Physics 127a: Class Notes

Lecture 15: Statistical Mechanics of Superfluidity

Elementary excitations/quasiparticles

In general, it is hard to list the energy eigenstates, needed to calculate the statistical mechanics of an interacting system. Indeed for a classical liquid, the statistical mechanics is very hard, and little is known except from numerical calculations. For many *quantum* systems at *low temperatures* it is possible to develop the statistical mechanics in terms of a dilute gas of *elementary excitations* or *quasiparticles*. For a translationally invariant system the excited states are labelled by the momentum $\hbar\mathbf{k}$. Then the energy expression of the excited states to be used in e.g. the partition function, is

$$E \simeq E_G + \sum_{\mathbf{k}} n_{\mathbf{k}} \varepsilon_{\mathbf{k}} \tag{1}$$

where $n_{\mathbf{k}} = 0, 1, 2 \cdots$ for Boson excitations (the present case) and $n_{\mathbf{k}} = 0, 1$ for Fermion excitations, and $\varepsilon_{\mathbf{k}}$ is the *excitation spectrum* which is to be determined, empirically or by hard calculation. Although, as in the ideal case, we consider plane wave momentum states, we could also localize the quasiparticles in space by forming wave packets of nearby wave vectors. Then at low temperatures where the number excited is small, they will on average be far apart and weakly interacting. Thus the *excitations* can be treated as an ideal gas, even though the particles themselves are strongly interacting, and terms in Eq. (1) that are higher order (e.g. excitation interaction terms in $n_{\mathbf{k}} n_{\mathbf{k}'}$) can be neglected.

The concept of quasiparticles is rather subtle. It is best approached using the idea of *switching on* the strength of the interactions from the noninteracting state. In the noninteracting state the energy states are certainly given by Eq. (1) with $E_G = 0$, $n_{\mathbf{k}}$ the number of particles transferred from the zero momentum state to the state with momentum $\hbar\mathbf{k}$, and the spectrum $\varepsilon_{\mathbf{k}}^{(0)} = \hbar^2 k^2/2m$. Now switch on the interactions between the particles. In a short time, which we would estimate to be about $\hbar/V$ with V a measure of the interaction strength, the particle at $\hbar\mathbf{k}$ becomes *dressed* by interacting with the surrounding particles, forming the elementary excitation or quasiparticle with the modified the energy spectrum $\varepsilon_{\mathbf{k}}$. Since the total momentum and particle number is conserved by the interactions, these quasiparticles have the *same quantum numbers* as the original ones. The quasiparticles are in one to one correspondence to the original excited particles, and so the *counting* of the states is the same as in the noninteracting system.

The quasiparticles we get by this process are *not* in general true eigenstates of the system: if we wait a longer time, the dressed particle state will decay into a mess of much more complicated states. We replace the difficult (or impossible) task of listing the *precise* eigenstates of the strongly interacting system (which by definition, never decay) by a discussion in terms of *quasi*eigenstates that have simple properties and counting, but have a finite lifetime. This idea only makes sense if the lifetime $\tau_{\mathbf{k}}$ is long enough for the energy of the elementary excitation to be well defined, i.e., by the uncertainty principle, $\hbar/\tau_{\mathbf{k}} \ll \varepsilon_{\mathbf{k}}$. That the quasiparticles satisfy this can be demonstrated empirically by measuring the lifetime of the quasiparticle. It is often true in low temperature quantum systems because there are not many states to decay into—i.e. there is a small *density of states* for the decay process, which reduces the decay rate as predicted by the Fermi Golden Rule.

The idea of quasiparticles is largely due to *Landau*, and is well understood in the context of interacting Fermi liquids, where the ideas can be justified by diagrammatic perturbation theory. Another example we will discuss is phonon excitations of solids.

