Pattern Formation in Spatially Extended Systems

Lecture 4

Chaos

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Chaos

with emphasis on Rayleigh-Bénard convection

Outline

- Small system chaos
  - ◇ Introduction
  - ◇ Ideas of chaos apply to continuum systems
- Large system chaos: spatiotemporal chaos
  - ◇ Definition and characterization
  - ◇ Transitions to spatiotemporal chaos and between different chaotic states
  - ◇ Coarse grained descriptions

Some Theoretical Highlights

Landau (1944) Turbulence develops by infinite sequence of transitions adding additional temporal modes and spatial complexity

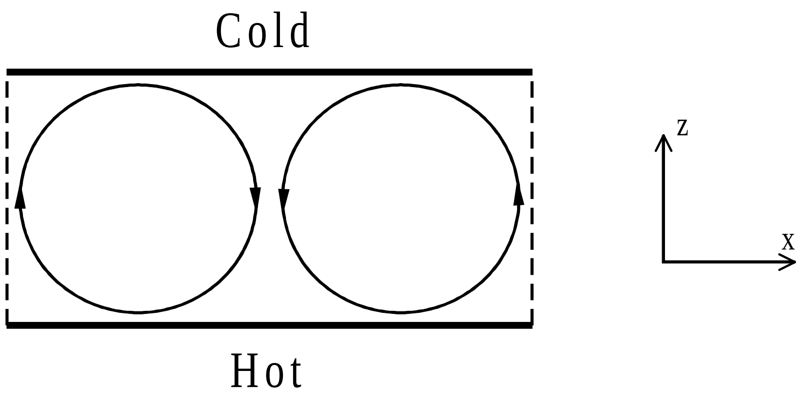
Lorenz (1963) Discovered chaos in simple model of convection

Ruelle and Takens (1971) Suggested the onset of aperiodic dynamics from a low dimensional torus (quasiperiodic motion with a small number N frequencies)

Feigenbaum (1978) Quantitative universality for period doubling route to chaos

...

Lorenz Chaos



$$T(x,z,t) \simeq -rz + 9\pi^3\sqrt{3}Y(t)\cos(\pi z)\cos\left(\frac{\pi}{\sqrt{2}}x\right) + \frac{27\pi^3}{4}Z(t)\sin(2\pi z)$$

$$\psi(x,z,t) = 2\sqrt{6}X(t)\cos(\pi z)\sin\left(\frac{\pi}{\sqrt{2}}x\right)$$

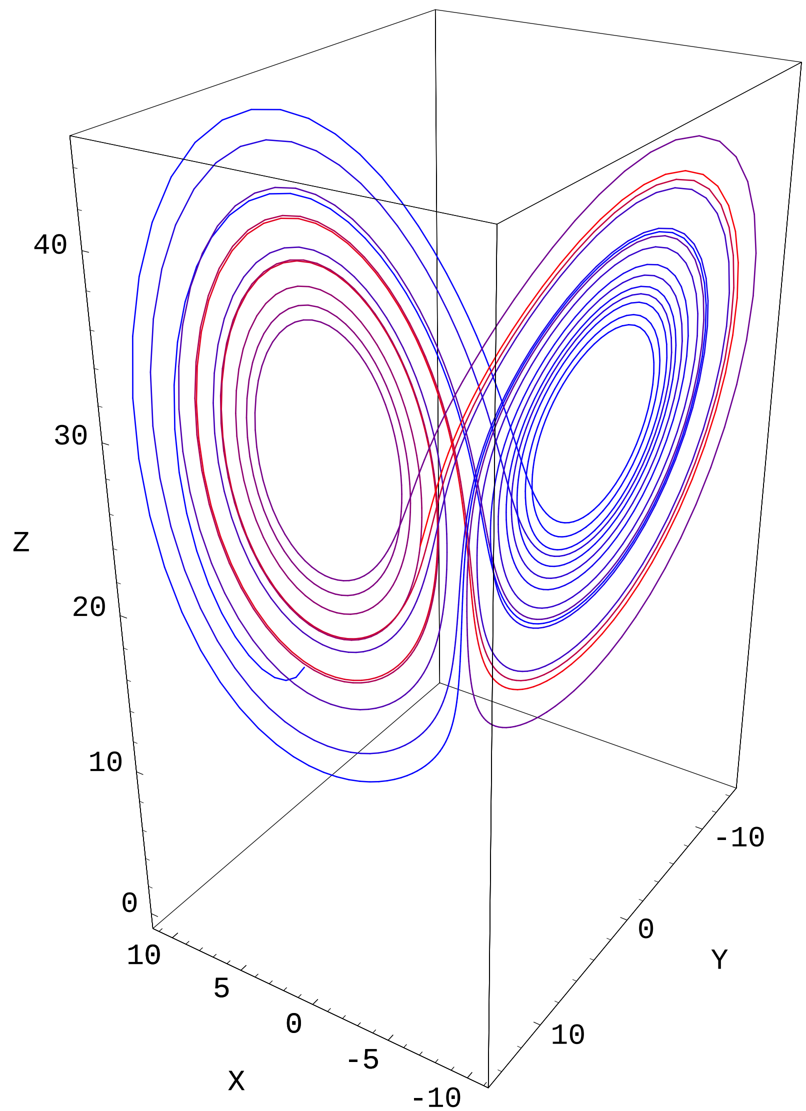
$$(u = -\partial\psi/\partial z,\ v = \partial\psi/\partial x,\ r = R/R_c)$$

Lorenz Model

$$\begin{aligned} \dot{X} &= -\sigma(X - Y) \\ \dot{Y} &= rX - Y - XZ \\ \dot{Z} &= b(XY - Z) \end{aligned}$$

$b = 8/3$ and σ is the Prandtl number.

“Classic” values are $\sigma = 10$ and $r = 27$.


40
30
Z
20
10
0
10
5
0
-5
X
-10
10
0
Y
-10

The Butterfly Effect

The "sensitive dependence on initial conditions" found by Lorenz is often called the "butterfly effect".

In fact Lorenz first said (Transactions of the New York Academy of Sciences, 1963)

