Physics 127b: Statistical Mechanics

Renormalization Group: Advanced Topics *Revised February 9, 2004*

First I discuss two extra details for the RNG procedure. The first is the notion of spin rescaling, an addition step in the RNG that was not needed in the $1d$ Ising model. The second is the idea of an infinitesimal scale factor change at each step, which often simplifies the calculation. Since it is not possible to implement the RNG exactly for most systems, I then introduce two general approximation methods often used to implement the RNG.

In general, the simple elimination procedure that was successful in the $1d$ Ising model leads to successively more complicated terms in the renormalized Hamiltonian, and the process cannot be followed analytically. Two approximation methods are commonly used.

The first approach is to formulate the problem in terms of Fourier modes. Rather than starting from a (perhaps) physical lattice model, the starting point is the Landau expansion of the free energy based on the idea of Taylor expansion and symmetry. However, this free energy is now interpreted as an "intermediate length scale" description: we imagine some steps of renormalization have been carried out, so that we are looking at scales larger than the atomic scale, but not so large that the singular effect of the fluctuations have accumulated. On these scales a Taylor expansion in coarse-grained fields might be reasonable. In this context the Landau expansion is known as the Landau-Wilson free energy. This free energy is then used as the Boltzmann factor for the probability of fluctuations, and the RNG is implemented by iteratively integrating out the highest wave number (shortest length scale) modes. Again, this will tend to introduce more and more complicated terms into the expansion, so that the procedure cannot be implemented exactly in general. A powerful procedure is to use an expansion in the dimensionality of space d, about a dimension d_c where this proliferation of terms does not occur. This is often known as the ε expansion, named after the small parameter $\varepsilon = d_c - d$. I will sketch the method in the main text, with details relegated to an appendix.

The second approach is some type of approximation along the lines of the direct elimination procedure, but truncating the renormalized Hamiltonian to take the original form after each step. This is an uncontrolled approximation, and various methods can be used to implement the procedure. These are called *real space renormalization* methods. They are often useful in getting qualitative behavior about novel phase transitions, but not quantitative values for exponents, etc. Kadanoff and Migdal were early developers of this approach.

Spin Rescaling

For the 1d Ising model the spin correlations for the remaining spins are unchanged by the RNG, only their separation is changed. Thus we have for the correlation function $G(r, \bar{H}) = \langle s(0)s(r)\rangle_{\bar{H}}$:

$$G(r/b, \bar{H}') = G(r, \bar{H}). \tag{1}$$

At the fixed point

$$G(r) = G(r/b) \tag{2}$$

which shows that G is a constant, independent of r. Remember generally we write

$$G(r, T_c) \sim r^{-d+2-\eta} \tag{3}$$

and so for the 1d Ising model $\eta = 1$.

The simple result of constant $G(r)$ at T_c is a special feature of the $1d$ Ising model. In general a power law dependence on r different from $2 - d$ arises. How this result appears in the formalism depends on the details

of the technique used in the RNG. Typically it turns out to be necessary to rescale the order parameter field, as well as the space scale, at each step of the renormalization procedure (otherwise there is no fixed point). In a lattice elimination process, such as in the $1d$ Ising model, where we simply calculate the correlation of the remaining spins, for example, it is necessary to rescale the spin at each step

$$\mathcal{R}_b(s) = s' = cs \tag{4}$$

and then

$$\eta - 2 + d = -\frac{2\ln c}{\ln b}. \tag{5}$$

In the $1d$ Ising model $c = 1$, there is no spin rescaling, and $\eta = 1$. In Fourier space methods there is a corresponding rescaling of the Fourier transform field $s_{\vec{q}}$.

We can see how η is related to the other exponents by the following argument. The correlation function can be constructed by imagining a *spatially varying* field $h(\vec{r})$ coupling to the order parameter

$$H = H_0(s) - \int h(r)s(r) \tag{6}$$

and then by the usual construction

$$G(r) = \left\langle \delta s(\vec{r})\delta s(\vec{0})\right\rangle = \left.\frac{\partial^2 \ln Q_N}{\partial h(\vec{r})\partial h(\vec{0})}\right|_{h=0}. \tag{7}$$

We denote quantities after one step of the RNG by primes. Then

$$G'(r') = \frac{\partial^2 \ln Q'_{N/2}}{\partial h'(\vec{r}')\partial h'(\vec{0})} = \frac{\partial^2 \ln Q_N}{\partial h'(\vec{r}')\partial h'(\vec{0})} \tag{8}$$

where the last equality is because the free energy is preserved by the RNG. The first expression is $G(r/b, H')$ the correlation function for Hamiltonian H' at separation $r' = r/b$. A uniform field scales as $h' = b^{\lambda_h}h$, and we suppose the same to be true for a slowly varying $h(r)$. A change h' coupling to the renormalized spin s' corresponds to changing the field h over b^d original spins. Thus the second expression is $b^{-2\lambda_h}\left\langle (s_1 + s_2 + \cdots)_{\vec{r}}(s_1 + s_2 + \cdots)_{\vec{0}}\right\rangle$ where the sums are over the b^d spins in the block averaged over in the RNG at $\vec{r}$ and $\vec{0}$. For separation $\vec{r}$ large compared to the block size this is $b^{2d-2\lambda_h}\left\langle s(\vec{r})s(0)\right\rangle$ .Thus

$$G(r/b, H') = b^{2(d-\lambda_h)}G(r, H). \tag{9}$$

At T_c where the behavior is determined by the fixed point Hamiltonian, and then $H' = H = H^*$ this gives

$$G(r/b) = b^{2(d-\lambda_h)}G(r). \tag{10}$$

This is consistent with the power law decay defined by η as $G \sim r^{-(d-2+\eta)}$ if

$$\eta = 2 + d - 2\lambda_h. \tag{11}$$

Thus η is determined by the field eigenvalue λ_h.