Excitation spectrum For the weakly interacting Bose gas the energy spectrum is found to be the *Bogoliubov spectrum*

$$\varepsilon_{\mathbf{k}} = \sqrt{(\hbar c k)^2 + (\hbar^2 k^2/2m)^2} \tag{2}$$

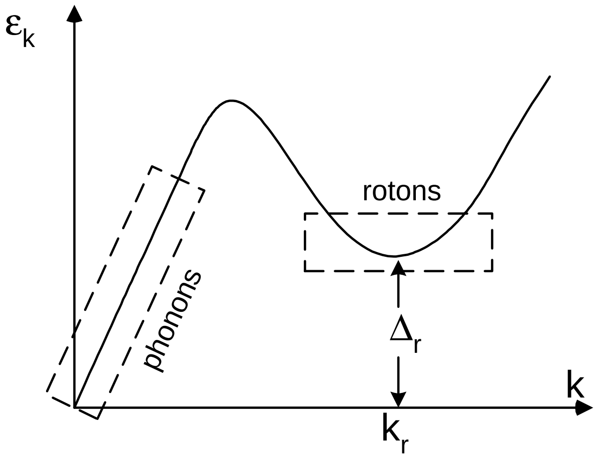

Figure 1: Photon-roton spectrum of He^4. The roton part may be parameterized as $\varepsilon_k = \Delta_r + \hbar^2(k - k_r)^2/2m_r$ with the roton gap $k_B\Delta_r = 8.65K$, the wave number of the roton mimimum $k_r = 0.192$nm, and the roton effective mass $m_r = 0.16m_{He}$.

Here $c = (ng/m)^{1/2}$ with g the interaction strength is the *speed of sound* in the liquid. The difference from the noninteracting spectrum is that at small k (about $\hbar k < mc$) the spectrum is *linear* $\varepsilon_{\mathbf{k}} \simeq (\hbar c)k$,and not quadratic. This turns out to be vital for superfluidity, as we will see later.

For He^4 the excitation spectrum may be measured by neutron scattering (although actually it was derived first by Landau by fitting to the measured thermodynamic quantities using the sort of calculation we will do below), and is sketched in the figure. Again the spectrum is linear at small k, with a slope given by the speed of sound in the liquid. The excitations here are therefore called *phonons*. There is a dip in the spectrum at larger k, at a wave number corresponding to a length scale of about the interatomic spacing. The excitations around the dip are important in the thermodynamics (the energy is relatively low, and there are a large number of states in the vicinity of the minimum). They are called *rotons*. (The name comes from an early model of the nature of the excitations, that is not thought to be very useful anymore.) Feynman presented a nice approximate formula for the spectrum (see his book, *Statistical Mechanics*)

$$\varepsilon_{\mathbf{k}} \simeq \frac{\hbar^2 k^2}{2mS(k)} \tag{3}$$

where $S(k)$ is the *structure factor* of the liquid. The structure factor $S(k)$ determines the X-ray scattering off the liquid, and has a peak at $k \simeq 2\pi/r_{nn}$ where r_{nn} is the nearest neighbor separation (defined precisely as the position of the peak in the density distribution function $g(r)$).

Statistical Mechanics

The canonical partition function is

$$Q_N = \sum_{\{n_{\mathbf{k}}\}} e^{-\beta(E_G + \sum_{\mathbf{k}} n_{\mathbf{k}} \varepsilon_{\mathbf{k}})} \tag{4}$$

$$= e^{-\beta E_G} \prod_{\mathbf{k}} \sum_{n=0}^{\infty} e^{-\beta n \varepsilon_{\mathbf{k}}} = e^{-\beta E_G} \prod_{\mathbf{k}} (1 - e^{-\beta \varepsilon_{\mathbf{k}}})^{-1}. \tag{5}$$

Note that the excitations of the noninteracting system are obtained by *moving* a particle from the zero momentum condensate to a finite momentum state, so that the total particle number is unchanged. Thus we can readily calculate in the canonical ensemble.

