Physics 127c: Statistical Mechanics

Fermi Liquid Theory: Collective Modes

Boltzmann Equation

The quasiparticle energy including interactions

$$\tilde{\varepsilon}_{\mathbf{p},\sigma} = \varepsilon_p + \frac{1}{V}\sum_{\mathbf{p}',\sigma'} f(\mathbf{p},\mathbf{p}';\sigma,\sigma')\delta n_{\mathbf{p}',\sigma'}, \tag{1}$$

with $\varepsilon_p \simeq \varepsilon_F + v_F(p - p_F)$, acts as an effective Hamiltonian for the quasiparticles. Since $\delta n_{\mathbf{p},\sigma}$ can depend on space and time, through the spatial gradient of $\tilde{\varepsilon}_{\mathbf{p},\sigma}$ this gives a force acting on the quasiparticles, and can lead to oscillations or collective modes. Pictorially, we can think of the modes as "oscillations of the Fermi sea", although since the Fermi sea is a momentum space construction, and perturbations lead to velocities, the modes must also involve spatial derivatives.

For wavelengths that are long compared to the interparticle spacing, corresponding to mode wave vectors $\mathbf{q}$ with $q \ll k_F$, and frequencies small compared with the Fermi energy $\omega \ll \varepsilon_F$, the modes can be calculated using a kinetic theory (Boltzmann equation) for the quasiparticles. This is a semiclassical approximation in which the phase space point $\mathbf{r},\mathbf{p}$ of a quasiparticle evolves driven by the effective Hamiltonian $\tilde{\varepsilon}_{\mathbf{p},\sigma}$. For $q \ll k_F, \omega \ll \varepsilon_F$ the uncertainty principle restrictions on the accuracy of a prescription of both of $\mathbf{r},\mathbf{p}$ is not important.

To set up the equation suppose there is an constant equilibrium distribution $n^{(0)}_{p,\sigma}$ and a perturbation $\delta n_{\mathbf{p},\sigma}(\mathbf{r},t)$

$$n_{\mathbf{p},\sigma}(\mathbf{r},t) = n^{(0)}_{p,\sigma} + \delta n_{\mathbf{p},\sigma}(\mathbf{r},t). \tag{2}$$

We will study zero temperature so that $n^{(0)}_{p,\sigma}$ is the Fermi sea. The Boltzmann equation is

$$\frac{\partial n_{\mathbf{p},\sigma}}{\partial t} + \frac{\partial \tilde{\varepsilon}_{\mathbf{p},\sigma}}{\partial \mathbf{p}}\cdot\frac{\partial n_{\mathbf{p},\sigma}}{\partial \mathbf{r}} - \frac{\partial \tilde{\varepsilon}_{\mathbf{p},\sigma}}{\partial \mathbf{r}}\cdot\frac{\partial n_{\mathbf{p},\sigma}}{\partial \mathbf{p}} = I(\{\delta n_{\mathbf{p},\sigma}\}), \tag{3}$$

where I is the collision term coming from the scattering of quasiparticles.

The Boltzmann equation (3) simplifies considerably if we only keep terms linear in $\delta n_{\mathbf{p},\sigma}$. The spatial dependence necessarily involves $\delta n_{\mathbf{p},\sigma}$, and so the multiplying terms in (3) can be replaced by their zeroth order values, to give

$$\frac{\partial \delta n_{\mathbf{p},\sigma}}{\partial t} + \mathbf{v}_{\mathbf{p}}\cdot\frac{\partial \delta n_{\mathbf{p},\sigma}}{\partial \mathbf{r}} - \frac{\partial n^{(0)}_{\mathbf{p},\sigma}}{\partial \mathbf{p}}\cdot\frac{1}{V}\sum_{\mathbf{p}',\sigma'} f(\mathbf{p},\sigma;\mathbf{p}',\sigma')\frac{\partial \delta n_{\mathbf{p}',\sigma'}}{\partial \mathbf{r}} = I(\{\delta n_{\mathbf{p},\sigma}\}), \tag{4}$$

with $\mathbf{v}_{\mathbf{p}} = \partial\varepsilon_p/\partial\mathbf{p} \simeq v_F\hat{\mathbf{p}}$. At zero temperature

$$\frac{\partial \delta n^{(0)}_{\mathbf{p},\sigma}}{\partial \mathbf{p}} \simeq \mathbf{v}_{\mathbf{p}}\frac{\partial n^{(0)}_{\mathbf{p},\sigma}}{\partial \varepsilon_p} = -\delta(\varepsilon_p - \varepsilon_F)\mathbf{v}_{\mathbf{p}}. \tag{5}$$