Equation (11) is consistent with our previous results. The comparison with the scaling result

$$\eta = 2 - \frac{\gamma}{\nu} \tag{12}$$

is easiest to see by writing the RNG form for the free energy as

$$f \sim |t|^{d/\lambda_t}\, W(h/\,|t|^{\lambda_h/\lambda_0}). \tag{13}$$

The correlation length exponent is $\nu = 1/\lambda_t$. The susceptibility $\chi \sim |t|^{-\gamma}$ is the second derivative of f with respect to the field giving

$$\gamma = \frac{2\lambda_h}{\lambda_t} - \frac{d}{\lambda_t}. \tag{14}$$

Thus the two expressions for η agree. For the $1d$ Ising model $\lambda_h = 1, d = 1$ giving via Eq. (11) $\eta = 1$, the result corresponding to no spin rescaling we found before.

Continuous RNG

Often it is convenient to perform a continuous rescaling instead of using discrete factors such as 2. The *continuous renormalization group* is defined as

$$\mathcal{G} = \lim_{b\to 1} \frac{\mathcal{R}_b - 1}{b-1}. \tag{15}$$

Writing $b = e^l$, the evolution of the Hamiltonian with scale parameter l is the differential equation

$$\frac{d\bar{H}}{dl} = \mathcal{G}[\bar{H}], \tag{16}$$

giving a continuous flow in Hamiltonian space.

Dimensionality Expansion

One of the important techniques used in RNG theory is to suppose that the dimension of space can be varied continuously, and then to expand in dimension about some value where the critical behavior becomes simple (this usually happens in high enough spatial dimension where the dangerous long wavelength fluctuations are quenched by phase space considerations). Since it is hard to imagine a spatial lattice in, for example, 3.3 dimensions, the dimension continuation is usually done in Fourier space, where the integral $\int d^d q \sim \int q^{d-1} dq$ can be generalized to d nonintegral. This method is too technical to go into in great detail here, but it is worth sketching a crude motivation. There are more details in the appendix.

In our calculations on the Ising model we looked at the fluctuation corrections to mean field theory in the quadratic approximation

$$\left\langle \left| m_{\vec{q}} \right|^2 \right\rangle = \frac{k_B T}{2V(a_1 + \gamma q^2)}. \tag{17}$$

Because $a_1 \propto |T - T_c|$, the fluctuations become large for $T \to T_c$ and for $q \to 0$. The probability distribution is

$$P(\{m_q\}) = \prod_{\vec{q}} \exp[-\beta V (a_1 + \gamma q^2) \left| m_{\vec{q}} \right|^2] \tag{18}$$

and so the approximation is called the Gaussian approximation. As we saw in a previous lecture, the corrections to the specific heat become important within a range of T_c given by the Ginzburg criterion. Integrating over the fluctuations it can be shown[1] gives a singular contribution to the specific heat at T_c corresponding to the exponent

$$\alpha = 2 - \frac{d}{2} \tag{19}$$

rather than $\alpha = 0$ as in mean field theory. All the other exponents ($\beta, \gamma \ldots$) are unchanged from the mean field values. This model is called the Gaussian model, and its critical point properties are determined by the Gaussian fixed point Hamiltonian.

[1] Basically, for each mode q we integrate $P(m_q)$ over all states m_q to give the q-mode partition function (dropping all uninteresting proportionality constants) $Q_{\vec{q}} \propto (a_1 + \gamma q^2)^{-1}$ and so the contribution to the free energy $-kT \ln Q_{\vec{q}} \propto \ln(a_1 + \gamma q^2)$. Differentiating the free energy twice with respect to temperature to get the specific heat, the singular contribution comes from the temperature dependence of $a_1 = a_1'(T - T_c)$. Summing over the modes then gives the most singular fluctuation contribution to the specific heat

$$\delta C \propto \int d^d q (a_1 + \gamma q^2)^{-2} \propto (a_1/\gamma)^{-(2-d/2)}.$$

Now putting $a_1 \propto t$ gives the result quoted. (See e.g. *Modern Theory of Critical Phenomena* by S-K Ma, chapter III for more details.)

When is the quadratic approximation leading to the Gaussian model accurate? To estimate the importance of the nonlinear terms in the free energy expansion, we would need to integrate over the fluctuations $\int d^d q \ldots \sim \int q^{d-1} dq \ldots$. In high enough dimension the q^{d-1} quenches the importance of the small q fluctuations, and the Gaussian model becomes an accurate description of the phase transition.

To study this further, let us slightly change our notation to match that typically used in the renormalization group literature (although where people put the factors of $\frac{1}{2}$ doesn't seem to be consistent), writing the Landau free energy density for some configuration $m(\vec{r})$ as

$$f = \frac{1}{2} r m^2 + u m^4 + c \left| \vec{\nabla} m \right|^2 \tag{20}$$

with $r \to 0$ at T_c, leading to the free energy

$$A(\{m_q\}) = V \left\{ \frac{1}{2} \sum_{\vec{q}} (r + cq^2) \left| m_{\vec{q}} \right|^2 + u \sum_{\vec{q}_1, \vec{q}_2, \vec{q}_3} m_{\vec{q}_1} m_{\vec{q}_2} m_{\vec{q}_3} m_{-\vec{q}_1 - \vec{q}_2 - \vec{q}_3} \right\}. \tag{21}$$

(To get the free energy of the *system* we must still integrate over these fluctuations.)