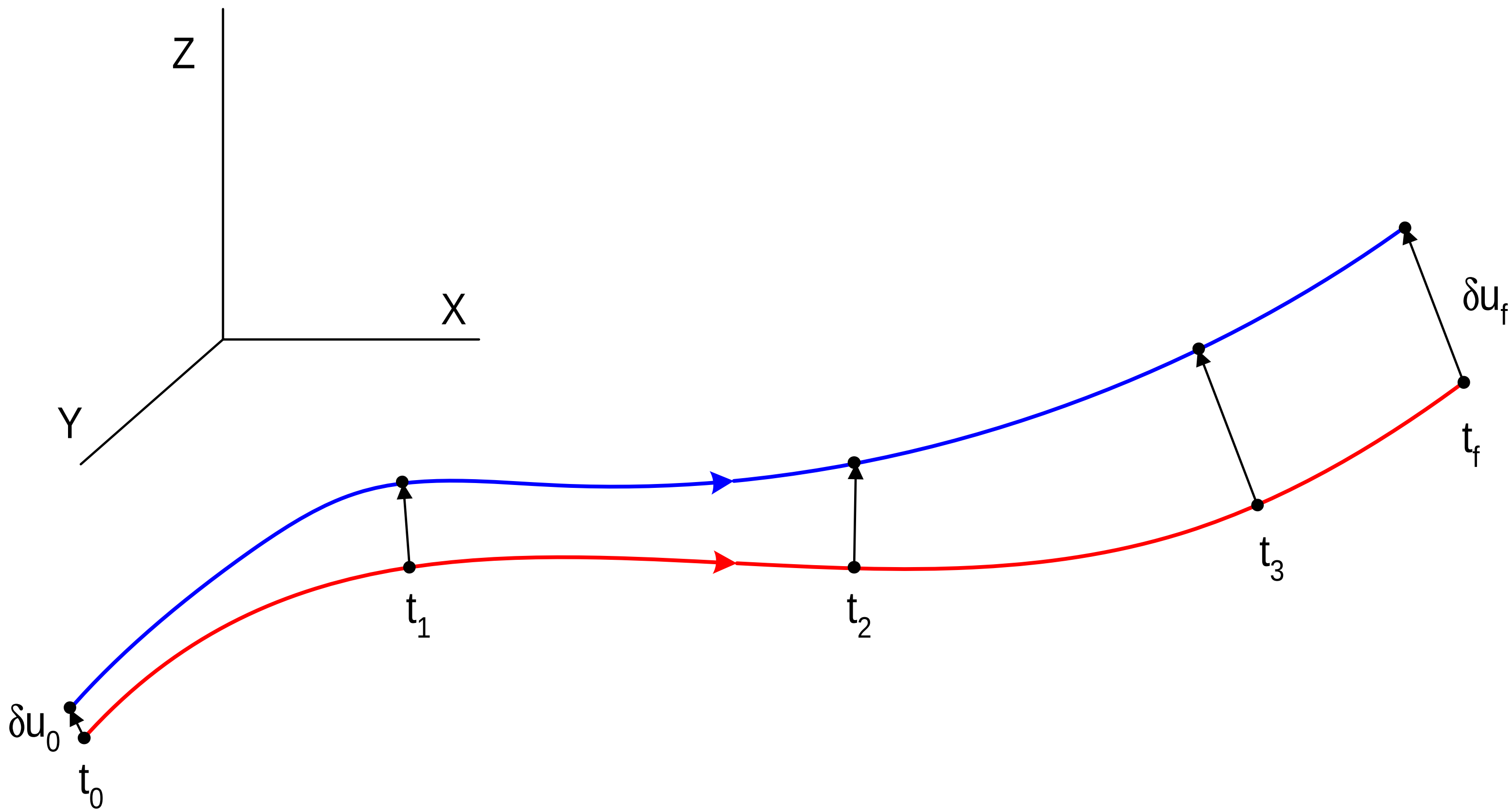
One meteorologist remarked that if the theory were correct, one flap of the sea gull's wings would be enough to alter the course of the weather forever.

By the time of Lorenz's talk at the December 1972 meeting of the American Association for the Advancement of Science in Washington, D.C. the sea gull had evolved into the more poetic butterfly - the title of his talk was

Predictability: Does the Flap of a Butterfly's Wings in Brazil set off a Tornado in Texas?

Lyapunov Exponents and Eigenvectors

Quantifying the sensitive dependence on initial conditions



exponent: $\lambda = \lim_{t_f \to \infty} \frac{1}{t_f - t_0} \ln \left| \frac{\delta \mathbf{u}_f}{\delta \mathbf{u}_0} \right|$; $\delta \mathbf{u}_f \to$ eigenvector

Dimension of the Attractor

- The fractal dimension of the attractor quantifies the number of chaotic degrees of freedom.
- There are many possible definitions. Most are inaccessible to experiment and numerics for high dimensional attractors.
- I will discuss the *Lyapunov dimension* which is conjectured to be the same as the *information dimension*

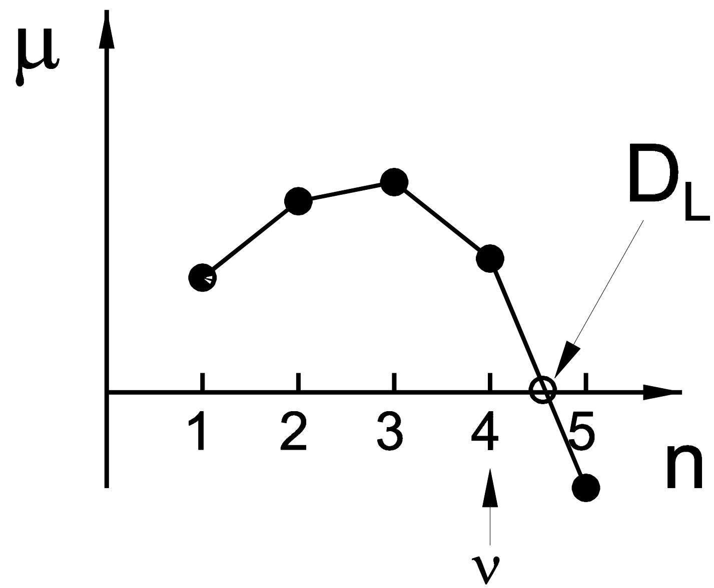
Line lengths $\rightarrow e^{\lambda_1 t}$, Areas $\rightarrow e^{(\lambda_1+\lambda_2)t}$, Volumes $\rightarrow e^{(\lambda_1+\lambda_2+\lambda_3)t}$, ...

Lyapunov Dimension:

$$D_L = \nu + \frac{1}{|\lambda_{\nu+1}|} \sum_{i=1}^{\nu} \lambda_i$$

where ν is the largest index such that the sum is positive.

Lyapunov dimension



Define $\mu(n) = \sum_{i=1}^{n} \lambda_i \quad (\lambda_1 \geq \lambda_2 \cdots)$ with λ_i the ith Lyapunov exponent.

D_L is the interpolated value of n giving $\mu = 0$ (the dimension of the volume that neither grows nor shrinks under the evolution)

Numerical Approach

Chaotic Boussinesq driving solution:

$$\sigma^{-1}\left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\,\mathbf{u}\right] = -\nabla P + RT\hat{\mathbf{z}} + \nabla^2\mathbf{u}$$

$$\frac{\partial T}{\partial t} + (\mathbf{u}\cdot\nabla)\,T = \nabla^2 T$$

$$\nabla\cdot\mathbf{u} = 0$$

Linearized equations (tangent space equations):

$(\mathbf{u}, P, T) \to (\mathbf{u}+\delta\mathbf{u}_k, P+\delta P_k, T+\delta T_k)$, for $k = 1, \ldots, n$

$$\sigma^{-1}\left[\frac{\partial \delta\mathbf{u}_k}{\partial t} + (\mathbf{u}\cdot\nabla)\,\delta\mathbf{u}_k + (\delta\mathbf{u}_k\cdot\nabla)\,\mathbf{u}\right] = -\nabla\delta P_k + R\delta T\hat{e}_z + \nabla^2\delta\mathbf{u}_k$$

$$\frac{\partial \delta T_k}{\partial t} + (\mathbf{u}\cdot\nabla)\,\delta T_k + (\delta\mathbf{u}_k\cdot\nabla)\,T = \nabla^2\delta T_k$$

$$\nabla\cdot\delta\mathbf{u}_k = 0$$

Small system chaos: some experimental highlights

The Lorenz model does not describe Rayleigh-Bénard convection. However the ideas of low dimensional models *do* apply to convection, fluids and other continuum systems.