The Helmholtz free energy $A = -kT \ln Q_N$ is

$$A = E_G + kT \ln\left(1 - e^{-\beta \varepsilon_{\mathbf{k}}}\right). \tag{6}$$

The internal energy $U = -\partial \ln Q_N / \partial \beta$ is

$$U = E_G + \sum_{\mathbf{k}} \varepsilon_{\mathbf{k}} \frac{1}{e^{\beta \varepsilon_{\mathbf{k}}} - 1}. \tag{7}$$

The finite temperature correction to the ground state energy is given by the average excitation energy, with the number distribution given by the Bose distribution (with zero μ because the excitations do not change the particle number). You can readily check by transforming the sum over **k** to an integral in the usual way, that for the noninteracting Bose gas with $\varepsilon_{\mathbf{k}} = \hbar k^2/2m$ at low temperatures the internal energy is $U \propto T^{5/2}$, so that the specific heat $C \propto T^{3/2}$ as found before.

Low temperature specific heat At low temperatures only the low energy portion of the spectrum contributes, which for both the weakly interacting Bose gas and He^4 is $\varepsilon_{\mathbf{k}} = \hbar c k$. The internal energy is then (transforming the sum $\sum_{\mathbf{k}}$ to an integral)

$$U = \frac{V}{2\pi^2} \int_0^{\infty} dk\, k^2\, \hbar c k \frac{1}{e^{\beta \hbar c k} - 1} \tag{8}$$

$$= V \frac{(kT)^4}{(\hbar c^3)} \left[\frac{1}{2\pi^2} \int_0^{\infty} dx\, \frac{x^3}{e^x - 1} \right]. \tag{9}$$

The quantity in the braces evaluates to $\pi^2/30$, so the low temperature specific heat is

$$\frac{C}{Nk} = \frac{2}{15} \frac{N}{V} \left(\frac{kT}{\hbar c} \right)^3. \tag{10}$$

For higher temperatures, the specific heat in Helium is dominated by the rotons, giving a specific heat varying roughly as $e^{-\Delta_r/kT}$.

Superfluidity

In superfluid He^4, a circulating flow in a torus continues indefinitely, without dissipation. We want to understand this, and the connection with Bose-Einstein Condensation.

Ideal Bose Gas Consider first flow states in the ideal Bose Gas at zero temperature when all the particles are condensed into a single state. We model the torus as a length L with periodic boundary conditions. A flowing state is given by Bose condensing into a single particle state with momentum $\hbar\mathbf{k}$ with $\mathbf{k} = (n2\pi/L, 0, 0)$ with n integral. The many particle wavefunction is

$$\Psi(\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_N) = \prod_{i=1}^{N} \psi(\mathbf{x}_i) \qquad \text{with} \qquad \psi(\mathbf{x}) = \frac{1}{\sqrt{V}} e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{11}$$

The total momentum is $\mathbf{P} = N\hbar\mathbf{k}$, which we can write in terms of a momentum density $\mathbf{g} = \mathbf{P}/V = \rho_s\mathbf{v}_s$ with $\mathbf{v}_s = \hbar\mathbf{k}/m$ the velocity and ρ_s called the superfluid density, which here is the total density $\rho_s = \rho = Nm/V$. The "condensate wavefunction" $\psi(\mathbf{x})$—the single particle wavefunction of the Bose condensation—can be written in magnitude-phase form $\psi = |\psi|\, e^{i\phi}$ and then the superfluid velocity is given by the gradient of the phase

$$\mathbf{v}_s = \frac{\hbar}{m}\nabla\phi. \tag{12}$$

This allows the results to be generalized to a more complicated flow fields that correspond to a $\psi(\mathbf{x})$ that is not simply a plane wave. Note that since the value of $\mathbf{k}$ is quantized, so to is the *circulation* (the line integral of the velocity around a closed loop)

$$\oint \mathbf{v}_s \cdot \mathbf{dl} = n \times \frac{h}{m} \tag{13}$$

with h/m the *quantum of circulation*. This is generally true, and follows from Eq. (12) and the single valuedness of the quantum wavefunction (ϕ increases by multiples of 2π around a loop).