Thus we finally get

$$\frac{\partial \delta n_{\mathbf{p},\sigma}}{\partial t} + \mathbf{v}_{\mathbf{p}}\cdot\frac{\partial \delta n_{\mathbf{p},\sigma}}{\partial \mathbf{r}} + \delta(\varepsilon_p - \varepsilon_F)\mathbf{v}_{\mathbf{p}}\cdot\frac{1}{V}\sum_{\mathbf{p}',\sigma'} f(\mathbf{p},\sigma;\mathbf{p}',\sigma')\frac{\partial \delta n_{\mathbf{p}',\sigma'}}{\partial \mathbf{r}} = I(\{\delta n_{\mathbf{p},\sigma}\}). \tag{6}$$

Note that although we postulated an "unperturbed distribution" of quasiparticles $n^{(0)}_{\mathbf{p},\sigma}$ as if it were well defined for all p, only its properties for $p \simeq p_F$ where the concept makes sense were involved in the derivation, and an explicit $\delta(\varepsilon_p - \varepsilon_F)$ occurs in the equation for $\delta n_{\mathbf{p},\sigma}$.

Hydrodynamic Equations

As in the Boltzmann equation for the classical gas, the collision term conserves the total particle number, momentum, and energy. Taking appropriate moments of the Boltzmann equation we could write down conservation equations for the mass density, momentum density, and energy density. These conservation equations would involve the divergence of correspond currents or fluxes, with expressions for these quantities given in terms of moments of $\delta n_{\mathbf{p},\sigma}$. In equilibrium these currents are zero. In the low frequency limit, the system is close to equilibrium, and we can approximately solve for $\delta n_{\mathbf{p},\sigma}$ by balancing the collision term with the driving terms coming from the gradients of $n^{(0)}_{p,\sigma}(T,\mu,\mathbf{v})$ where T, μ and the velocity $\mathbf{v}$ are space and time dependent. This gives $\delta n_{\mathbf{p},\sigma}$ proportional to $\omega\tau$ or $qv_F\tau$ with τ the collision time characterizing the collision integral. The currents can be characterized by kinetic coefficients such as the thermal conductivity, viscosity etc. This is the *hydrodynamic limit* valid for $\omega\tau, qv_F\tau \ll 1$, when the system is everywhere close to a (local) thermodynamic equilibrium. These calculations are very similar to the ones for the classical gas, discussed in ??, except that the collision integral must take into account the Fermi properties of the quasiparticles and the exclusion principle etc.

Collisionless Limit

A more novel limit is the collisionless limit $\omega\tau \gg 1$, when the dynamics is dominated by the evolution of the state in phase space, and the collisions redistributing the quasiparticles amongst the different states can be ignored. This is where thinking of the modes as oscillations of the Fermi sea becomes useful. Indeed, if we imagine a displacement $u_{\hat{\mathbf{p}},\sigma}$ of the Fermi surface of the spin σ component at direction $\hat{\mathbf{p}}$ we can write

$$\delta n_{\mathbf{p},\sigma} = \delta(\varepsilon_p - \varepsilon_F) v_F u_{\hat{\mathbf{p}},\sigma} \tag{7}$$

