and the result is actually $H_{c1} = (2\kappa^{-1})(\ln C\kappa)$ with C a number of order unity. Near H_{c1} the density of vortices d^{-2} is determined by balancing the repulsive energy of the interacting vortices against the magnetic energy gained. Since the interaction is exponential in the separation $\sim e^{-d}$, whereas the magnetic energy gained per vortex is proportional to $(H - H_{c1})$ there is a rapid increase in the density of vortices $\sim [\ln(H - H_{c1})]^{-2}$. The average magnetic field B scales in the same way. We would expect the repulsive interaction to lead to a lattice structure, perhaps a close packed triangular lattice.

Intermediate Fields

As the external field is increased, the density of vortices increases, and the average magnetic field over the superconductor grows, initially rapidly since the flux lines interact weakly. When the separation d becomes comparable with the penetration depth λ the supercurrent and field regions begin to extend over the whole superconductor, the flux lines interact more strongly, and the growth of B with H is slower. When $d \sim \xi$ the normal cores of the flux lines overlap, and the state becomes normal. In scaled units this is $d \sim \kappa^{-1}$ and the magnetic field is one flux quantum per d^2 or of order $\kappa^{-1}/\kappa^{-2} \sim \kappa$. In the normal state $B = H$ so this gives us the estimate of the upper critical field $H_{c2} = \kappa$ (i.e. $\sqrt{2}\kappa H_c$ in unscaled units). We will see this is actually the exact result.

Upper Critical Field

At the upper critical field the vortex cores overlap suppressing the order parameter close to zero. This means we can linearize Eq. (30) in Ψ to give

$$\left(-i\kappa^{-1}\nabla + \mathbf{A}\right)^2 \Psi - \Psi = 0 \tag{40}$$

neglect the feedback of the supercurrents on the magnetic field so that for an applied field $H\hat{\mathbf{z}}$ we can take $\mathbf{A} = Hx\hat{\mathbf{y}}$ and then

$$\left[-\kappa^{-2}\nabla^2 - 2i\kappa^{-1}Hx\frac{\partial}{\partial y} + H^2x^2\right]\Psi = \Psi. \tag{41}$$

This is the same as Schrodinger's equation for particles in a constant magnetic field, giving Landau levels. Note that although the physical problem is symmetric in $x \to y$, by choice of gauge we have formulated the problem in a way that does not respect this symmetry. This is also what is usually done for the Landau level problem. Assuming a z-independent solution, we can write

$$\Psi = e^{ik_y y}u(x), \tag{42}$$

with $u(x)$ satisfying

$$-u'' + (H\kappa x - k_y)^2 u = \kappa^2 u. \tag{43}$$

This is the same as Schrodinger's equation for a harmonic oscillator about $x_0 = k_y/\kappa H$ and there are bounded solutions for

$$\kappa = (2n+1)H. \tag{44}$$

The largest critical field corresponds to $n = 1$ giving

$$H_{c2} = \kappa. \tag{45}$$

The corresponding eigenfunction is

$$\Psi = e^{ik_y y}e^{-\kappa^2(x-\kappa^{-2}k_y)^2/2}. \tag{46}$$

If we suppose a periodic solution in the y direction with period L_y then the physical solution can be any linear combination of the solutions with $k_y = n \times 2\pi/L_y$

$$\Psi = \sum_n C_n e^{i(2n\pi/L_y)y}e^{-\kappa^2\left[x-\kappa^{-2}(2n\pi/L_y)\right]^2/2}. \tag{47}$$

First consider the case $C_n = C$. Then increasing x by $\kappa^{-2}(2\pi/L_y)$ simply corresponds to a phase change of Ψ by $2\pi y/L_y$ so that $|\Psi|^2$ is periodic in x with period L_x satisfying

$$L_x L_y = 2\pi\kappa^{-2} \tag{48}$$

The individual values of L_x and L_y (and the competition with other solutions with different choices of C_n) is not given by this linear calculation. Keeping the lowest order nonlinear terms in $|\Psi|^2$, in both Eqs. (30) and (31) is quite involved, and is the subject of Abrikosov's original paper. Not surprisingly, for the case of $C_n = C$ the free energy is minimized for $L_x = L_y$ which actually gives a structure with square symmetry (not immediately obvious, since our the representation of the x and y dependence is quite different). The structure consists of a square lattice of points where $|\Psi|$ goes to zero, about which the phase winds by 2π—the structure of a rudimentary lattice of vortices. Actually it turns out that the lowest free energy solution is for a two parameter solution $C_{2n} = C_0$, $C_{2n+1} = C_1$ with $C_1 = iC_0$ and a ratio of L_x/L_y which turns out to correspond to a *triangular lattice*. (In Abrikosov's first paper, he made a numerical error in the evaluation of the energy of this state, and erroneously suggested that the square lattice was the stable one.) Since the same structure is expected in the dilute flux line limit near H_{c1} this flux lattice structure is expected over the whole range of fields $H_{c1} < H < H_{c2}$. The lattice structure is dramatically confirmed by experiments where the points of large field on the surface where flux lines emerge are decorated with small magnetic particles.

The flux-lattice structure in type II superconductors is enormously important in technological applications, allowing fields enhanced by the factor κ to be sustained. Unlike the case of superfluids, where vortex lines are immediately mobile and dissipative, flux lines are typically pinned to impurities in the lattice (κ tends to be large in "dirty superconductors"). In high-T_c superconductors the thermal fluctuations of the flux lines becomes important, and indeed the superconducting transition in a magnetic field must be thought of as the melting of the flux line lattice, since in a disordered flux-line state there is no long range phase order.

Further Reading

Abrikosov's original paper is Soviet Physics JETP **5**, 1174 (1957). I've made a copy that you can find here. His Nobel Prize lecture at

http://www.nobel.se/physics/laureates/2003/abrikosov-lecture.html

is well worth listening to or reading. Probably the best review of type II superconductors is the theory article by *Fetter and Hohenberg*, in *Superconductivity*, vol 2, edited by Parks (Caltech Library QC612.S8 P28). The following article by *Serin* discusses the experimental situation in 1969. A decoration experiment visualizing the flux lattice in a high-T_c superconductor is *Gammel et al*, Phys. Rev. Lett. **59,** 2592 (1987), available online