Physics 127b: Statistical Mechanics

Scaling Hypothesis

The scaling hypothesis allows us to relate all the power laws for the static, bulk thermodynamic quantities and the correlation function in terms of two basic exponents. The hypothesis was first arrived at empirically by Widom, and then using the phenomenological idea that a *single, divergent correlation length* determines the behavior near the transition temperature by Kadanoff and others.

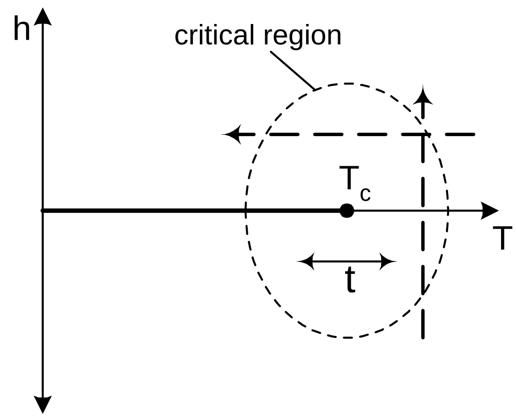

Figure 1: Critical region (encircled by short-dashed line) where the scaling hypothesis is supposed true. Behavior along the paths denoted by long-dashed arrows—which avoid the critical point—should be smooth.

The hypothesis can be introduced by an expression for the order parameter as a function of $t = T - T_c$ in the critical regime and the symmetry breaking field h. We will use the language of a magnetic phase transition for concreteness, but the results are quite general:

$$m = D\,|t|^{\beta}\,W_{\pm}\left(E\,\frac{h}{|t|^{\Delta}}\right). \tag{1}$$

Here β and Δ are *universal* exponents, and the functions W_+ for $t > 0$ and W_- for $t < 0$ are also *universal*, whereas D and E (which depend on the units chosen) are non-universal. The form Eq. (1) determines the behavior approaching the critical point $t = h = 0$, i.e. t, h both small) but is assumed valid for *arbitrary* ratios of t/h. The only singular behavior is at the critical point: elsewhere the physical behavior should be (essentially) smooth (e.g. along the dashed lines in Fig. (1). This allows us to determine $\alpha, \beta, \gamma, \delta$ in terms of β and Δ (and also can be used to show that the exponents above and below the critical temperature are the same).

First consider the variation with h for $T > T_c$, i.e. some nonzero value of $t > 0$. Since the trajectory does not pass through the critical point, the dependence on h should be analytic

$$W_+(x) = b_+ x + c_+ x^3 + \cdots. \tag{2}$$

There are no even terms by the symmetry in h. The susceptibility is

$$\chi = \partial m/\partial h|_{h=0} = |t|^{\beta-\Delta}\; D\,E\,W_+'(0), \tag{3}$$

which determines the susceptibility exponent

$$\gamma = \Delta - \beta. \tag{4}$$

For $T < T_c$ the argument is similar except there is a finite m as $h \to 0$, so

$$W_-(x) = a_-\text{sign}(x) + b_- x + c_- x^3 + \cdots . \tag{5}$$

This gives as $h \to 0$

$$m(h \to 0) = |t|^\beta \ D\, W_-(0). \tag{6}$$

showing that β is indeed the order parameter exponent. The susceptibility below T_c can still be defined in terms of the change of the magnetization away from the zero field value, and the exponent is easily seen to be the same as above T_c.

For $h \neq 0$ the behavior will be smooth as a function of t, even when t passes through zero. In particular the t dependence of m should vanish here at $t = 0$. This fixes the behavior of $W_\pm(x)$ for $x \to \infty$

$$W_\pm(x \to \infty) \sim |x|^{\beta/\Delta}\, \text{sign} x, \tag{7}$$

(we know the behavior must be odd in h) so that

$$m(h, t = 0) \sim |h|^{\beta/\Delta}\, \text{sign} h \tag{8}$$

giving

$$\delta = \Delta/\beta. \tag{9}$$

The smooth behavior as t passes through zero also relates W_+ and W_-.

We can also look at the scaling behavior of the free energy density

$$f = A\, |t|^{2-\alpha}\, Y_\pm \left(C \frac{h}{|t|^\Delta} \right). \tag{10}$$

Two remarks are in order. First, the prefactor is explicitly written so that the specific heat has the exponent α. Secondly the argument of the scaling function Y is again $h/\,|t|^\Delta$—this is not a separate assumption since we can get the magnetization from the free energy.

The specific heat for $t \to \pm 0$ at $h = 0$ is simply

$$C \propto |t|^{-\alpha} \ Y_\pm(0). \tag{11}$$

The magnetization is

$$m = \frac{\partial f}{\partial h} \sim |t|^{2-\alpha-\Delta}\, Y'_\pm \left(C \frac{h}{|t|^\Delta} \right) \tag{12}$$

so that comparing with Eq. (1) fixes

$$\alpha = 2 - \beta - \Delta \tag{13}$$

and shows the function $W_\pm$ is just $Y'_\pm$. Note that corresponding to the jump in the order parameter across $h = 0$ for $t < 0$, the function $Y_-(x)$ for small x will contain a term proportional to $x\text{sign}x$, as well as (even) analytic terms.

We now look at the correlation function, and introduce the correlation exponent ν giving the divergence of the correlation length at $h = 0$

$$\xi \sim |t|^{-\nu}. \tag{14}$$

In non zero field we would then have

$$\xi(t, h) \sim |t|^{-\nu}\, X \left(\frac{h}{|t|^\Delta} \right). \tag{15}$$

The scaling form for the correlation function is

$$G(r,t,h) = \frac{1}{r^{d-2+\eta}} g(r\,|t|^{\nu}, h/|t|^{\Delta}) \tag{16}$$

with the exponent η giving the power law decay at $t = h = 0$. Just as the mean square fluctuating temperature can be related to the specific heat, the mean square magnetization $(\left|m_{\vec{q}\to 0}\right|^2)$ gives us the susceptibility

$$\chi = \int G(r,t,h=0) d^d r \tag{17}$$

$$\sim |t|^{-(2-\eta)\nu} \int y^{2-d-\eta} g(y,0) dy, \tag{18}$$

relating a combination of η and ν to the susceptibility exponent γ

$$(2-\eta)\nu = \gamma. \tag{19}$$

So far we have related the exponents $\alpha, \beta, \gamma, \delta, \nu, \eta$ in terms of some choice of three "fundamental" exponents. All these results are satisfied by the mean field theory exponents, and are well satisfied by experimentally measured exponents too. An additional relationship known as *hyperscaling*, or strong scaling, is *not* satisfied by the mean field exponents, and also not for some rare experimental situations, such as systems with long range forces, but is otherwise generally found to be true. It results from the argument that the fluctuations dominate the (singular) contribution to the free energy density, or equivalently that the free energy per correlation volume ξ^d should be of order $k_B T_c$. This gives at $h = 0$

$$f \sim t^{2-\alpha} \sim \xi^{-d} \tag{20}$$

and so

$$d\nu = 2 - \alpha. \tag{21}$$

We don't expect this to be satisfied in mean field theory, since there the *mean* value of the magnetization gives the singular free energy, and the fluctuations are explicitly ignored. Indeed the criterion Eq. (20) gave us the Ginzburg criterion for when mean field theory breaks down, i.e. the boundary of the critical region—hyperscaling supposes this relationship is then satisfied throughout the critical region.

The scaling hypothesis is empirical/phenomenological, but sets the stage for a more complete understanding—if we can understand the scaling behavior we have a quite complete account of the critical region.