Physics 127a: Class Notes

Lecture 17: Ideal Fermi Gas

General Properties

For an ideal Fermi gas

$$\Omega = -kT \ln \mathcal{Q} = -kT \sum_s \ln\left(1 + z e^{-\beta\varepsilon_s}\right) \tag{1}$$

$$N = \sum_s \langle n_s \rangle = \sum_s \frac{1}{z^{-1} e^{\beta\varepsilon_s} + 1} \tag{2}$$

where the sums run over single particle states s which are usually labelled by the wave vector $\mathbf{k}$ and the spin. The fugacity $z = e^{\beta\mu}$ may run over the whole range $0 < z < \infty$, corresponding to $-\infty < \mu < \infty$.

For a nonrelativistic gas in 3-dimensional free space $\varepsilon_k = \hbar^2 k^2/2m$ and $\sum_s \to \int d\varepsilon\, \rho(\varepsilon)$ with the density of states

$$\rho(\varepsilon) = g \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2}\right)^{3/2} \varepsilon^{1/2}, \tag{3}$$

with g the spin degeneracy, usually $2s + 1$. Proceeding as in the Boson case leads to expressions

$$\frac{P}{kT} = \frac{g}{\lambda^3} f_{5/2}(z) \qquad \frac{N}{V} = \frac{g}{\lambda^3} f_{3/2}(z) \tag{4}$$

with λ the thermal length $h/\sqrt{2\pi mkT}$ and $f_\nu(z)$ *Fermi functions* (their properties are discussed in *Pathtria* appendix E)

$$f_\nu(z) = \frac{1}{(\nu-1)!} \int_0^\infty \frac{y^{\nu-1}}{z^{-1} e^y + 1} dy. \tag{5}$$

High temperature expansions may be developed by expanding the integrands in small z as in the Bose Case.

"Zero Temperature" Behavior

The *Fermi occupation function*

$$n_F(\varepsilon) = \frac{1}{e^{\beta(\varepsilon-\mu)} + 1} = \frac{1}{2} + \frac{1}{2} \tanh\left[\frac{1}{2}\beta\left(\varepsilon - \mu\right)\right] \tag{6}$$

tells us the occupation number of a single particle state at energy ε. The function is 1 for ε many kT less than μ, becomes $\frac{1}{2}$ at $\varepsilon = \mu$, and goes exponentially to zero for ε many kT greater than μ. At low temperatures the chemical potential is the dividing energy between occupied and empty states. The second form in Eq. (6) shows the symmetry

$$n_F(\mu + \delta) = 1 - n_F(\mu - \delta). \tag{7}$$

At zero temperature the states are filled up to an energy called the *Fermi energy* $\epsilon_F = \mu(T = 0)$ and empty above. The value of ε_F is fixed by demanding the total number of states with $\varepsilon < \varepsilon_F$ be equal to the number of Fermions N. For free particles, $\varepsilon = \varepsilon_F$ defines a spherical surface in k-space $|\mathbf{k}| = k_F$ with k_F the *Fermi wave vector*. Counting the states inside the Fermi surface (volume $(2\pi)^3/V$ per wave vector) gives

$$\frac{g\frac{4}{3}\pi k_F^3}{(2\pi)^3/V} = N \qquad \text{i.e.} \qquad k_F = \left(\frac{6\pi^2}{g}\frac{N}{V}\right)^{1/3}. \tag{8}$$

The order of magnitude of k_F is the reciprocal of the interparticle spacing. The corresponding Fermi energy $\varepsilon_F = \hbar^2 k_F^2/2m$ is of order the zero point energy of a particle confined to this scale. This is typically a large energy, e.g. $10^4 K$ for electrons in a metal. Note that $\varepsilon_F \propto (N/V)^{2/3}$ so that $\rho(\varepsilon_F) = \partial N/\partial\varepsilon_F|_V = (3N/2\varepsilon_F)$, a result we will use below in the low temperature calculations.

