Physics 127a: Statistical Mechanics

Diamagnetism of the Electron Gas

The Hamiltonian coupling the electron current to the magnetic field $\vec{B}$ is

$$H = \sum_i \frac{1}{2m}[\vec{p}_i + \frac{e}{c}\vec{A}(\vec{x}_i)]^2 \tag{1}$$

summing over the electrons i with position $\vec{x}_i$ and momentum $\vec{p}_i$. (I will consider "spinless Fermions" so that I can ignore the paramagnetic terms.) For a uniform field $\vec{B} = B\hat{z}$ it is convenient to choose the vector potential $\vec{A} = (-By, 0, 0)$.

In the grand canonical approach we sum over single particle energy levels, so we first want to understand the one-particle Hamiltonian:

$$H_1 = \frac{1}{2m}\left[(p_x - \frac{eB}{c}y)^2 + p_y^2 + p_z^2\right]. \tag{2}$$

Incidentally, from this Hamiltonian we can immediately see that the canonical partition function for the classical gas is *independent of the magnetic field*, since B is eliminated by shifting the p_x integration variable to $\bar{p}_x = p_x - (eB/c)y$. This is known as van Leeuwen's theorem.

The one particle eigenstates are given by solving $H\psi = \varepsilon\psi$. Since the Hamiltonian is independent of the x, z coordinates the eigenfunctions are plane waves in this direction, and we can write the wave function as

$$\psi = e^{i(k_x x + k_z z)} f(y) \tag{3}$$

where $f(y)$ must satisfy the one-dimensional Schrodinger equation (by substitution)

$$\left[\frac{1}{2m}p_y^2 + \frac{1}{2}m\omega_0^2(y - y_0)^2\right]f(y) = \varepsilon' f(y) \tag{4}$$

with $\hbar\omega_0 = e\hbar B/mc = 2\mu_B B$ the cyclotron frequency and $\mu_B = e\hbar/2mc$ the Bohr magneton, and y_0 a constant determined by the value of the x-momentum, $y_0 = (\hbar c/eB)k_x$. The one particle energy is

$$\varepsilon = \varepsilon' + \frac{\hbar^2 k_z^2}{2m}. \tag{5}$$

Equation (4) is Schrodinger's equation for a simple harmonic oscillator, so we can immediately find the energies

$$\varepsilon' = (j + \frac{1}{2})\hbar\omega_0 \quad \text{with } j \text{ a positive integer.} \tag{6}$$

The eigenfunctions $f(y)$ are Hermite polynomials in $y - y_0$, which are Gaussians multiplied by a polynomial. Thus the wave functions are localized around y_0. The energy levels of the full problem depend on k_z and j

$$\varepsilon(j, k_z) = (j + \frac{1}{2})\hbar\omega_0 + \frac{\hbar^2 k_z^2}{2m}. \tag{7}$$

The discrete energy levels for the motion transverse to the field are known as Landau levels.

Since the energies are independent of k_x, the levels are degenerate, and to do the counting we need to find the degeneracy factor. As usual we imagine the system in a finite box with periodic boundary conditions over distances L. The allowed values of k_x are then $l(2\pi/L)$ with l integral. Since k_x determines the center of the y-wavefunctions

y_0, and increasing y_0 by L will, by the periodic boundary conditions, simply duplicate an already-counted eigenstate, the finite size in the y direction yields constraints on l

$$0 < l < \frac{eB}{hc}L^2. \tag{8}$$

The energy is independent of k_x and this gives the degeneracy factor $g = (eB/hc)L^2$. This counting of the degeneracy is a little suspect: what happens, for example, if L is not a multiple of hc/eB? More sophisticated arguments give the same result for the large L we are interested in, however.

The solution to the quantum problem seems rather strange, since classically we expect the electrons to be in circular orbits, i.e. oscillating in both the x and y directions. The harmonic oscillator frequency ω_0 *does* correspond to the classical rotation frequency, so this is comforting. The strangeness of the quantum results can partially be understood from the degeneracy: we are free to take any orthonormal set given by linear combinations of the degenerate states, and we have chosen a combination that is convenient for counting the states, if not very intuitive

We can now calculate the grand canonical potential by summing over the single particle states in the usual way

$$-\frac{\Omega}{kT} = \int_{-\infty}^{\infty} \frac{L}{2\pi\hbar} dp_z \sum_{j=0}^{\infty} \left(\frac{eB}{hc}L^2\right) \ln\left\{1 + z\exp\left[-\beta\left(2\mu_B B(j+1/2) + \frac{p_z^2}{2m}\right)\right]\right\} \tag{9}$$

with $z = e^{\beta\mu}$ the fugacity. For small B the sum can be evaluated using the Euler summation formula (e.g. see *Handbook of Mathematical Function* by Abramowitz and Stegun, §23.1.32, p806 in my edition)

$$\sum_{j=0}^{\infty} f(j+1/2) = \int_0^{\infty} f(x)\,dx + \frac{1}{24}f'(0) + \cdots \tag{10}$$

where the $\cdots$ will involve higher derivative of f which in our case will bring down extra factors of $e\hbar B/mc$. This gives (also collecting constants and introducing the Bohr magneton $\mu_B = e\hbar/2mc$)

$$-\frac{\Omega}{kT} \simeq \frac{VeB}{h^2c}\left\{\int_{-\infty}^{\infty} dp_z \int_0^{\infty} dx \ln\left\{1 + z\exp\left[-\beta\left(2\mu_B Bx + \frac{p_z^2}{2m}\right)\right]\right\}\right.$$
$$\left. -\frac{1}{12}\beta\mu_B B \int_{-\infty}^{\infty} dp_z \frac{1}{e^{\beta(p_z^2/2m-\mu)}+1}\right\} \tag{11}$$

Remarkably, the first term in Eq. (11), which is the classical result, is independent of B, e, c, as can be seen by introducing the integration variable $y = 2\mu_B Bx$, and is an obscure way of writing the expression for the grand potential of a Fermi gas in zero field. The second term gives a B^2 contribution from which we can calculate the susceptibility

$$\chi = -\frac{1}{V}\left[\frac{1}{B}\frac{\partial\Omega}{\partial B}\right]_{B=0} = -\frac{1}{6}\mu_B^2 \frac{4\pi m}{h^3} \int_{-\infty}^{\infty} dp_z \frac{1}{e^{\beta(p_z^2/2m-\mu)}+1}. \tag{12}$$

For high temperatures

$$\chi \to -\frac{1}{6}\mu_B^2 \frac{4\pi m}{h^3} e^{\beta\mu} \int_{-\infty}^{\infty} dp_z\, e^{-\beta p_z^2/2m}. \tag{13}$$

The integral is just $\sqrt{2\pi m k_B T}$ and using the classical-gas expression for μ gives (with $n = N/V$)

$$\chi \to -\frac{n\mu_B^2}{3k_B T}. \tag{14}$$

At low temperatures the Fermi function is unity for $-p_F < p_z < p_F$ and zero otherwise, so that

$$\chi \to -\frac{1}{6}\mu_B^2 \frac{4\pi m}{h^3} 2p_F = -\frac{n\mu_B^2}{2\varepsilon_F}. \tag{15}$$

Notice χ is negative, corresponding to diamagnetism (known as Landau diamagnetism). For both high and low temperatures, χ is of comparable size, but of opposite sign, to the Pauli paramagnetic susceptibility coming from the coupling of the field to the spins. Both susceptibilities are proportional to $\hbar^2$, and so disappear in the classical limit, in agreement with the van Leeuwen's theorem.