Ahlers (1974) Transition from time independent flow to aperiodic flow at $R/R_c \sim 2$ (aspect ratio 5)

Gollub and Swinney (1975) Onset of aperiodic flow from time-periodic flow in Taylor-Couette

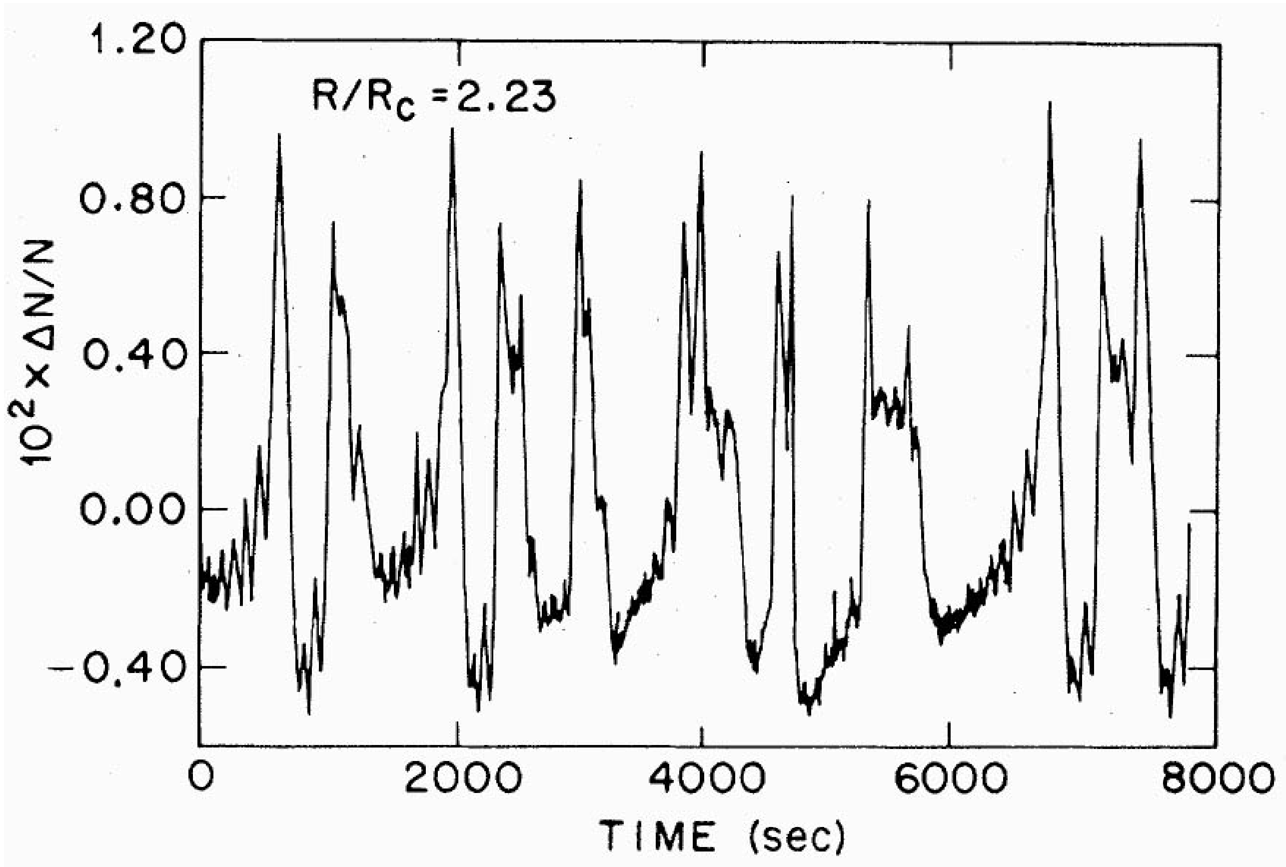
Maurer and Libchaber, Ahlers and Behringer (1978) Transition from quasiperiodic flow to aperiodic flow in small aspect ratio convection

Lichaber, Laroche, and Fauve (1982) Quantitative demonstration of the Fiegenbaum period doubling route to chaos

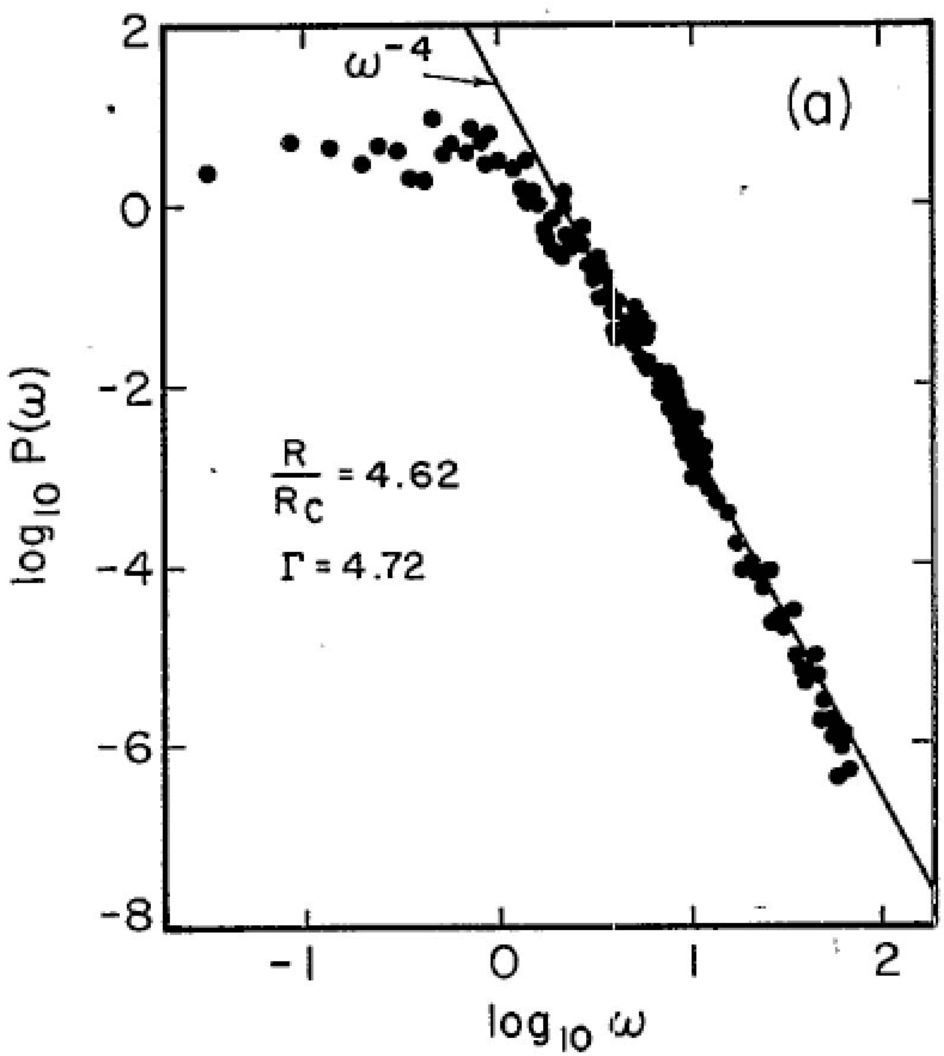
Revisit Ahlers (1974), Ahlers and Behringer (1978)

- First experiments: $\Gamma = 5.27$ cell, cryogenic (normal) liquid He^4 as fluid. High precision heat flow measurements (no flow visualization).
- Onset of aperiodic time dependence in low Reynolds number flow: relevance of chaos to "real" (continuum) systems.
- Power law decrease of power spectrum $P(f) \sim f^{-4}$
- Aspect ratio dependence of the onset of time dependence (Ahlers and Behringer, 1978)

Γ	2	5	57
R_t	$10R_c$	$2R_c$	$1.1R_c$


R/R$_c$ =2.23
1.20
0.80
0.40
0.00
−0.40
10^2 x ΔN/N
0
2000
4000
6000
8000
TIME (sec)

(Ahlers 1974)