Thermodynamic Properties The total internal energy at zero temperature (all kinetic!) is given by integrating over the sphere up to radius k_F. Clearly $U = cN\varepsilon_F$ with c a constant of order unity. This shows us that $U(T = 0)$ scales with the thermodynamic variables as

$$U \propto N^{5/3}V^{-2/3}. \tag{9}$$

So we have

$$\mu(T = 0) = \varepsilon_F = \left(\frac{\partial U}{\partial N}\right)_V = \frac{5}{3}\frac{U}{N} \qquad \text{or} \qquad U = \frac{3}{5}N\varepsilon_F \tag{10}$$

(fixing the constant $c = 3/5$), and

$$P = -\left(\frac{\partial U}{\partial V}\right)_N = \frac{2}{3}\frac{U}{V} \tag{11}$$

(as we know from general considerations). This *degeneracy pressure* is important in diverse areas of physics, such as solid state physics and astrophysics.

Degeneracy pressure and white dwarf stars White dwarf stars are old stars of mass similar to our sun, that have used up most of their nuclear fuel. A simple model of these stars supposes that they consist of fully ionized Helium, i.e. a mixture of alpha-particles and electrons. For simplicity we consider the density to be uniform, so that there are N electrons and $N/2$ alpha particles in the volume of the star of radius R. The electrons from a low temperature ($kT \ll \varepsilon_F$) degenerate electron gas, and the degeneracy pressure balances the inward gravitational force. The Fermi energy turns out to be comparable to the electron rest mass, and so we must consider a relativistic Fermi gas. The alpha particles provide charge neutrality, and dominate the mass and so the gravitational attraction. Their kinetic energy (and contribution to the degeneracy pressure) is small by the mass ratio $m_e/4m_p$.

First lets suppose the electrons are strongly relativistic, so the spectrum is $\varepsilon_p = cp$. The total kinetic energy is of order $U_{kin} \sim N\varepsilon_F \sim N\hbar c(N/V)^{1/3} \sim \hbar c N^{4/3}/R$. This provides the repulsive degeneracy pressure. The gravitational potential energy is $U_{pot} \sim -G(Nm_p)^2/R$. These expressions imply that for a star mass $M \sim Nm_p$ satisfying

$$M \gtrsim \frac{(\hbar c/G)^{3/2}}{m_p} \tag{12}$$

the attractive forces overcome the repulsive one, so that so solution can be found—the star *collapses* in this model. This limit is known as the *Chandrasekhar limit*, and indeed more massive stars are believed to collapse to neutron stars or black holes.

If we instead suppose the classical limit $\varepsilon_p = p^2/2m$ the kinetic energy scales as $U_{kin} \sim N^{5/3}(\hbar^2/2m)/R^2$ and so collapse never occurs. Clearly we need to consider the full expression for the kinetic energy

$$\varepsilon_p = \sqrt{m^2c^4 + p^2c^2} - mc^2. \tag{13}$$

The pressure is calculated from the grand potential

$$P = gkTh^{-3}\int_0^\infty \ln[1 + e^{-\beta(\varepsilon_p - \mu)}]4\pi p^2 dp \tag{14}$$

which becomes on integrating by parts

$$P = \frac{g}{3h^3}\int_0^\infty \frac{1}{e^{\beta(\varepsilon_p - \mu)} + 1} p\frac{d\varepsilon_p}{dp} 4\pi p^2 dp. \tag{15}$$

We can now take the $T \to 0$ limit

$$P = \frac{4\pi g}{3h^3}\int_0^{p_F} p\frac{d\varepsilon_p}{dp} 4\pi p^2 dp, \tag{16}$$

and finally put in the form for ε_p, use the spin degeneracy $g = 2$, and introduce the scaled integration variable $y = p/mc$ to get

$$P = \frac{\pi}{3}\frac{m^4c^5}{h^3}A(p_F/mc) \tag{17}$$

where the function $A(x)$ is defined by

$$A(x) = 8\int_0^x \frac{y^4}{(1+y^2)^{1/2}}dy. \tag{18}$$

The function A is rather boring. For small x it varies as $A(x) \simeq \frac{8}{5}x^5$ (and then c drops out of P and $P \propto R^{-5}$, the classical limit) and for for large x as $A(x) \simeq 2x^4$(so that $P \propto cR^{-4}$, the strongly relativistic limit.