We might estimate when the fourth order term is important by approximately evaluating

$$u \sum_{\vec{q}_1, \vec{q}_2, \vec{q}_3} m_{\vec{q}_1} m_{\vec{q}_2} m_{\vec{q}_3} m_{-\vec{q}_1 - \vec{q}_2 - \vec{q}_3} \sim \text{terms like } u \sum_{\vec{q}_1, \vec{q}_2, \vec{q}_3} \left\langle m_{\vec{q}_1} m_{\vec{q}_2} \right\rangle m_{\vec{q}_3} m_{-\vec{q}_1 - \vec{q}_2 - \vec{q}_3} \tag{22}$$

where the average is calculated from using the Gaussian distribution

$$\left\langle m_{\vec{q}_1} m_{\vec{q}_2} \right\rangle = \left\langle \left| m_{q_1} \right|^2 \right\rangle \delta_{\vec{q}_1, -\vec{q}_2} \tag{23}$$

so that

$$u \sum_{\vec{q}_1, \vec{q}_2, \vec{q}_3} m_{\vec{q}_1} m_{\vec{q}_2} m_{\vec{q}_3} m_{-\vec{q}_1 - \vec{q}_2 - \vec{q}_3} \sim \text{terms like } u \sum_{\vec{q}_1} \left\langle \left| m_{\vec{q}_1} \right|^2 \right\rangle \sum_{\vec{q}} \left| m_{\vec{q}} \right|^2. \tag{24}$$

This effectively gives a correction to the first term in Eq. (21). Thus to evaluate the importance of the quartic term we compare $u \sum_{\vec{q}_1} \left\langle \left| m_{\vec{q}_1} \right|^2 \right\rangle$ with r as $r \to 0$. Now in the Gaussian approximation

$$\sum_{\vec{q}} \left\langle \left| m_{\vec{q}} \right|^2 \right\rangle_G = \frac{k_B T}{(2\pi)^d} \int \frac{d^d q}{r + cq^2} \sim r^{d/2 - 1} \tag{25}$$

where in the last expression we are focusing just on the r dependence. Thus the ratio of the two terms as $r \to 0$ is

$$\frac{u \sum_{\vec{q}_1} \left\langle \left| m_{q_1} \right|^2 \right\rangle}{r} \sim r^{d/2 - 2} \tag{26}$$

For $d > 4$ the correction terms are small, and this estimate suggests that the Gaussian model is an accurate description of the critical region. On the other hand for $d < 4$ the "correction terms" diverge towards T_c and the Gaussian model fails. This crossover dimension is called the *upper critical dimension* d_u. We have evaluated this for a simple scalar order parameter. The value d_u may depend on the symmetry of the order parameter.

The RNG in dimensions greater than the upper critical dimension shows that the critical properties are determined by a fixed point Hamiltonian for which the Gaussian model is exact: this is called the Gaussian fixed point. It is attracting on the critical surface, and has one unstable direction corresponding to the temperature variable, Fig. 1. For $d < d_u$ the Gaussian fixed point still *exists* (if u is zero it remains zero

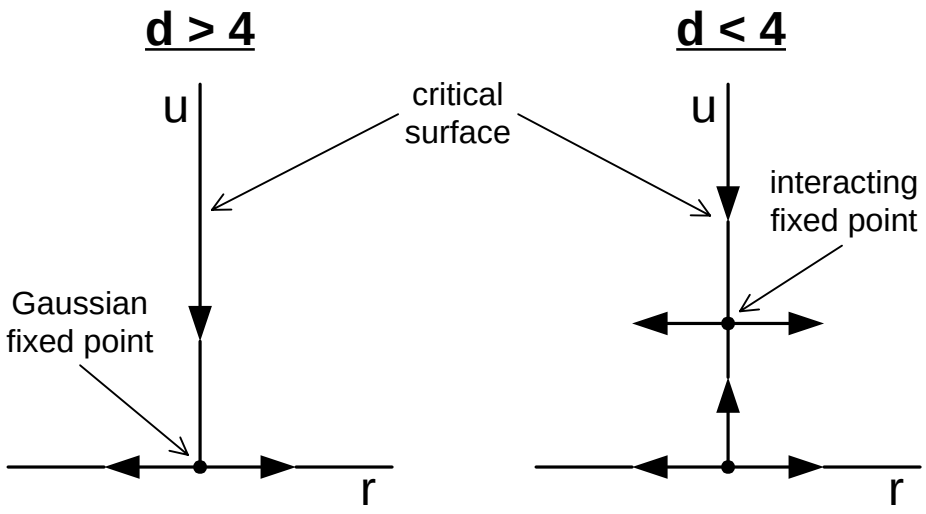

Figure 1: Schematic picture of the critical surface of the Ising transition as a function of varying the space dimension. For a more accurate picture, see the appendix.

under $\mathcal{R}_b$) but it becomes *unstable on the critical surface* towards a new fixed point Hamiltonian representing the physical, $u \neq 0$ non-Gaussian, fixed point. This fixed point *grows continuously out of the Gaussian fixed point* as $\varepsilon = d_u - d$ increases, and in $d = 3$ becomes the standard $3d$ Ising fixed point in the case we are considering. Since for small ε the new fixed point is near the Gaussian one, its properties (eigenvalues, eigenvectors etc.) can be calculated by perturbing around the "easy" Gaussian fixed point. This is the basis of the ε *expansion*, where $\varepsilon = d_u - d$ is often $4 - d$ since the upper critical dimension is often 4, as we found here. In the appendix I describe some of the methodology of the ε-expansion for a scalar order parameter. This is definitely an advanced topic, and it is only included for those interested in a brief dip into this vast field!