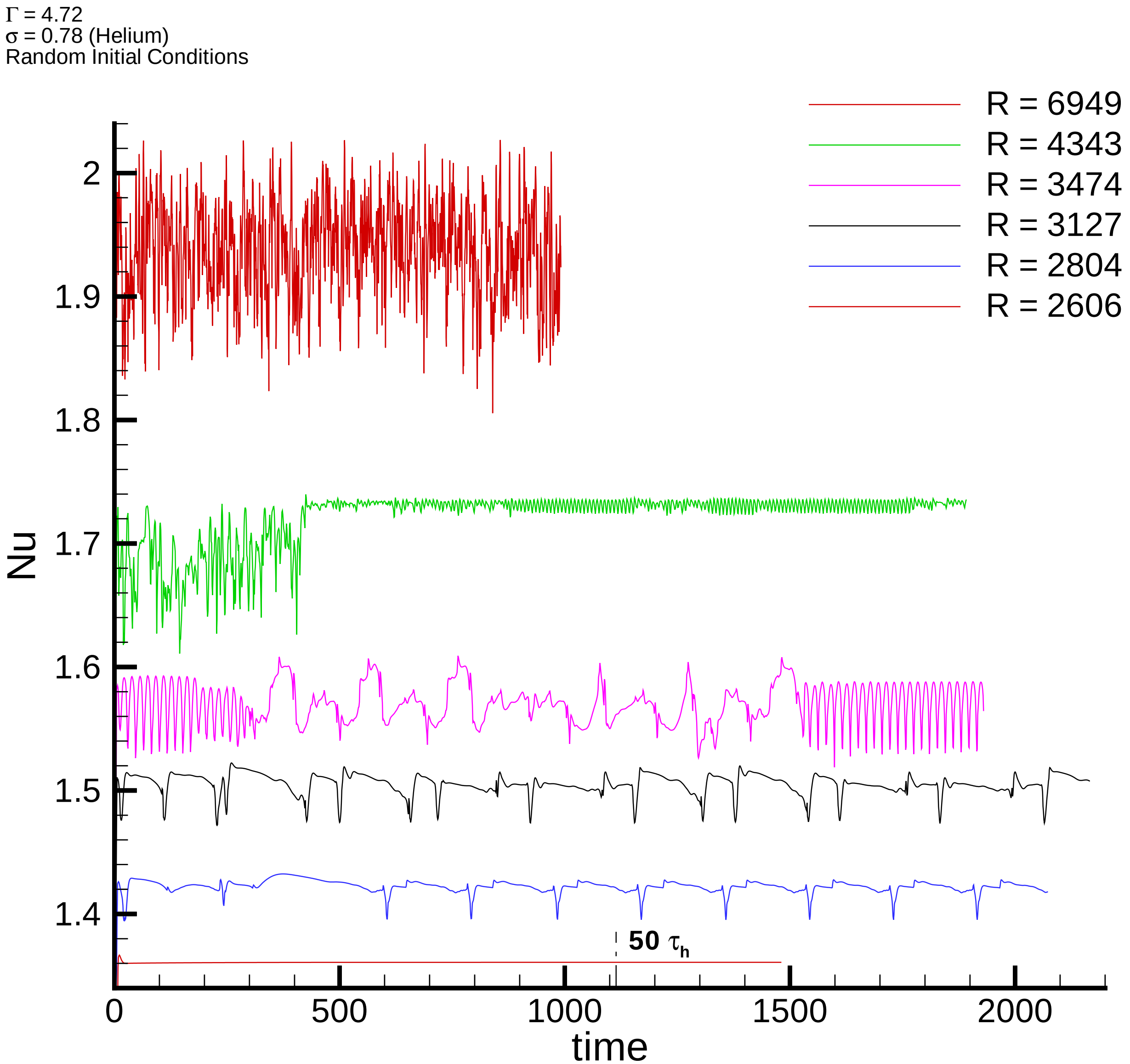


(Ahlers and Behringer 1978)

Numerical Simulations

(Paul, MCC, Fischer and Greenside)

- $\Gamma = 4.72$, $\sigma = 0.78$, $2600 \lesssim R \lesssim 7000$
- Conducting sidewalls
- Random thermal perturbation initial conditions
- Simulation time $\sim 100\tau_h$
  - Simulation time $\sim$ 12 hours on 32 processors
  - Experiment time $\sim$ 172 hours or $\sim$ 1 week


Γ = 4.72
σ = 0.78 (Helium)
Random Initial Conditions
R = 6949
R = 4343
R = 3474
R = 3127
R = 2804
R = 2606
Nu
2
1.9
1.8
1.7
1.6
1.5
1.4
50 τh
0
500
1000
1500
2000
time