To determine the equilibrium radius of the star we minimize the total energy with respect to the radius $dE/dR = 0$. For the kinetic energy

$$\frac{dE_{\text{kin}}}{dR} = \frac{3V}{R}\frac{dE}{dV} = -\frac{3V}{R}P = -4\pi R^2 P \tag{19}$$

and for the potential energy

$$\frac{dE_{\text{pot}}}{dR} = \frac{d}{dR}\left(-\frac{\alpha GM^2}{R}\right) = \frac{\alpha GM^2}{R^2} \tag{20}$$

where α is a fudge factor that depends on the density distribution. Minimizing the total energy gives

$$4\pi R^2 P = \frac{\alpha GM^2}{R^2} \tag{21}$$

balancing the outward pressure force on the surface with the inward gravitational force. Now it simply remains to collect all the factors, using $p_F = \hbar(3\pi^2 n)^{1/3}$, $n = N/(\frac{4}{3}\pi R^3)$, $M = 4m_p \times \frac{N}{2}$. Defining the mass scale

$$M_0 = \left(\frac{9}{64}\right)\left(\frac{3\pi}{\alpha^3}\right)^{1/2}\frac{(\hbar c/G)^{3/2}}{m_p^2} \sim \frac{(\text{Planck mass})^{3/2}}{(\text{Proton mass})^2} \tag{22}$$

and the length scale

$$R_0 = \frac{3}{8}\left(\frac{3\pi}{\alpha}\right)^{1/2}\frac{(\hbar c/G)^{1/2}}{m_p}\frac{\hbar}{mc} \sim \frac{(\text{Planck mass})}{(\text{Proton mass})} \times (\text{Compton wavelength of electron}) \tag{23}$$

gives the final expression

$$\left.\frac{1}{2x^4}A(x)\right|_{x=\left(\frac{M}{M_0}\right)^{1/2}\frac{R_0}{R}} = \left(\frac{M}{M_0}\right)^{2/3}.$$

You can find a figure of the resulting solution for R/R_0 as a function of M/M_0 in *Pathria* §8.4. Since the left hand side of this equation approaches 1 for small R/R_0 this shows there is a maximum $M = M_0$ for which solutions can be found, the Chandrasekhar limit.

Pauli spin susceptibility In a magnetic field, for charged particles such as the electron, the single particle energies depends on the spin. For an electron, a spin-$\frac{1}{2}$particle

$$\varepsilon_{k,\sigma} = \varepsilon_k - \sigma\mu_B B, \tag{24}$$

where $\sigma = \pm 1$ for the ↑ and ↓ spin respectively, and $\mu_B = e\hbar/2mc$ is the Bohr magneton. The statistical mechanics is now easily generalized to this case by replacing $2\sum_{\mathbf{k}}$ by $\sum_{\mathbf{k},\sigma}$.

The number of up or down spin particles is then

$$N_\sigma(T,\mu,B)=\sum_{\mathbf{k}}\frac{1}{e^{\beta(\varepsilon_k-\sigma\mu_B B-\mu)}+1}. \tag{25}$$

If we define N_0 as the number of particles of *one* spin in zero field

$$N_0(T,\mu)=\sum_{\mathbf{k}}\frac{1}{e^{\beta(\varepsilon_k-\mu)}+1} \tag{26}$$

then by inspection

$$N_\sigma(T,\mu,B)=N_0(T,\mu+\sigma\mu_B B). \tag{27}$$

Then for small B we have

$$N=N_\uparrow+N_\downarrow=N_0(T,\mu+\mu_B B)+N_0(T,\mu-\mu_B B)=2N_0(T,\mu)+O(B^2) \tag{28a}$$

$$M=\mu_B(N_\uparrow-N_\downarrow)=N_0(T,\mu+\mu_B B)-N_0(T,\mu-\mu_B B)\simeq 2\left(\frac{\partial N_0}{\partial\mu}\right)_{T,B=0}\mu_B^2 B \tag{28b}$$

where the second equalities are true for small B. Equation (28a) is the same as in zero field, and shows that the chemical potential for fixed N is unchanged at $O(B)$. Equation (28b) then gives us the susceptibility

$$\chi=\mu_B^2\frac{1}{V}\left(\frac{\partial N}{\partial\mu}\right)_T \tag{29}$$