The interaction term in the Boltzmann equation (6) can be evaluated as

$$\frac{1}{V}\sum_{\mathbf{p}',\sigma'} f(\mathbf{p},\sigma;\mathbf{p}',\sigma')\frac{\partial \delta n_{\mathbf{p}',\sigma'}}{\partial \mathbf{r}} = N(0)\int d\varepsilon_p \delta(\varepsilon_p - \varepsilon_F)\int \frac{d\Omega'}{4\pi}\sum_{l,\sigma'}\frac{F_{l,\sigma\sigma'}}{2N(0)}P_l(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}')v_F\frac{\partial u_{\hat{\mathbf{p}},\sigma}}{\partial \mathbf{r}} \tag{8}$$

$$= v_F\int \frac{d\Omega'}{8\pi}\sum_{l,\sigma'} F_{l,\sigma\sigma'}P_l(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}')\frac{\partial u_{\hat{\mathbf{p}}',\sigma'}}{\partial \mathbf{r}}, \tag{9}$$

for the moment expanding the interaction parameter in Legendre polynomials but not using the spin symmetric and antisymmetric notation $F_{l,\sigma\sigma'} = F^{(s)}_l + \sigma\sigma' F^{(a)}_l$. Supposing a single mode disturbance so that $u_{\hat{\mathbf{p}},\sigma} \propto e^{i(\mathbf{q}\cdot\mathbf{r}-\omega t)}$ The Boltzmann equation becomes

$$(v_F\hat{\mathbf{p}}\cdot\mathbf{q} - \omega)u_{\hat{\mathbf{p}},\sigma} + v_F\hat{\mathbf{p}}\cdot\mathbf{q}\int\frac{d\Omega'}{8\pi}\sum_{l,\sigma'}F_{l,\sigma\sigma'}P_l(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}')u_{\hat{\mathbf{p}}',\sigma'} = 0. \tag{10}$$

Introducing spin symmetric and antisymmetric displacements

$$u_{\hat{\mathbf{p}},\uparrow} = u^{(s)}_{\hat{\mathbf{p}}} + u^{(a)}_{\hat{\mathbf{p}}} \tag{11}$$

$$u_{\hat{\mathbf{p}},\downarrow} = u^{(s)}_{\hat{\mathbf{p}}} - u^{(a)}_{\hat{\mathbf{p}}} \tag{12}$$

the two components decouple. Then dividing through by qv_F and writing λ for the dimensionless speed of the collective mode ω/qv_F and θ for the angle between $\hat{\mathbf{p}}$ and $\mathbf{q}$, gives (for either s or a)

$$(\cos\theta - \lambda)u^{(s,a)}_{\hat{\mathbf{p}}} + \frac{\cos\theta}{8\pi}\int d\Omega' \sum_l F^{(s,a)}_l P_l(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}')u^{(s,a)}_{\hat{\mathbf{p}}'} = 0. \tag{13}$$

Equation (13) is the dynamical equation for spin symmetric and antisymmetric modes. The paremter $\lambda = \omega/qv_F$ is determined as the eigenvalues of the equation. We may expand $u_{\hat{\mathbf{p}}}^{(s,a)}$ on spherical harmonics $Y_{l,m}(\theta,\phi)$. The different m modes decouple. The equations couple different l however, leading in general to complicated mode equations that can only be solved numerically. We can gain intuition about the modes by making simplifying assumptions for the interaction parameters $F_l^{(s,a)}$.

Zero Sound

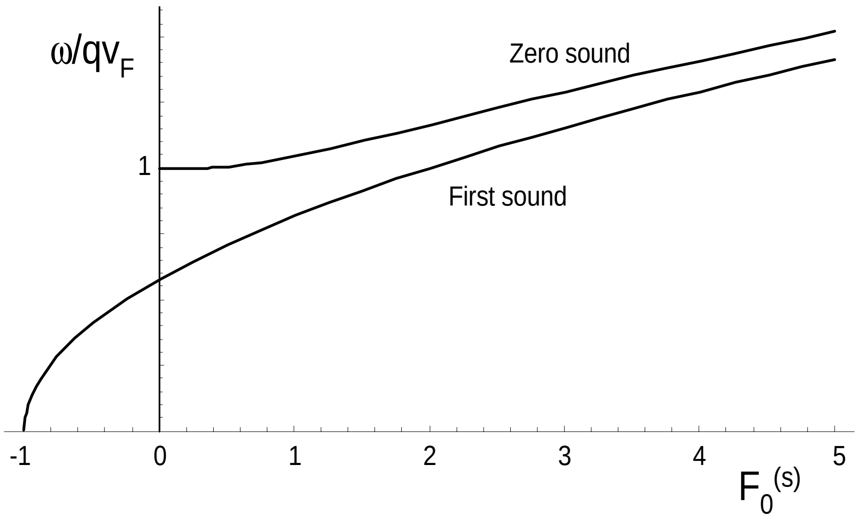

Figure 1: Speed of zero sound and first sound as a function of $F_0^{(s)}$ in the model where only this Fermi liquid parameter is nonzero.