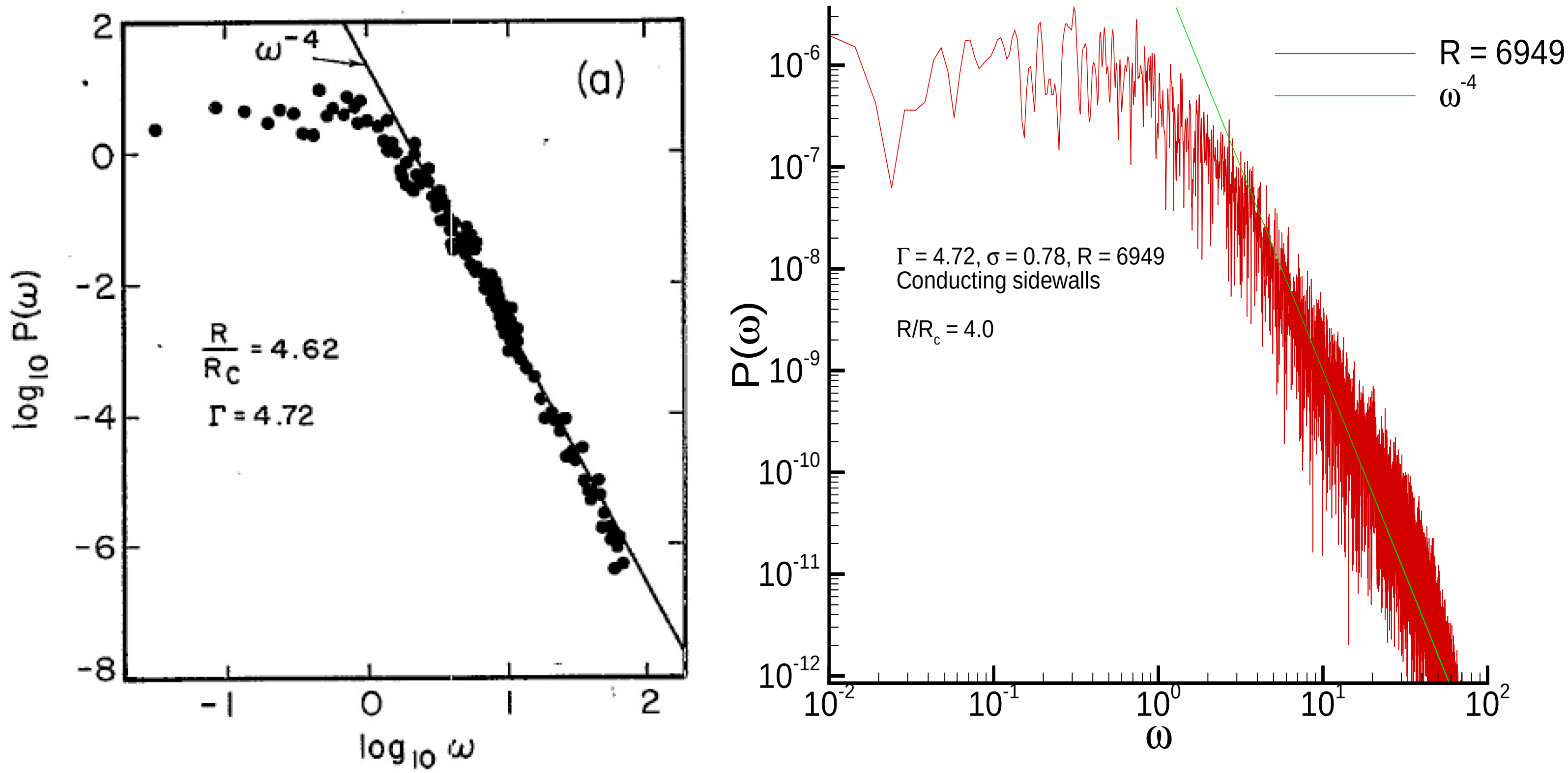
$R = 3127$ $R = 6949$

Power Spectrum

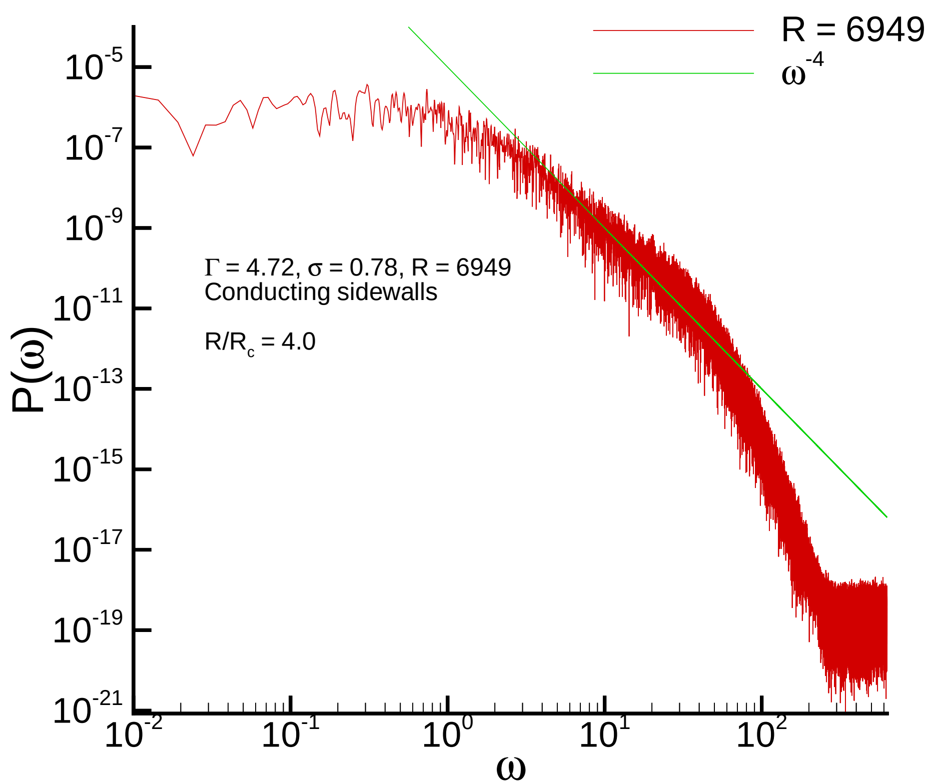
- Simulations of low dimensional chaos (e.g. Lorenz model) show exponential decaying power spectrum
- Power law power spectrum easily obtained from stochastic models (white-noise driven oscillator, etc.)

$$\text{Deterministic Chaos} \quad ?\Rightarrow? \quad \text{Exponential decay}$$
$$\text{Stochastic Noise} \quad ?\Rightarrow? \quad \text{Power law decay}$$

Simulation yields a power law over the range accessible to experiment

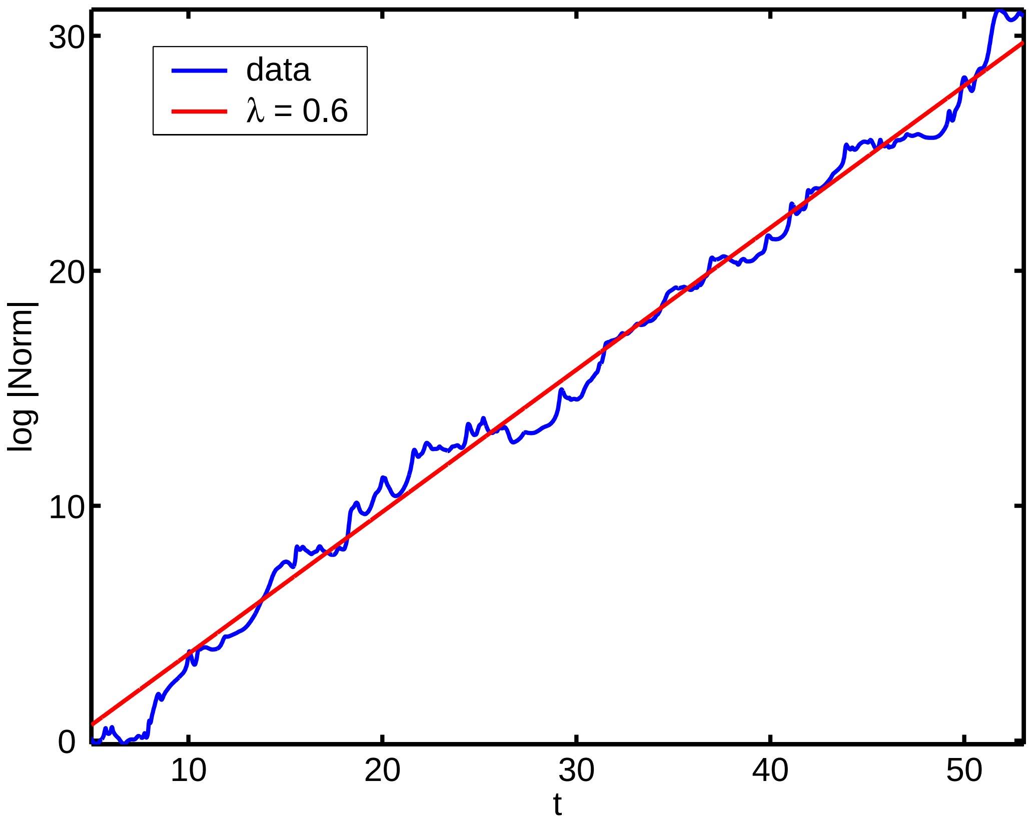


When larger frequencies are included an exponential tail is found



- Power law arises from quasi-discontinuous changes in the slope of $N(t)$ on a $t = 0.1 - 1$ time scale associated with roll pinch-off events.

Lyapunov Exponent



Lyapunov eigenvector

Spatiotemporal chaos

Rough definition: the dynamics, disordered in time and space, of a large aspect ratio system (i.e. one that is large compared to the size of a basic chaotic element)

- Natural examples
  - ◇ The atmosphere and ocean (weather, climate etc.)
  - ◇ Heart fibrillation
- Examples from convection
  - ◇ Spiral Defect Chaos (experiment , simulations)
  - ◇ Domain Chaos (model simulations: stripes , orientations , walls ; convection simulations)

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Spatiotemporal chaos is a new paradigm of unpredictable dynamics. (What effect does the butterfly really have?)