(using $N=2N_0$ for the total number of electrons). At zero temperature $\mu=\varepsilon_F$, so

$$\chi=\mu_B^2\rho(\varepsilon_F)=\frac{3}{2}n\frac{\mu_B^2}{\varepsilon_F} \tag{30}$$

where the first expression in quite general, and the second evaluated for nonrelativistic electrons $N\propto k_f^3\propto\varepsilon_F^{2/3}$. This is the *Pauli susceptibility*, and is independent of temperature. At high temperatures $\mu=kT\ln(N\lambda^3/V)$ and so $\partial N/\partial\mu=N/kT$ and

$$\chi=n\frac{\mu_B^2}{kT} \tag{31}$$

the usual *Curie susceptibility* of free spins. These susceptibilities are *positive*, i.e. *paramagnetic*.

Charged particles also couple to a magnetic field via their charge, given by the Hamiltonian

$$H=\sum_i\frac{1}{2m}[\mathbf{p}_i+\frac{e}{c}\mathbf{A}(\mathbf{x}_i)]^2 \tag{32}$$

with $\mathbf{B}=\nabla\times\mathbf{A}$. This turns out to lead to a diamagnetic contribution to the magnetic susceptibility coming from circulating currents (diamagnetic because of Lenz's law): at high temperatures

$$\chi_D\to-\frac{1}{3}n\frac{\mu_B^2}{k_BT}, \tag{33}$$

and at low temperatures (known as Landau diamagnetism)

$$\chi_D\to=-\frac{1}{2}n\frac{\mu_B^2}{\varepsilon_F}. \tag{34}$$

For both high and low temperatures, the diamagnetic susceptibility is a factor of 3 smaller, but of opposite sign, than the paramagnetic susceptibilities Eqs. (30) and (31) coming from the coupling of the field to the spins. The calculation of the diamagnetic component of the susceptibility involves the rather tricky solution of the quantum mechanics of the Hamiltonian Eq. (32) giving "Landau levels", which I will not describe here.

Low-T Fermi Integrals

We are often interested in the low temperature thermodynamic properties at temperature $kT \ll \epsilon_F$. Only particle near within a few kT of the Fermi surface may be thermally excited, because of the Pauli exclusion principle (cf. the form of $n_F(\varepsilon)$). This means that the *temperature dependence of thermodynamic properties for $kT \ll \varepsilon_F$ should only depend on properties near the Fermi surface*, and in particular only on $\rho(\varepsilon \simeq \varepsilon_F)$, not on the whole density of states. This is important because in some applications of the results, e.g. to electrons in metals, the function $\rho(\varepsilon)$ might be quite complicated or even unknown in full. The following method to evaluate low temperature Fermi integrals stresses this result. (The result is equivalent to Eq. (16) of Appendix E of *Pathria*, although his derivation is rather obscure.)

Thermodynamic sums over the single particle (momentum) states lead to integrals over the single particle energy ε of the form

$$I = \int_0^\infty d\varepsilon\, \phi(\varepsilon)\, n_F(\varepsilon) \tag{35}$$

where $n_F(\varepsilon) = (e^{\beta(\varepsilon-\mu)} + 1)^{-1}$ is the Fermi occupation function, and ϕ will depend on the quantity being calculated, e.g. $\phi = \rho(\varepsilon)$ for $I = N$, $\phi = \varepsilon\rho(\varepsilon)$ for $I = U$ (the internal energy), with $\rho(\varepsilon)$ the density of states. For the low temperature expansion of these expression, since all the action is for an energy range of order kT about the Fermi energy ε_F, we try to isolate this region of the energy integral.