For physical systems we need answers of course in $d = 3$ (or less) where $\varepsilon = 1$ (or larger!). Although the expansion in ε is not convergent for $\varepsilon = 1$, it is possible to "resum" the (asymptotic) series, using methods such as Padé approximants, to get accurate values of the exponents. Finally let us remark that once the Gaussian fixed point has become unstable the Landau free energy no longer directly gives an accurate description of the long length scale properties near T_c because of the large, non-Gaussian fluctuations. This makes the $m^4, m^6 \ldots$ terms all become important. However, taking advantage of universality, we may use it as an *intermediate scale* initial approximation, and then use the RNG to establish the long length scale properties. Used in this context the expression is known as the Landau-Wilson free energy.

Migdal-Kadanoff Method

The Migdal-Kadanoff is a real space renormalization method. I will use the method on the d-dimensional Ising model to illustrate many of the ideas introduced above: approximations to truncate the Hamiltonian; continuous scale factor RNG; and dimensionality expansion.

Consider first the $2d$ Ising model. We cannot implement the elimination procedure exactly in two dimensions, so we first a bold approximation to make the elimination step look one dimensional! Label the sites by (i, j). The approximation is simply to "move" every other y-bond so that the y bonds connecting spins with even i have strength $2K$, and there are no y bonds linking spins with odd i. The hope is that this bond moving preserves the physical behavior, but it is clearly an uncontrolled approximation. Since the odd i spins now

only have interactions to their neighbors in the x direction, we can apply the $1d$ elimination procedure which is the recursion function $\mathcal{R}_b^{(1d)}(K)$ we found for the $1d$ Ising model. This gives

$$K'_y = bK_y \qquad \text{(bond moving)}, \tag{27}$$

$$K'_x = \mathcal{R}_b^{(1d)}(K_x) \qquad \text{(spin elimination)}. \tag{28}$$

Now repeat the procedure interchanging x and y bond directions

$$K''_x = bK'_x = b\mathcal{R}_b^{(1d)}(K_x), \tag{29}$$

$$K''_y = \mathcal{R}_b^{(1d)}(K'_y) = \mathcal{R}_b^{(1d)}(bK_y). \tag{30}$$

This procedure has eliminated half the spins, again leaving a square lattice. The final step is to rescale the lattice by a factor of b, defining one step of the approximate $2d$ RNG. The asymmetry in x and y can be eliminated by alternating which bonds are moved and which are eliminated at each step.

Generalizing the procedure to d dimensions gives after one complete cycling through all d directions

$$K'_p = b^{d-p}\mathcal{R}_b^{(1d)}(b^{p-1}K_p) \tag{31}$$

where $p = 1 \ldots d$ labels $x, y, z, \ldots$. We could also do the procedure for general b. The convenient expression of the scale-factor b elimination in $1d$ is

$$K' = \mathcal{R}_b^{(1d)}(K) = \tanh^{-1}[(\tanh K)^b]. \tag{32}$$

(Check that this reproduces our $b = 2$ result!).

The asymmetry in direction index p can again be eliminated by rotating through the direction for the elimination, rather than the bond moving, procedure. However the asymmetry is actually eliminated if we look at the continuous renormalization for $b \to 1$. Write $b = e^{\delta l} \simeq 1 + \delta l$ with δl small. Then for Eq. (31) we need

$$K'_p \simeq (1+\delta l)^{d-p}\mathcal{R}_{1+\delta l}^{(1d)}[(1+\delta l)^{p-1}K_p] = [1+(d-p)\delta l]\tanh^{-1}\left\{\left(\tanh[K_p + (p-1)K_p\delta l]\right)^{1+\delta l}\right\} + O(\delta l^2). \tag{33}$$

Carrying through all the rather messy Taylor expansions gives

$$K'_p = K_p + [(d-1)K_p + \frac{1}{2}\sinh(2K_p)\ln(\tanh K_p)]\delta l + O(\delta l^2). \tag{34}$$

Note that this result is independent of p, so we can write down a single result for the flow of the coupling constant

$$\frac{dK}{dl} = (d-1)K + \frac{1}{2}\sinh(2K)\ln(\tanh K). \tag{35}$$

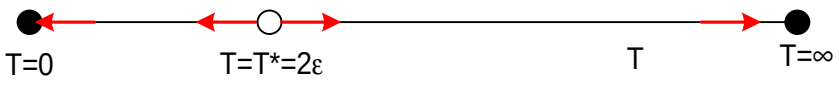

Figure 2: Flows of the temperature T in the $d = 1 + \varepsilon$ Ising model. The $T = T^*$ fixed point controls the critical behavior.

We know that $d = 1$ is a special dimension where we can solve the Ising model exactly and the RNG is simple, so we might expect that expanding in space dimension around $d = 1$ might be useful. For $d = 1$ we also know that the fixed point is at zero temperature, $K \to \infty$, and so it is useful to work with $K^{-1} = k_B T/J$

which we will just call T (the temperature in units of the interaction strength). Now expand Eq. (35) in small T, using $dT/dl = -K^{-2}dK/dl$. We actually only have to evaluate the second term in Eq. (35) to its zeroth order value -1 at $K \to \infty$ to give up to order T^2

$$\frac{dT}{dl} = -\varepsilon T + \frac{1}{2}T^2 \tag{36}$$

introducing $\varepsilon = d - 1$.

For $d = 1$ we see $T = 0$ is a marginally stable fixed point (eigenvalue zero). For $\varepsilon > 0$ this fixed point becomes stable, and there is a new, unstable fixed point at $T = T^* = 2\varepsilon$, see Fig. 2. The critical behavior is determined by the *unstable* fixed point, and so is given by this new fixed point. The $T = 0$ fixed point becomes the *ordered* fixed point. There is also a high temperature *disordered* fixed point. Linearizing about the T^* fixed point, $T = T^* + \delta T$, gives

$$\frac{d\delta T}{dl} = -\varepsilon\delta T + T^*\delta T = \varepsilon\delta T, \tag{37}$$

and so the eigenvalue for the temperature variable is $\lambda_t = \varepsilon$.