Challenges

- System-specific questions
- Definition and characterization
- Transitions to spatiotemporal chaos and between different chaotic states
- Coarse grained descriptions
- Control

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Ideas and methods from dynamical systems, statistical mechanics, phase transition theory …

Systems

- Coupled Maps

$$x_{\mathbf{i}}^{(n+1)} = f(x_{\mathbf{i}}^{(n)}) + D \times \frac{1}{n} \sum_{\delta=\text{n.n.}} [f(x_{\mathbf{i}+\delta}^{(n)}) - f(x_{\mathbf{i}}^{(n)})]$$

  with e.g. $f(x) = ax(1-x)$

- PDE simulations
  - ◇ Kuramoto-Sivashinsky equation

$$\partial_t u = -\partial_x^2 u - \partial_x^4 u - u\partial_x u$$

  - ◇ Amplitude equations, e.g. Complex Ginzburg-Landau Equation

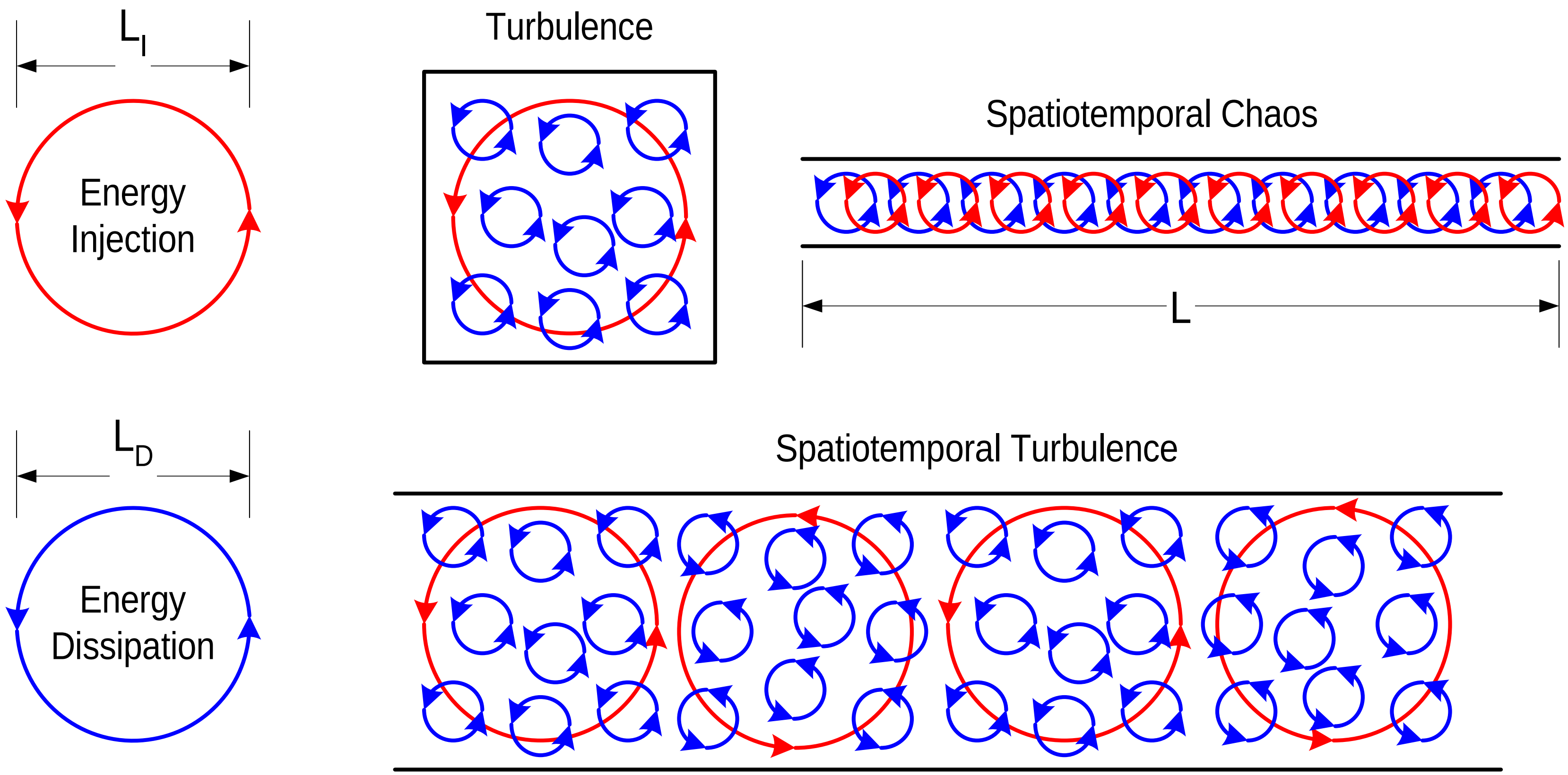
$$\partial_t A = \varepsilon A + (1 + ic_1)\nabla^2 A - (1 - ic_3)\,|A|^2\, A$$

- Physical systems (experiment and numerics)

Definition and characterization

- Narrow the phenomena
- Decide if theory, simulation, and experiment match

Spatiotemporal Chaos v. Turbulence



Characterizing spatiotemporal chaos

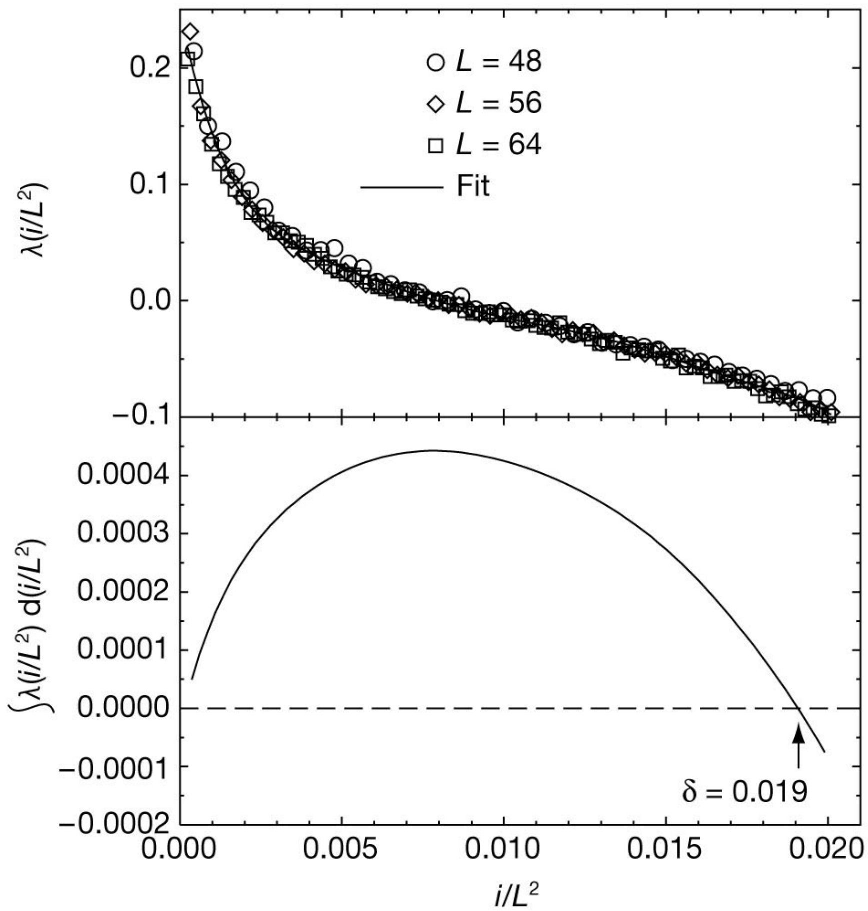
Methods from statistical physics: Correlation lengths and times, etc.

- Easy to measure, but perhaps not very insightful

Methods from dynamical systems: Lyapunov exponents and attractor dimensions.