First split up the integral

$$I = \int_0^\mu d\varepsilon\, \phi(\varepsilon) - \int_0^\mu d\varepsilon\, \phi(\varepsilon)\,[1 - n_F(\varepsilon)] + \int_\mu^\infty d\varepsilon\, \phi(\varepsilon)\, n_F(\varepsilon). \tag{36}$$

In the second two integrals, shift the integration variable $\varepsilon = \mu + \delta$

$$I = \int_0^\mu d\varepsilon\, \phi(\varepsilon) - \int_{-\mu}^0 d\delta\, \phi(\mu+\delta)\,[1 - n_F(\mu+\delta)] + \int_0^\infty d\delta\, \phi(\mu+\delta)\, n_F(\mu+\delta). \tag{37}$$

For low temperatures the lower limit in the second integral can be replaced by $-\infty$, with an error of order $\varepsilon^{-\beta\mu}$ (and $\mu \simeq \varepsilon_F$). Then in this integral put $\delta \to -\delta$, and use the symmetry of the n_F function

$$1 - n_F(\mu - \delta) = n_F(\mu + \delta) \tag{38}$$

to give

$$I = \int_0^\mu d\varepsilon\, \phi(\varepsilon) + \int_0^\infty d\delta\, [\phi(\mu+\delta) - \phi(\mu-\delta)]\, n_F(\mu+\delta) + O(\varepsilon^{-\beta\mu}). \tag{39}$$

Now expanding $\phi(\mu+\delta) - \phi(\mu-\delta)$ as a Taylor expansion in δ and integrating over δ gives a series expansion in $(kT/\varepsilon_F)^2$. We will just go up to quadratic order

$$I \simeq \int_0^\mu d\varepsilon\, \phi(\varepsilon) + \frac{\pi^2}{6}(kT)^2 (d\phi/d\varepsilon)\Big|_{\varepsilon=\mu} \tag{40}$$

where we have used

$$\int_0^\infty d\delta\, \frac{\delta}{e^{\beta\delta}+1} = \frac{\pi^2}{12}(kT)^2. \tag{41}$$

In the second term, which is already of quadratic order we can replace μ by the zero temperature value $\mu \simeq \varepsilon_F$, and we split up the first term into two pieces

$$I \simeq \int_0^{\varepsilon_F} d\varepsilon\, \phi(\varepsilon) + \int_{\varepsilon_F}^\mu d\varepsilon\, \phi(\varepsilon) + \frac{\pi^2}{6}(kT)^2 \frac{d\phi}{d\varepsilon}\Big|_{\varepsilon=\varepsilon_F}. \tag{42}$$

Note that the first term is temperature independent. The second two terms give the temperature dependence through $\mu(T)$ and the explicit temperature dependence.

Number of Particles This fixes $\mu(T, N)$. We use $\phi = \rho(\varepsilon)$:

$$N \simeq \int_0^{\varepsilon_F} d\varepsilon\, \rho(\varepsilon) + \int_{\varepsilon_F}^{\mu} d\varepsilon\, \rho(\varepsilon) + \frac{\pi^2}{6}(kT)^2 \frac{d\rho}{d\varepsilon}\bigg|_{\varepsilon=\varepsilon_F}. \tag{43}$$

Subtracting the zero temperature expression

$$N \simeq \int_0^{\varepsilon_F} d\varepsilon\, \rho(\varepsilon) \tag{44}$$

gives

$$\int_{\varepsilon_F}^{\mu} d\varepsilon\, \rho(\varepsilon) \simeq -\frac{\pi^2}{6}(kT)^2 (d\rho/d\varepsilon)\bigg|_{\varepsilon=\varepsilon_F} \tag{45}$$

so that

$$(\mu - \varepsilon_F)\rho(\varepsilon_F) \simeq -\frac{\pi^2}{6}(kT)^2 (d\rho/d\varepsilon)\bigg|_{\varepsilon=\varepsilon_F} \tag{46}$$

or

$$\mu \simeq \varepsilon_F - \frac{\pi^2}{6}(kT)^2 (d\ln\rho/d\varepsilon)\bigg|_{\varepsilon=\varepsilon_F} \tag{47}$$