Interacting Bose System For the interacting system at zero temperature the many particle wavefunction is *not* simply a product state, and in general it is too hard to calculate exactly. We can again learn about *carefully chosen* aspects of the interacting system by the notion of *switching on* the interactions from the noninteracting state (sometimes called *continuation*). The idea is that we set up a situation of interest in the noninteracting system, and then imagine the physical processes that occur when we slowly switch on the interaction (turn a "knob" that increases the interparticle potential). Of particular interest are the *conserved quantities* that cannot change during the switching on.

We do this for the flowing noninteracting state of the last section. As we turn up the interparticle potential, the many particle state may change by interparticle scattering processes that take particles out of the state $\hbar\mathbf{k}$ into other momentum states: this *reduces* the condensate fraction, but since the *interparticle interaction* conserves the total momentum (two particles from the state $\mathbf{k}$ must scatter into $\mathbf{k} + \mathbf{q}$ and $\mathbf{k} - \mathbf{q}$), the *total momentum of the system remains* $\mathbf{P} = N\hbar\mathbf{k}$. Thus in the interacting system we again have a flow state with momentum density $\mathbf{g} = \rho_s\mathbf{v}_s$ with $\rho_s = \rho$ and $\mathbf{v}_s = \hbar\mathbf{k}/m$ as before, even though the condensate fraction $f = N_{\mathbf{k}}/N$ is reduced from unity.

Another way of understanding the flow state in the interacting system is to start from the interacting system at rest, and then "run along" at a velocity $-\mathbf{v}_s$: the Galilean transformation to the runners rest frame gives in this new frame of reference precisely the state we have just described.

Now we must determine if this flow state persists, even in the presence of interactions with fixed bodies that do not necessarily preserve the total momentum (the walls of the container or a porous medium often used in experiment): i.e. is it superfluid.

There are two ways the momentum $\mathbf{g} = \rho_s\mathbf{v}_s$ may decay: $\mathbf{v}_s$ may decrease; or ρ_s may decrease from the value ρ. The first cannot happen continuously, since the circulation is quantized. This means v_s can only change in jumps of $(h/m)L^{-1}$, which can only occur through macroscopic events that will not occur in large systems. This leaves us with the question of whether scattering off the walls etc. can continuously reduce ρ_s to zero. In the noninteracting flow state this is indeed possible, since a particle in the $\hbar\mathbf{k}$ state can

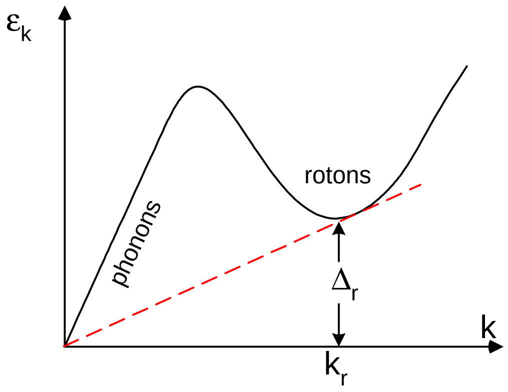

Figure 2: The slope of the dashed line—the shallowest tangent to the excitation curve—is $\hbar v_c$ defining the Landau critical velocity for superflow v_c. For He^4 the construction gives $v_c \simeq \Delta_r/\hbar k_r$ which is about 60m s^{-1}.

be scattered to the zero momentum state reducing the total energy by $\hbar^2 k^2/2m$. In the interacting system we might expect that the scattering will occur *if there are excitations from the flowing state that lower the total energy*. This is now easy to calculate. Using arguments of Galilean transformation the energy of an excitation of momentum $\hbar\mathbf{k}$ from the flowing state (which has energy $E_G + \frac{1}{2}\rho v_s^2$ with E_G the nonflowing ground state energy) is

$$\varepsilon'_{\mathbf{k}} = \varepsilon_{\mathbf{k}} + \hbar\mathbf{k}\cdot\mathbf{v}_s \tag{14}$$

with $\varepsilon_{\mathbf{k}}$ the excitation energy from the nonmoving state discussed above. For the flow state to persist, we must have $\varepsilon'_{\mathbf{k}} > 0$ which requires

$$\varepsilon_k > \hbar k v_s \qquad \text{for any } k. \tag{15}$$

Thus flows up to a critical velocity v_c given by the $\hbar^{-1}$ times the *slope of the lowest tangent to the* ε_k *excitation curve are superfluid*. This is known as the *Landau critical velocity*. For the weakly interacting Bose gas $v_c = c$, and for superfluid He^4 $v_c \simeq \Delta_r/\hbar k_r \sim 60$m s^{-1}.