To show how to calculate physical results, consider the correlation length. Starting our RNG procedure at a temperature $T(0)$ near the fixed point $T(0) = T^* + t$, where the correlation length is $\xi(t)$, we can now iterate the RNG by integrating the continuous flow

$$\delta T(l) = te^{\lambda_t l}, \tag{38}$$

$$b(l) = e^l, \tag{39}$$

$$\xi(l) = \xi(t)e^{-l}. \tag{40}$$

In the usual way, choose l so that $\delta T(l) = \delta T_f$ independent of $\delta T(0)$

$$e^l = \left(\frac{\delta T_f}{t}\right)^{1/\lambda_t} \tag{41}$$

and then

$$\xi(0) = \xi(\delta T_f)\left(\frac{\delta T_f}{t}\right)^{1/\lambda_t} \propto t^{-1/\lambda_t}. \tag{42}$$

Thus we have evaluated the correlation length exponent $\nu = \lambda_t^{-1} = \varepsilon^{-1}$.

Appendix: Details of the ε-expansion

In this appendix I will describe the details of the first order ε-expansion for the scalar order parameter case. The starting point is $(k_B T)^{-1}$ times the Landau-Wilson free-energy which plays the role of the effective Hamiltonian

$$\bar{H} = \bar{H}_0 + \int d^d x \left\{\frac{1}{2}\left[rm^2 + c(\nabla m)^2\right] + um^4\right\}. \tag{43}$$

It seems to be conventional when doing this calculation to define the Fourier transform with a different normalization than I usually use

$$m_{\vec{q}} = \int d^d x\, m(\vec{x})e^{-i\vec{q}\cdot\vec{x}}, \tag{44}$$

$$m(\vec{x}) = V^{-1}\sum_{\vec{q}} m_{\vec{q}}e^{i\vec{q}\cdot\vec{x}} = \int \frac{d^d q}{(2\pi)^d} m_{\vec{q}}e^{i\vec{q}\cdot\vec{x}}. \tag{45}$$

To save writing I will write

$$\int \frac{d^d q}{(2\pi)^d} = \int_{\vec{q}}. \tag{46}$$

Then in terms of the modes $m_{\vec{q}}$

$$\bar{H} = \bar{H}_0 + \frac{1}{2}\int_{\vec{q}} (r + cq^2) \left|m_{\vec{q}}\right|^2 + u \int_{\vec{q}_1,\vec{q}_2,\vec{q}_3} m_{\vec{q}_1} m_{\vec{q}_2} m_{\vec{q}_3} m_{-\vec{q}_1-\vec{q}_2-\vec{q}_3}. \tag{47}$$

First consider the *Gaussian model* $u = 0$

$$\bar{H} = \bar{H}_0 + \frac{1}{2}\int_0^\Lambda \frac{d^d q}{(2\pi)^d} (r + cq^2) \left|m_{\vec{q}}\right|^2, \tag{48}$$

where the integration is up to some maximum cutoff wave number Λ to take into account the fact that the Landau-Wilson form is only good for length scales large compared to the atomic scale.

The RNG proceeds by splitting the integration into an integration over small wave numbers and large wave numbers using a scale factor b. It is useful to split the real-space field $m(\vec{x}) = m^<(\vec{x}) + m^>(\vec{x})$ into a long wavelength component $m^<(\vec{x})$ with Fourier components $m^<_{\vec{q}}$ with $q < \Lambda/b$, and a short wavelength component $m^>(\vec{x})$ with Fourier components $m^>_{\vec{q}}$ for q in the range $\Lambda/b < q < \Lambda$. Then

$$\bar{H} = \bar{H}_0 + \frac{1}{2}\int_0^{\Lambda/b} \frac{d^d q}{(2\pi)^d} (r + cq^2) \left|m^<_{\vec{q}}\right|^2 + \frac{1}{2}\int_{\Lambda/b}^\Lambda \frac{d^d q}{(2\pi)^d} (r + cq^2) \left|m^>_{\vec{q}}\right|^2. \tag{49}$$

The scheme now is to integrate over the fluctuations of $m^>_{\vec{q}}$ in the partition function

$$Q = \int \mathcal{D}m_{\vec{q}}\, e^{-\bar{H}}, \tag{50}$$

(where the symbol $\mathcal{D}m_{\vec{q}}$ denotes integrating over all values of *all* $m_{\vec{q}}$) and see how this affects the terms in the effective Hamiltonian of the remaining degrees of freedom.

For the Gaussian model, the different $m_{\vec{q}}$ decouple, and the only effect of the integration over $m^>_{\vec{q}}$ is to change the additive constant $\bar{H}_0$, which we do not need to follow. Thus

$$\bar{H}(\{m^<_{\vec{q}}\}) = \bar{H}'_0 + \frac{1}{2}\int_0^{\Lambda/b} \frac{d^d q}{(2\pi)^d} (r + cq^2) \left|m^<_{\vec{q}}\right|^2. \tag{51}$$

Now we rescale lengths by the factor b^{-1} and so wave numbers by b, i.e. $q' = bq$, and $m^<_{\vec{q}}$ by the *spin rescaling factor* ζ, i.e. $m'^<_{\vec{q}} = \zeta^{-1} m^<_{\vec{q}}$ to give

$$\bar{H}' = \bar{H}'_0 + \frac{1}{2}\int_0^\Lambda \frac{d^d q'}{(2\pi)^d} (r' + c'q'^2) \left|m'^<_{\vec{q}'}\right|^2 \tag{52}$$

with

$$r' = \zeta^2 b^{-d} r, \tag{53}$$

$$c' = \zeta^2 b^{-(d+2)} c. \tag{54}$$

We want the chance of finding a fixed point Hamiltonian with r playing the role of the (relevant) temperature variable, and so we want to choose the spin rescaling factor to keep the coefficient of the gradient term the same, $c' = c$. Hence we choose

$$\zeta = b^{d/2+1} \tag{55}$$

and then

$$r' = b^2 r. \tag{56}$$

(Then since $c = \text{const}$, we might as well put it to unity $c = 1$ by choosing our unit of length.)