Zero sound is a spin symmetric mode that involves $l = m = 0$ distortions, amongst others, and so couples to the total density. A simple discussion of zero sound is given by assuming only $F_0^{(s)}$ is nonzero. Then in Eq. (13) for the distortion of the Fermi sea $\sum_l F_l^{(s)} P_l(\hat{\mathbf{p}} \cdot \hat{\mathbf{p}}')$ reduces to $F_0^{(s)}$ and the equation becomes

$$(\cos\theta - \lambda)u_{\hat{\mathbf{p}}}^{(s)} + F_0^{(s)} \frac{\cos\theta}{2} \int \frac{d\Omega'}{4\pi} u_{\hat{\mathbf{p}}'}^{(s)} = 0. \tag{14}$$

Clearly

$$u_{\hat{\mathbf{p}}}^{(s)} = C \frac{\cos\theta}{\cos\theta - \lambda} \tag{15}$$

and then substituting this form gives

$$1 + \frac{1}{2} F_0^{(s)} \int \frac{d\Omega'}{4\pi} \frac{\cos\theta'}{\cos\theta' - \lambda} = 0. \tag{16}$$

For $\lambda > 1$ the integral is easily done

$$\int \frac{d\Omega'}{4\pi} \frac{\cos\theta'}{\cos\theta' - \lambda} = \frac{1}{2} \int_{-1}^{1} \frac{x}{x - \lambda} dx = 1 + \frac{\lambda}{2} \ln\left(\frac{\lambda - 1}{\lambda + 1}\right) \tag{17}$$

to give the implicit equation for λ

$$\frac{1}{F_0^{(s)}} = \Phi(\lambda) = \frac{\lambda}{2} \ln\left(\frac{\lambda + 1}{\lambda - 1}\right) - 1. \tag{18}$$

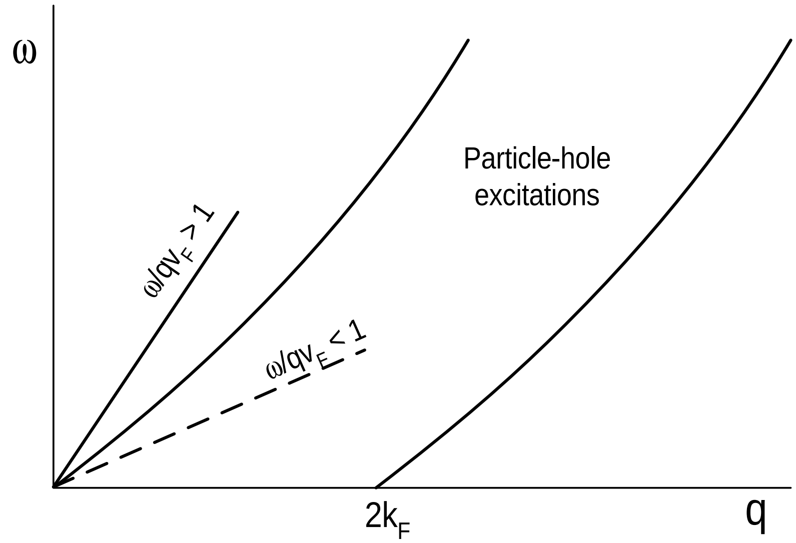

Figure 2: Particle-hole excitation spectrum. A collective mode with $\omega/qv_F < 1$ (dashed line) is strongly damped by Landau damping.