Specific Heat Use $C_V = \partial U/\partial T|_{N,V}$, and for the internal energy U we use $\phi = \varepsilon\rho(\varepsilon)$:

$$U \simeq \int_0^{\mu} d\varepsilon\, \varepsilon\rho(\varepsilon) + \frac{\pi^2}{6}(kT)^2 \frac{d(\varepsilon\rho)}{d\varepsilon}\bigg|_{\varepsilon=\varepsilon_F}. \tag{48}$$

Differentiating with respect to T

$$C_V \simeq \mu\rho(\mu)\frac{d\mu}{dT} + \frac{\pi^2}{3}k^2 T \frac{d(\varepsilon\rho)}{d\varepsilon}\bigg|_{\varepsilon=\varepsilon_F}. \tag{49}$$

To evaluate $d\mu/dT$ we differentiate Eq. (43) using $dN/dT = 0$

$$0 \simeq \rho(\mu)\frac{d\mu}{dT} + \frac{\pi^2}{3}k^2 T \frac{d\rho}{d\varepsilon}\bigg|_{\varepsilon=\varepsilon_F} \tag{50}$$

and substituting into Eq. (49) and then putting $\mu \simeq \varepsilon_F$ gives

$$C_V \simeq \frac{\pi^2}{3}k^2 T\, \rho(\varepsilon_F). \tag{51}$$

For the free particle spectrum $\varepsilon = \hbar^2 k^2/2m$ and then $\rho(\varepsilon_F) = 3N/2\varepsilon_F$, so that

$$C_V \simeq \frac{1}{2}Nk\pi^2\left(\frac{kT}{\varepsilon_F}\right). \tag{52}$$

Internal Energy Another way to calculate the temperature dependence of the internal energy is to use $\phi(\varepsilon) = (\varepsilon - \varepsilon_F)\rho(\varepsilon)$. This gives

$$U - N\varepsilon_F = \int_0^{\varepsilon_F} (\varepsilon - \varepsilon_F)\rho(\varepsilon) + \int_{\varepsilon_F}^{\mu} (\varepsilon - \varepsilon_F)\rho(\varepsilon) + \frac{\pi^2}{6}(kT)^2 \left.\frac{d[(\varepsilon - \varepsilon_F)\rho]}{d\varepsilon}\right|_{\varepsilon=\varepsilon_F} . \tag{53}$$

The second term is of order $(\mu - \varepsilon_F)^2$ and so is negligible at order T^2, and similarly in the ε derivative we can neglect the term $(\varepsilon - \varepsilon_F)d\rho/d\varepsilon$, so that

$$U = N\varepsilon_F + \int_0^{\varepsilon_F} (\varepsilon - \varepsilon_F)\rho(\varepsilon) + \frac{\pi^2}{6}(kT)^2\rho(\varepsilon_F) \tag{54}$$

and all the temperature dependence is in the second term. For the free, nonrelativistic gas

$$U = \frac{3}{5}N\varepsilon_F\,[1 + \frac{5\pi^2}{12}\left(\frac{kT}{\varepsilon_F}\right)^2 + \cdots]. \tag{55}$$

Other Results Some other results are

$$PV = \frac{2}{3}U = \frac{2}{5}N\varepsilon_F\,[1 + \frac{5\pi^2}{12}\left(\frac{kT}{\varepsilon_F}\right)^2 + \cdots]. \tag{56}$$

The Helmholtz free energy A is given by $A = N\mu - PV$

$$A = \frac{3}{5}N\varepsilon_F\,[1 - \frac{5\pi^2}{12}\left(\frac{kT}{\varepsilon_F}\right)^2 + \cdots]. \tag{57}$$

The entropy is derived as $\partial A/\partial T|_{N,V}$ or from $C_V = T\ \partial S/\partial T|_{N,V}$

$$S = Nk\,\frac{\pi^2}{2}\frac{kT}{\varepsilon_F} + \cdots . \tag{58}$$

Note that the free energy spectrum $\varepsilon = \hbar^2 k^2/2m$ has been used in these expressions: the quadratic temperature dependences could more generally be written in terms of $\rho(\varepsilon_F)$ as in Eq. (51).