Equation (56) is the recursion relation for r under the RNG. There is a fixed point Gaussian Hamiltonian with $r = r^* = 0$. Linearizing about this fixed point, Eq. (56) gives the eigenvalue $\lambda_t = 2$. Thus the Gaussian model has the mean field exponent $\nu = \lambda_t^{-1} = 1/2$ for the correlation length. The correlation function at T_c for the Gaussian model is proportional to the Fourier transform of q^{-2} and so varies as $r^{-(d-2)}$. This corresponds to $\eta = 0$.

Now we ask the question about the role of the interaction term, fourth order in $m_{\vec{q}}$. We phrase the question in terms of whether u is relevant or irrelevant at the Gaussian fixed point, i.e. by calculating the effect of a small u on what we have done so far. Integrating out the $m_{\vec{q}}^<$ in Eq. (50) now takes the form

$$\int \mathcal{D}m_{\vec{q}}^> \, e^{-(r+cq^2)\left|m_{\vec{q}}^>\right|^2} e^{-\bar{H}^{(4)}(m^< + m^>)} \tag{57}$$

with $H^{(4)}$ the fourth order term with coefficient u. This expression is (up to constant factors) the Gaussian average over the large wave number fluctuations in $e^{-\bar{H}^{(4)}}$ which we denote $\left\langle e^{-\bar{H}^{(4)}} \right\rangle_>$. Since $\bar{H}^{(4)}$ is small, we can expand the exponential

$$\left\langle e^{-\bar{H}^{(4)}} \right\rangle_> = 1 - \left\langle \bar{H}^{(4)} \right\rangle_> + \frac{1}{2}\left\langle \left(\bar{H}^{(4)}\right)^2 \right\rangle_> + \cdots \tag{58}$$

$$= \exp\left\{ -\left\langle \bar{H}^{(4)} \right\rangle_> + \frac{1}{2}\left[\left\langle \left(\bar{H}^{(4)}\right)^2 \right\rangle_> - \left(\left\langle \bar{H}^{(4)} \right\rangle_> \right)^2 \right] + \cdots \right\}. \tag{59}$$

For terms first order in small u we need just $\left\langle \bar{H}^{(4)} \right\rangle_>$. It is convenient to write this as the real space integral

$$\left\langle \bar{H}^{(4)} \right\rangle_> = \left\langle u \int d^d x (m^<(\vec{x}) + m^>(\vec{x}))^4 \right\rangle_> . \tag{60}$$

This gives

$$\left\langle \bar{H}^{(4)} \right\rangle_> = u \int d^d x (m^<(\vec{x})^4 + 6u \int d^d x (m^<(\vec{x}))^2 \left\langle (m^>(\vec{x}))^2 \right\rangle_> + u \int d^d x \left\langle (m^>(\vec{x}))^4 \right\rangle_> , \tag{61}$$

where we have used the fact that only the averages of even powers are nonzero for the Gaussian distribution, and the factor of 6 is simply the coefficient of the cross term in the expansion of the fourth power.

The first term in Eq. (61) is just the (unchanged) u term for the modes $m_{\vec{q}}^<$ that have not been eliminated. The last term adds to the constant $\bar{H}_0$ and is not important. The all important second term gives an additional contribution to the coefficient $\frac{1}{2}r$ of $(m^<)^2$ so that the coefficient becomes

$$r^< = r + 12u \int d^d x \left\langle (m^>(\vec{x}))^2 \right\rangle_> , \tag{62}$$

$$= r + 12u \int_{\Lambda/b}^{\Lambda} \frac{d^d q}{(2\pi)^d} \frac{1}{r + q^2}. \tag{63}$$

After rescaling we then get

$$r' = b^2 \left[r + 12u \int_{\Lambda/b}^{\Lambda} \frac{d^d q}{(2\pi)^d} \frac{1}{r + q^2} \right]. \tag{64}$$

Note that $\left\langle (m^>(\vec{x}))^2 \right\rangle_>$ is independent of $\vec{x}$, so that there are no corrections to the coefficient of the gradient term from the elimination. Thus the choice of the spin rescaling factor to keep c constant is not changed at this order.

The integral in Eq. (64) is still messy to evaluate in general. It is easiest to go to the continuous renormalization procedure $b = e^{\delta l}$ with small δl, so that $\Lambda/b \simeq \Lambda(1-\delta l)$ and $b^2 \simeq 1 + 2\delta l$, and to order δl

$$r' = r + 2\delta l r + 12u \int \frac{d\Omega_d}{(2\pi)^d} \int_{\Lambda - \Lambda\delta l}^{\Lambda} dq\, q^{d-1} \frac{1}{r+q^2}. \tag{65}$$

The integral is now trivial to evaluate (it is just the integrand evaluated at $q = \Lambda$, multiplied by the range of integration $\Lambda\delta l$), to give

$$r' = r + 2r\,\delta l + 12u K_d \frac{\Lambda^{d-1}}{r+\Lambda^2} \Lambda \delta l, \tag{66}$$

where

$$K_d = \frac{1}{\Gamma(\frac{1}{2}d) 2^{d-1} \pi^{d/2}} \tag{67}$$

is just a geometrical factor, $(2\pi)^{-d}$ times the surface area of a sphere in d dimensions. Converting to differential form

$$\frac{dr}{dl} = 2r + 12 K_d \frac{\Lambda^d}{r+\Lambda^2} u. \tag{68}$$

We also need to find out how u evolves, unfortunately up to second order in u. For this we need to evaluate the *cumulant*

$$C^{(4)} = \left\langle \left(\bar{H}^{(4)}\right)^2 \right\rangle_> - \left(\left\langle \bar{H}^{(4)} \right\rangle_> \right)^2. \tag{69}$$