For $\lambda > 1$ the function $\Phi(\lambda)$ ranges over all positive values, and so for any positive $F_0^{(s)}$ Eq. (18) can be inverted to find $\lambda(F_0^{(s)})$. For $-1 < F_0^{(s)} < 0$ there are no real solutions to Eq. (18) for λ. Indeed returning to the integral expression Eq. (16) shows that λ must be complex. Redoing the integrals yields solutions for λ with real and imaginary parts comparable, yielding a strongly damped collective mode (decay time comparable to frequency). This damping coming not from collisions but from a resonant interaction of the collective mode with particle-hole excitations is known as *Landau damping*. What is going on can be understood by first considering the range of energies for exciting a particle and hole with total momentum $\mathbf{q}$

$$\hbar\omega_{\mathbf{q}} = \varepsilon_{\mathbf{p}+\mathbf{q}} - \varepsilon_{\mathbf{p}} \tag{19}$$

for all $\mathbf{p}$ with $|\mathbf{p}| > p_F$ and $|\mathbf{p}+\mathbf{q}| < p_F$. This is sketched in Fig. 1. The upper boundary of the region of particle-hole energies for small q is $\omega = qv_F$. Now consider the collective mode with $\omega/qv_F = \mathrm{Re}\,\lambda$. For $\mathrm{Re}\,\lambda > 1$ the mode frequency is outside of the particle-hole band. However for $0 < \mathrm{Re}\,\lambda < 1$, the collective mode is immersed in the excitation sea, can resonantly excite particle hole pairs, and becomes strongly damped.

For $F_0^{(s)} < -1$ the solution for λ corresponds to an exponentially *growing* time dependence. This signals the instability of the system.

The mode we have calculated has nonzero values of $\sum_{\mathbf{p},\sigma} \delta n_{\mathbf{p},\sigma}$ and $\sum_{\mathbf{p},\sigma} \mathbf{p}\delta n_{\mathbf{p},\sigma}$ (i.e. total number and momentum), as well as other moments, and so experimentally would be detected as *sound*. Since the mode results from the coherent interaction of the quasiparticles in the collisionless limit, rather than the near equilibrium behavior at low frequencies, it is known as *zero sound*, and the hydrodynamic sound is called *first sound*. The speed for first sound in the model where only $F_0^{(s)}$ is nonzero can be expressed as

$$\frac{\omega}{qv_F} = \sqrt{\frac{1+F_0^{(s)}}{3}}. \tag{20}$$

This sound exists as a propagating mode for all $F_0^{(s)} > -1$, whereas zero sound only propagates for $F_0^{(s)} > 0$. (Note the instability $F_0^{(s)} < -1$ also appears as a *negative* compressibility.) For our simple model of only

$F_0^{(s)}$ important, the speed of zero sound (where it exists) is always greater than the speed of first sound, with the two speeds becoming equal for $F_0^{(s)} \to \infty$.

First Sound and Zero Sound

First sound occurs in the low frequency limit $\omega\tau \ll 1$, and zero sound in the high frequency limit $\omega\tau \gg 1$. For first sound, the collisions restore the quasiparticle distribution to equilibrium, and the damping is proportional to the deviation from equilibrium. Thus the damping relative to the propagation (e.g. $\mathrm{Im}\, q / \mathrm{Re}\, q$ for an experiment with driving at some frequency ω) is proportional to $\omega\tau$. On the other hand zero sound is a collective motion of the quasiparticles, and is disrupted by collisions. The relative dissipation from collisions is proportional to $(\omega\tau)^{-1}$. If the drive frequency of the experiment is increased, the mode will cross over from first to zero sound at a frequency $\omega \sim \tau^{-1}$, and there will be a dissipation peak at this frequency. Alternatively, the experimentalist might lower the temperature (always with $k_B T \ll \varepsilon_F$). Since the collision rate τ^{-1} is proportional to T^2 by the usual phase space arguments, the product $\omega\tau$ increases as the temperature is lowered. Experiments in liquid He^3 for example [see *Abel, Anderson, and Wheatley* Phys. Rev. Lett. **17**, 74 (1966)] show a crossover from first sound at high temperatures to zero sound at low temperatures, signalled by an increase in the speed of propagation and a dissipation peak.

Further Reading

The Theory of Quantum Liquids, Vol. I by *Nozieres and Pines* §1.4 and §1.7-1.10 and *Statistical Mechanics, part 2* of by *Lifshitz and Pitaevskii* §4 discuss the dynamics of Fermi liquids.

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