Superflow at nonzero temperature At nonzero temperature the thermally excited excitations may redistribute and change the total momentum. Consider the situation where the condensed state flows at the superfluid velocity $\mathbf{v}_s$, and the walls of the container are moving with velocity $\mathbf{v}_n$. The walls provide a *momentum bath*. Equilibrium in contact with a momentum bath moving at velocity $\mathbf{v}$ is given by the Boltzmann factor $e^{-\beta(E-\mathbf{P}\cdot\mathbf{v})}$. Thus the number of excitations at $\hbar\mathbf{k}$ with excitation energy $\varepsilon'_{\mathbf{k}} = \varepsilon_{\mathbf{k}} + \hbar\mathbf{k}\cdot\mathbf{v}_s$ is

$$n_{\mathbf{k}} = \frac{1}{e^{\beta(\varepsilon'_{\mathbf{k}} - \hbar\mathbf{k}\cdot\mathbf{v}_n)}} = \frac{1}{e^{\beta[\varepsilon_{\mathbf{k}} + \hbar\mathbf{k}\cdot(\mathbf{v}_s - \mathbf{v}_n)]}} = n_B(\varepsilon_k + \hbar\mathbf{k}\cdot(\mathbf{v}_s - \mathbf{v}_n)) \tag{16}$$

with $n_B(\varepsilon)$ the Bose function $(e^{\beta\varepsilon} - 1)^{-1}$. The total momentum is the momentum of the flowing ground state plus the momentum of the excitations

$$\mathbf{P} = Nm\mathbf{v}_s + \sum_{\mathbf{k}} \hbar\mathbf{k}\, n_B(\varepsilon_k + \hbar\mathbf{k}\cdot(\mathbf{v}_s - \mathbf{v}_n)). \tag{17}$$

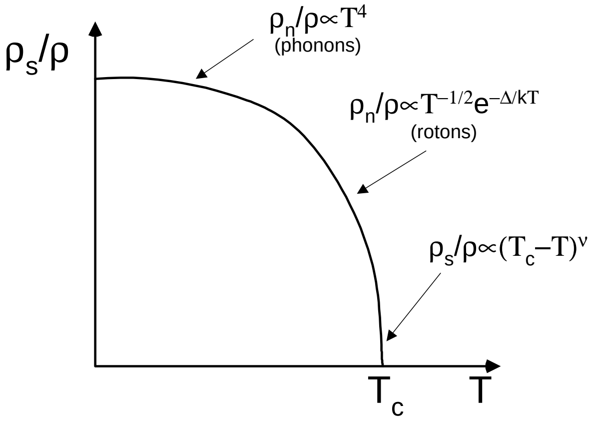

Figure 3: Superfluid density of He^4. Three regions of behavior can be roughly identified: at low temperatures ρ_n is dominated by the phonon region of the spectrum giving $\rho_n \propto T^4$; at intermediate temperatures rotons are important; near T_c critical fluctuations dominate, and universality tells us ρ_s goes to zero as $(T_c - T)^\nu$ with $\nu \simeq 0.669$