The first term is

$$\left\langle \left(\bar{H}^{(4)}\right)^2 \right\rangle_> = \left\langle \left(u \int_{\vec{q}_1,\vec{q}_2,\vec{q}_3} m_{\vec{q}_1} m_{\vec{q}_2} m_{\vec{q}_3} m_{-\vec{q}_1-\vec{q}_2-\vec{q}_3} \right) \left(u \int_{\vec{q}_4,\vec{q}_5,\vec{q}_6} m_{\vec{q}_4} m_{\vec{q}_5} m_{\vec{q}_6} m_{-\vec{q}_4-\vec{q}_5-\vec{q}_6} \right) \right\rangle_> . \tag{70}$$

For the Gaussian average over the large wave number fluctuations we must pick pairs of $m^>_{\vec{q}}$ and then use

$$\left\langle m^>_{\vec{q}_1} m^>_{\vec{q}_2} \right\rangle_> = \frac{1}{r+q_1^2} (2\pi)^d \delta(\vec{q}_1 + \vec{q}_2), \tag{71}$$

which is Eq. (23) in the revised notation and normalization. We call this averaging a pairing or contraction of the two $m_{\vec{q}}$. If we average over all 4 pairs of the $m_{\vec{q}}$ appearing in Eq. (70), we get an additional contribution to the constant term, averaging over 3 pairs gives an $O(u^2)$ correction to r and c which we do not need to evaluate to go to $O(\varepsilon)$, etc. We only need the contribution to u itself, which is given by averaging over 2 pairs, leaving four fluctuating (not averaged) $m^<_{\vec{q}}$ fields. In general the coefficient of the product of four $m^<_{\vec{q}_i}$ will depend on the $\vec{q}_i$; however we may evaluate the coefficient for uniform $m(\vec{x})$, i.e. for $\vec{q}_i \to 0$, since corrections will only contribute to gradient terms that are higher order than we are going.

Now to average over two pairs of $m^>_{\vec{q}}$ in Eq. (70). The terms paired $m_{\vec{q}}$ in the same ( ) are cancelled by the second term in the cumulant. So we get 72 terms (choose the two $m^>_{\vec{q}}$ in $4 \times 3/2 = 6$ ways from each bracket and then pair them up in two different ways) like (use Eq. (71))

$$u^2 \int_{\vec{q}_1,\vec{q}_2,\vec{q}_3,\vec{q}_4} \left\langle \left| m^>_{\vec{q}_1} \right|^2 \right\rangle_> \left\langle \left| m^>_{\vec{q}_2} \right|^2 \right\rangle_> m^<_{\vec{q}_3} m^<_{-\vec{q}_1-\vec{q}_2-\vec{q}_3} m^<_{\vec{q}_4} m^<_{\vec{q}_1+\vec{q}_2-\vec{q}_4}. \tag{72}$$

Evaluating this for the wave vectors of the $m^<_{\vec{q}}$ equal to zero sets $\vec{q}_1 = \vec{q}_2$ and so gives for the new value of u after eliminating the modes between Λ/b and Λ (remember the factor of $\frac{1}{2}$ and the minus sign in front of the cumulant, and that we are taking $c = 1$)

$$u^< = u - 36u^2 \int_{\Lambda/b}^{\Lambda} \frac{d^d q}{(2\pi)^d} \left(\frac{1}{r+q^2} \right)^2. \tag{73}$$

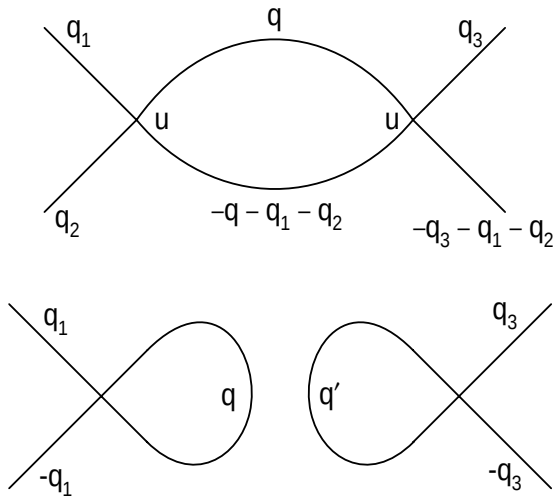

Figure 3: Diagrams for $O(u^2)$ terms renormalizing u. The top diagram is the term contributing to the result. Each vertex gives a factor of u, and the closed links correspond to $\left\langle \left| m_{\vec{q}}^{>} \right|^2 \right\rangle_{>} = (r+q^2)^{-1}$. The open lines correspond to $m_{\vec{q}}^{<}$. As you might guess, the lower diagram of disconnected terms from $\left\langle \left(H^{(4)}\right)^2 \right\rangle$ is cancelled by terms from $\left(\left\langle H^{(4)} \right\rangle\right)^2$ in the cumulant.