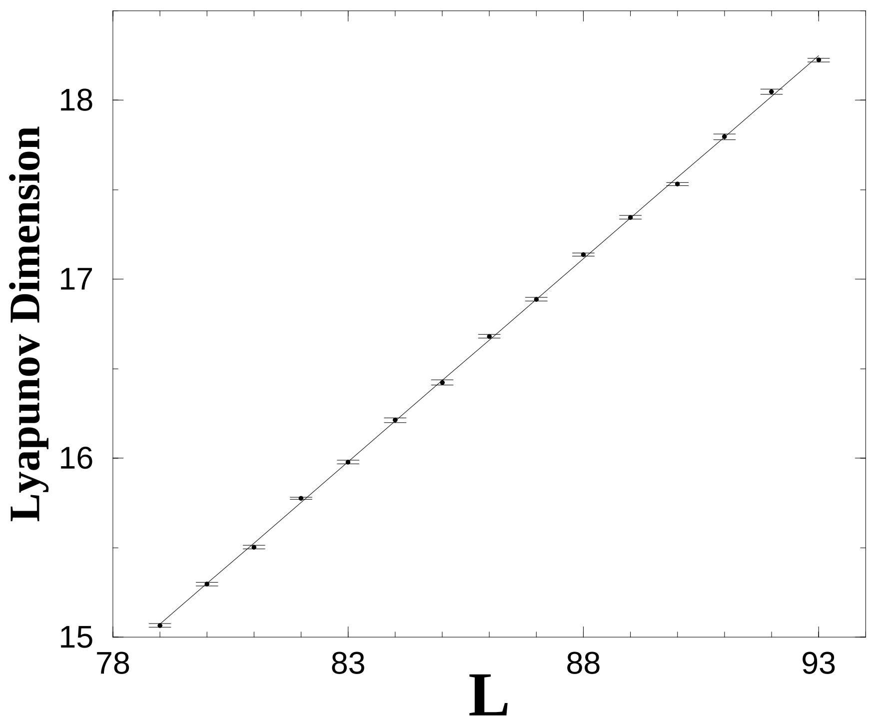
- Inaccessible in experiment, but can be measured in simulations
- Ruelle suggested that Lyapunov exponents should be intensive, and the dimension should be extensive $\propto L^d$

Lyapunov spectrum and dimension for spiral defect chaos



(Egolf et al. 2000)

Microextensivity for the 1d Kuramoto-Sivashinsky equation



(from Tajima and Greenside 2000)

Microextensivity

Interesting questions:

- Are there (tiny) windows of periodic orbits or chaotic orbits of non-scaling dimension so that smooth variation is only in the $L \to \infty$ limit, or is the variation smooth at finite L?
- Can we use this to define spatiotemporal chaos for finite L?
- Does spatiotemporal chaos in Rayleigh-Bénard Convection show microextensive scaling of the Lyapunov dimension?

Lyapunov vector for spiral defect chaos

(from Keng-Hwee Chiam, Caltech thesis 2003)

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Transitions

In thermodynamic equilibrium systems the behavior may be simpler near phase transitions.

Is there universal behavior near transitions in spatiotemporal chaos (transition to STC, transitions within STC)?

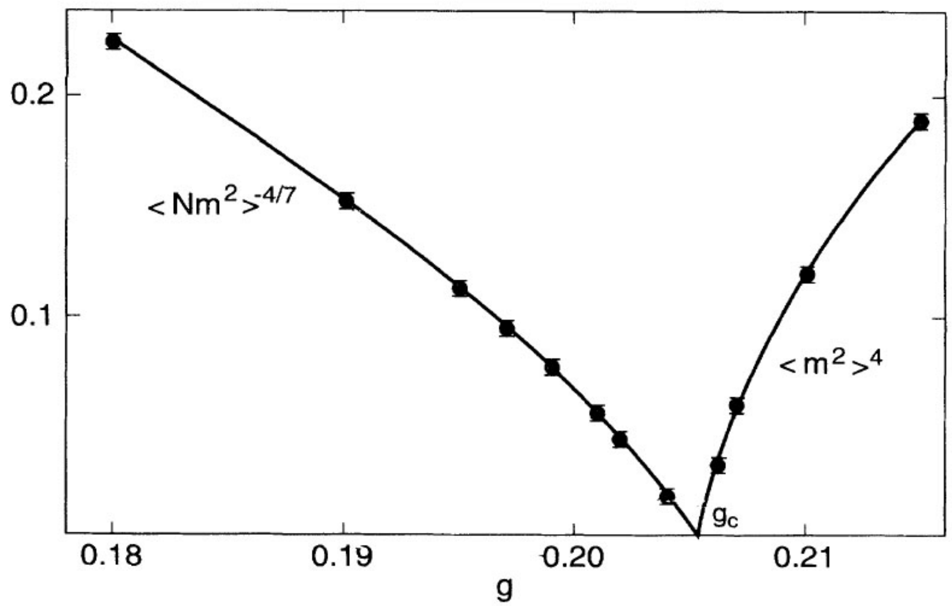
If so, is the universality the same as in corresponding equilibrium systems?

Examples:

- Chaotic Ising map
- Rotating convection

Chaotic Ising map

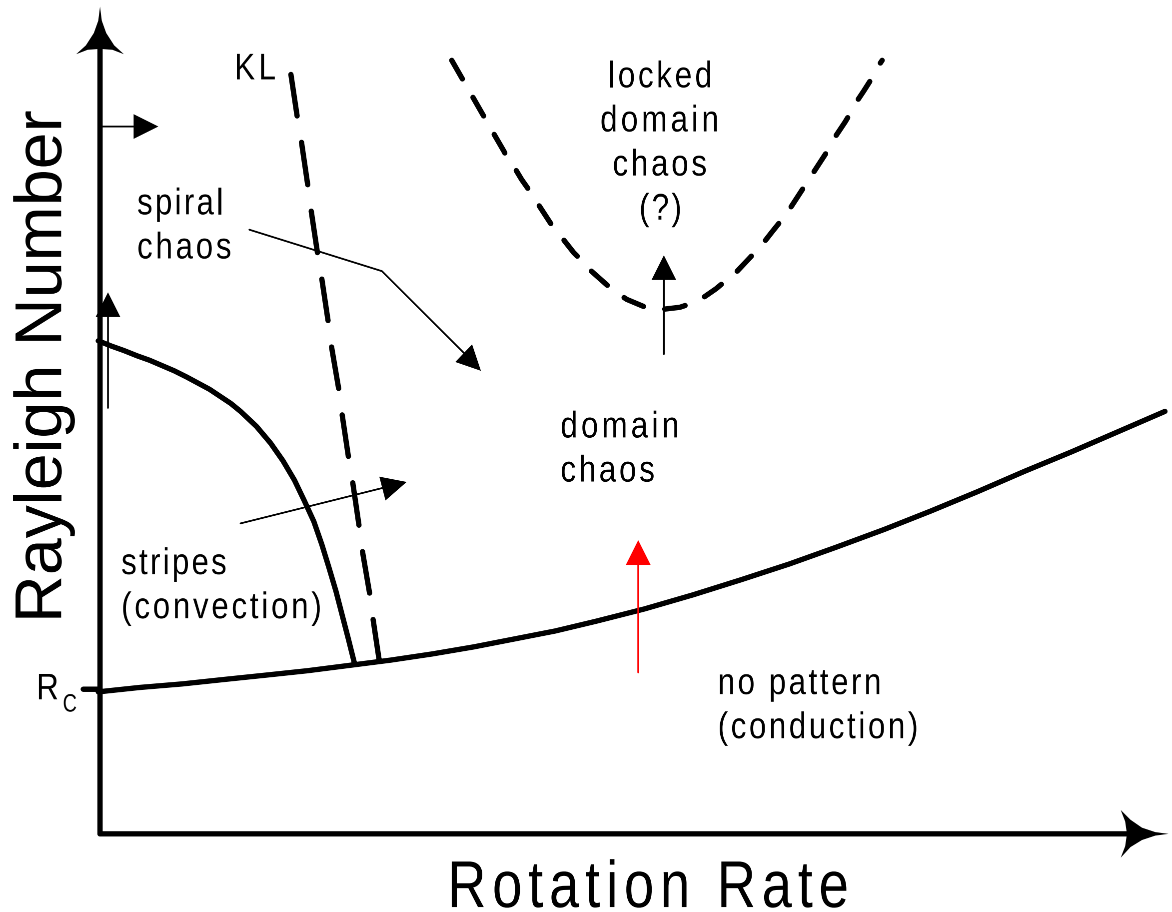
Universality apparently the same as for thermodynamic Ising system:



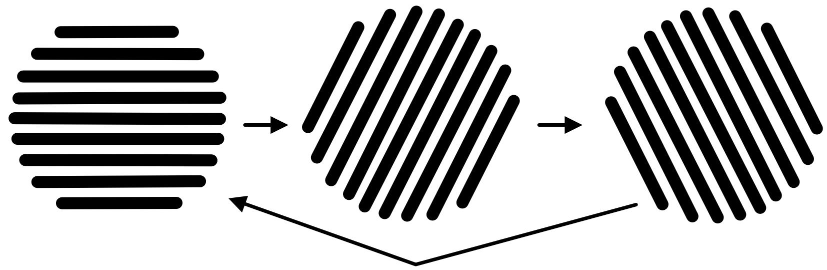
[Miller and Huse, Phys. Rev. **E48**, 2528 (1993)]

Also RNG treatment suggests non-equilibrium correction terms are *irrelevant* [Bennett and Grinstein, Phys. Rev. Lett. **55**, 657 (1985)]

Scaling near onset of domain chaos



Amplitude equation description (Tu and MCC, 1992)]



Amplitudes of rolls at 3 orientations $A_i(\mathbf{r}, t),\ i = 1 \ldots 3$

$$\partial_t A_1 = \varepsilon A_1 + \partial_{x_1}^2 A_1 - A_1(A_1^2 + g_+ A_2^2 + g_- A_3^2)$$

$$\partial_t A_2 = \varepsilon A_2 + \partial_{x_2}^2 A_2 - A_2(A_2^2 + g_+ A_3^2 + g_- A_1^2)$$

$$\partial_t A_3 = \varepsilon A_3 + \partial_{x_3}^2 A_3 - A_3(A_3^2 + g_+ A_1^2 + g_- A_2^2)$$

where $\varepsilon = (R - R_c(\Omega)/R_c(\Omega)$

Rescale space, time, and amplitudes:

Rescale $X = \varepsilon^{1/2}x,\ T = \varepsilon t,\ \bar{A} = \varepsilon^{-1/2}A$

$$\partial_T \bar{A}_1 = \bar{A}_1 + \partial^2_{X_1}\bar{A}_1 - \bar{A}_1(\bar{A}_1^2 + g_+\bar{A}_2^2 + g_-\bar{A}_3^2)$$
$$\partial_T \bar{A}_2 = \bar{A}_2 + \partial^2_{X_2}\bar{A}_2 - \bar{A}_2(\bar{A}_2^2 + g_+\bar{A}_3^2 + g_-\bar{A}_1^2)$$
$$\partial_T \bar{A}_3 = \bar{A}_3 + \partial^2_{X_3}\bar{A}_3 - \bar{A}_3(\bar{A}_3^2 + g_+\bar{A}_1^2 + g_-\bar{A}_2^2)$$

Numerical simulations show chaotic dynamics

Therefore in unscaled (physical) units

Length scale $\xi \sim \varepsilon^{-1/2}$

Time scale $\tau \sim \varepsilon^{-1}$

Summary of Tests

- Simulations [MCC and Meiron (1994)] of generalized Swift-Hohenberg equations in periodic geometries show results consistent with predictions
- [Hu et al. (1995) + others] Experiments give results that are consistent either with finite values of ξ, τ at onset, of much smaller power laws $\xi \sim \varepsilon^{-0.2}$, $\tau \sim \varepsilon^{-0.6}$
- [MCC, Louie, and Meiron (2001)] Simulations of generalized Swift-Hohenberg equations in circular geometries of radius Γ gave results consistent with finite size scaling
$$\xi_M = \xi f(\Gamma/\xi) \qquad \text{with} \qquad \xi \sim \varepsilon^{-1/2}$$
- [Scheel and MCC, preprint (2005)] Simulations of Rayleigh-Bénard convection with Coriolis forces give $\tau \sim \varepsilon^{-1}$ for small enough ε. For larger ε a slower growth is seen consistent with $\tau \sim \varepsilon^{-0.7}$.

Possible explanations for discrepancies?

- Finite size effects?
- Dislocation glide important (not in Tu-Cross model)?
- Centrifugal force important in experimental geometry?
- … or critical-like fluctuation effects important?

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Coarse grained description

- Can we find simplified descriptions of spatiotemporal chaotic systems at large length scales?
  - ◇ conserved quantity (cf. hydrodynamics)
  - ◇ near continuous transition
  - ◇ collective motion such as defects
- Is the simplified description analogous to a thermodynamic equilibrium system?

Rough argument

Rough argument

Expect a Langevin description at large scales

$$\partial_t \mathbf{y} = \mathbf{D}(\mathbf{y}) + \eta$$

$\mathbf{y}$ is vector of large length scale variables, $\mathbf{D}$ is some effective deterministic dynamics, and η is noise coming from small scale chaotic dynamics.

Rough argument

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Since η represents the effect of many small scale fast chaotic degrees of freedom acting on the large scales we might expect it to be Gaussian and white

$$\left\langle \eta_i(\mathbf{r}, t)\eta_j(\mathbf{r}', t')\right\rangle = \Omega_{ij}\delta(\mathbf{r} - \mathbf{r}')\delta(t - t')$$

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In systems deriving from a microscopic Hamiltonian dynamics *constraints* relate the noise Ω_{ij} and the deterministic terms **D** (the fluctuation-dissipation theorem).

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In systems deriving from a microscopic Hamiltonian dynamics *constraints* relate the noise Ω_{ij} and the deterministic terms **D** (the fluctuation-dissipation theorem).

In systems based on a dissipative small scale dynamics, *if* the dominant macroscopic dynamics is sufficiently simple, or sufficiently constrained by symmetries, these relationships may *happen* to occur.

Examples

- Chaotic Kuramoto-Sivashinsky dynamics reduces to noisy Burgers equation [Zaleski (1989)]
- Chaotic Ising map model near the transition
  - ◇ Langevin equation for dynamics of domain walls same as in equilibrium system [Miller and Huse, (1993)]
  - ◇ Coarse grained configurations satisfy detailed balance and have a distribution given by an effective free energy [Egolf, Science **287**, 101 (2000)]
- Defect dynamics description of 2D Complex Ginzburg Landau chaos [Brito et al., Phys. Rev. Lett. **90**, 063801 (2003)]

Conclusions

In this lecture I introduced some of the basic ideas of chaos, and discussed the application of these ideas to continuum systems.

I reviewed some old experiments on chaos in continuum systems, together with recent numerical results on these systems.

I then introduced spatiotemporal chaos, which remains a poorly characterized and understood phenomenon. The careful comparison between experiment, theory, and numerics has been and will continue to be important in increasing our understanding.