For small $\mathbf{v}_s - \mathbf{v}_n$ we can Taylor expand the n_B term. The zeroth order term gives zero since $n_B(\varepsilon_k)$ is spherically symmetric, leaving

$$\mathbf{P} \simeq Nm\mathbf{v}_s + \sum_{\mathbf{k}} \hbar\mathbf{k} \left(-\frac{\partial n_B}{\partial \varepsilon_k}\right) \hbar\mathbf{k} \cdot (\mathbf{v}_n - \mathbf{v}_s) \tag{18}$$

$$= Nm\mathbf{v}_s + \left[\frac{1}{3}\sum_{\mathbf{k}} \hbar^2 k^2 \left(-\frac{\partial n_B}{\partial \varepsilon_k}\right)\right] (\mathbf{v}_n - \mathbf{v}_s) \tag{19}$$

on doing the angular average. Thus we can write the momentum density

$$\mathbf{g} = \frac{\mathbf{P}}{V} = \rho\mathbf{v}_s + \rho_n(\mathbf{v}_n - \mathbf{v}_s) \tag{20}$$

with the *normal fluid density*

$$\rho_n = \frac{1}{3V}\sum_{\mathbf{k}} \hbar^2 k^2 \left(-\frac{\partial n_B}{\partial \varepsilon_k}\right). \tag{21}$$

Alternatively we can write this as

$$\mathbf{g} = \rho_s\mathbf{v}_s + \rho_n\mathbf{v}_n \tag{22}$$

with $\rho_s = \rho - \rho_n$. Note that at zero temperature $\rho_s = \rho$, and *is not related to the condensate fraction*. As the temperature increases ρ_n increases, and we can identify the transition temperature T_c as where $\rho_s \to 0$, $\rho_n \to \rho$. The temperature dependence of ρ_s for He^4 is plotted in the figure. Note that near T_c the excitations are no longer dilute and Eq. (21) for ρ_n no longer applies. Arguments based on the *universality* of the behavior near second order phase transitions show that ρ_s goes to zero as a *power law* $\rho_s \propto (T_c - T)^\nu$ with $\nu \simeq 0.669$ (close, but not equal, to $2/3$).

Thus we can think of the superfluid state in two ways:

1. A "flowing ground state" with momentum $\rho\mathbf{v}_s$ together with excitations giving momentum $\rho_n(\mathbf{v}_n - \mathbf{v}_s)$

2. A *two fluid model* where the fluid appears to behave as having two components, a *superfluid* component with "density" ρ_s and velocity $\mathbf{v}_s$ and a *normal fluid* component with "density" ρ_n and velocity $\mathbf{v}_n$. (The densities are in quotes, because they relate momentum to velocity, and do not determine the mass.) With a little more work we could also calculate the total energy of the flowing state. It is found to be $E = \frac{1}{2}\rho_s v_s^2 + \frac{1}{2}\rho_n v_n^2$, also consistent with this two fluid way of thinking. The "normal fluid" is called normal, since it is made up of a gas of excitations which can be scattered, leading to the usual properties of a fluid such as viscosity: the normal fluid will come into equilibrium with the container walls; the superfluid velocity persists independently.

Fundamental Description The most fundamental description of the superfluid state is that there is Bose condensation into a condensate wavefunction $\psi(\mathbf{x}) = |\psi| e^{i\phi}$. The superfluid velocity is $\mathbf{v}_s = (\hbar/m)\nabla\phi$, and the energy in the rest frame of the momentum bath is

$$E = E_G + \frac{1}{2}\rho_s(\hbar/m)^2(\nabla\phi)^2. \tag{23}$$

The parameter ρ_s is a *stiffness constant* that gives the energy cost of a nonuniform condensate wavefunction. It is quite analogous to the elastic constants of a crystal or the spin-wave stiffness of a magnet. The superfluid momentum is $\mathbf{g}_s = (\partial E/\partial \mathbf{v}_s) = \rho_s \mathbf{v}_s$. We have estimated ρ_s based on a theory of a dilute gas of excitations. The momentum in the presence of walls (a momentum bath) with velocity $\mathbf{v}_n$ is given by a Galilean boost

$$\mathbf{g} = \rho_s \mathbf{v}_s + \rho \mathbf{v}_n = \rho_s \mathbf{v}_s' + \rho_n \mathbf{v}_n \tag{24}$$

with $\mathbf{v}_s' = \mathbf{v}_s + \mathbf{v}_n$ the superfluid velocity in the new frame of reference.