The rescaling gives a factor of (3 wave vector integrals, 4 powers of m) $\zeta^4 b^{-3d} = b^{4-d}$, so that to $O(u^2)$ we have

$$u' = b^{4-d}\left[u - 36u^2 \int_{\Lambda/b}^{\Lambda} \frac{d^d q}{(2\pi)^d} \left(\frac{1}{r+q^2}\right)^2\right]. \tag{74}$$

For $b = e^{\delta l}$ with small δl this gives

$$\frac{du}{dl} = (4-d)u - 36u^2 K_d \frac{\Lambda^d}{(r+\Lambda^2)^2}. \tag{75}$$

As with most such expansions taken beyond first order, it is most economical to formulate the terms diagrammatically—certainly this is essential for calculations to high order. As an indication of the diagrammatic approach the diagrams corresponding to the argument in the last paragraph are shown in Fig. 3. The diagrams are very useful for constructing the various possibilities for the pairing or contraction corresponding to the average $\left\langle \left| m_{\vec{q}}^{>} \right|^2 \right\rangle_{>}$ and keeping track of momentum conservation etc.

Equations (68) and (75) are the recursion relations for r and u. It is convenient to rescale these variables by powers of the original cutoff $\bar{r} = r/\Lambda^2$ and $\bar{u} = u\Lambda^{d-4}$ to give our final form for the recursion relations

$$\frac{d\bar{r}}{dl} = 2\bar{r} + 12K_d \frac{\bar{u}}{1+\bar{r}}, \tag{76}$$

$$\frac{d\bar{u}}{dl} = \varepsilon\bar{u} - 36K_d \frac{\bar{u}^2}{(1+\bar{r})^2}. \tag{77}$$

with $\varepsilon = 4-d$. We see again that u is irrelevant for $d > 4$, and therefore we expect u to be small near the fixed point for ε small. The flows given by these recursion relations are shown in Fig. 4.

For $\varepsilon > 0$ the Gaussian fixed point is unstable in two directions, and a new fixed point with a single unstable direction develops an $O(\varepsilon)$ distance away. The new fixed point, found by setting $d\bar{r}/dl = d\bar{u}/dl = 0$ in the recursion relations, is

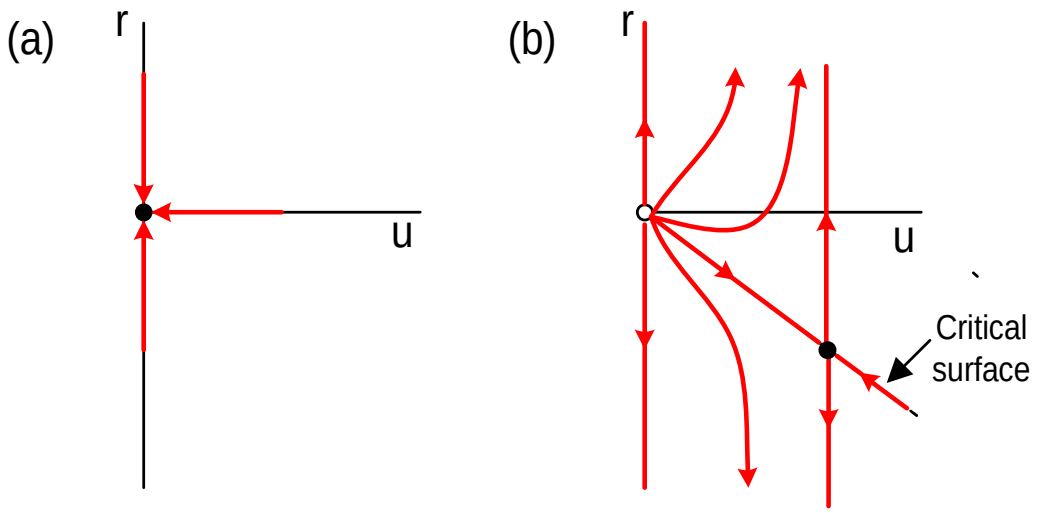

Figure 4: Flows for (a) $d > 4$ and (b) $d = 4 - \varepsilon$ with ε small. Note that the critical surface—the surface of parameters that flows into the critical fixed point—is no longer precisely $r = 0$. This corresponds physically to a change in T_c coming from the interaction terms.

$$\bar{r}^* = -\frac{1}{6}\varepsilon, \tag{78}$$

$$\bar{u}^* = \frac{\varepsilon}{36K_d}. \tag{79}$$

Linearizing about this fixed point

$$\begin{bmatrix} d\delta\bar{r}/dl \\ d\delta\bar{u}/dl \end{bmatrix} = \begin{bmatrix} 2 - \frac{\varepsilon}{3} & \frac{12K_d}{1+\bar{r}^*} \\ 0 & -\varepsilon \end{bmatrix} \begin{bmatrix} \delta\bar{r} \\ \delta\bar{u} \end{bmatrix} \tag{80}$$

gives eigenvalues at $O(\varepsilon)$

$$\lambda_t = 2 - \frac{\varepsilon}{3}, \qquad \lambda_u = -\varepsilon. \tag{81}$$

These are the first terms in an expansion in small $\varepsilon = 4 - d$. (You can check that indeed we needed to go to $O(u^2)$ in the u equation, but only to $O(u)$ in the r equation to get results to first order in ε).

The thermodynamic exponents can now be evaluated to $O(\varepsilon)$. The correlation length exponent $\nu = \lambda_t^{-1}$ is

$$\nu = \frac{1}{2} + \frac{\varepsilon}{12}. \tag{82}$$

The value of η follows from the spin rescaling, which was unchanged from the Gaussian result at $O(\varepsilon)$

$$\eta = 0 + O(\varepsilon^2). \tag{83}$$

The remaining exponents can be found from the scaling identities to be

$$\alpha = \frac{\varepsilon}{6}, \tag{84}$$

$$\beta = \frac{1}{2} - \frac{\varepsilon}{6}, \tag{85}$$

$$\gamma = 1 + \frac{\varepsilon}{6}, \tag{86}$$

$$\delta = 3 + \varepsilon. \tag{87}$$