Pattern Formation in Spatially Extended Systems

Lecture 3: Oscillatory Instabilities

Oscillatory Instabilities

If $\omega_c = -\,\mathrm{Im}\,\sigma_{q_c} \neq 0$ we have an instability to

- a nonlinear oscillator for $q_c = 0$ which also supports travelling waves
- a wave pattern (standing or travelling) for $q_c \neq 0$

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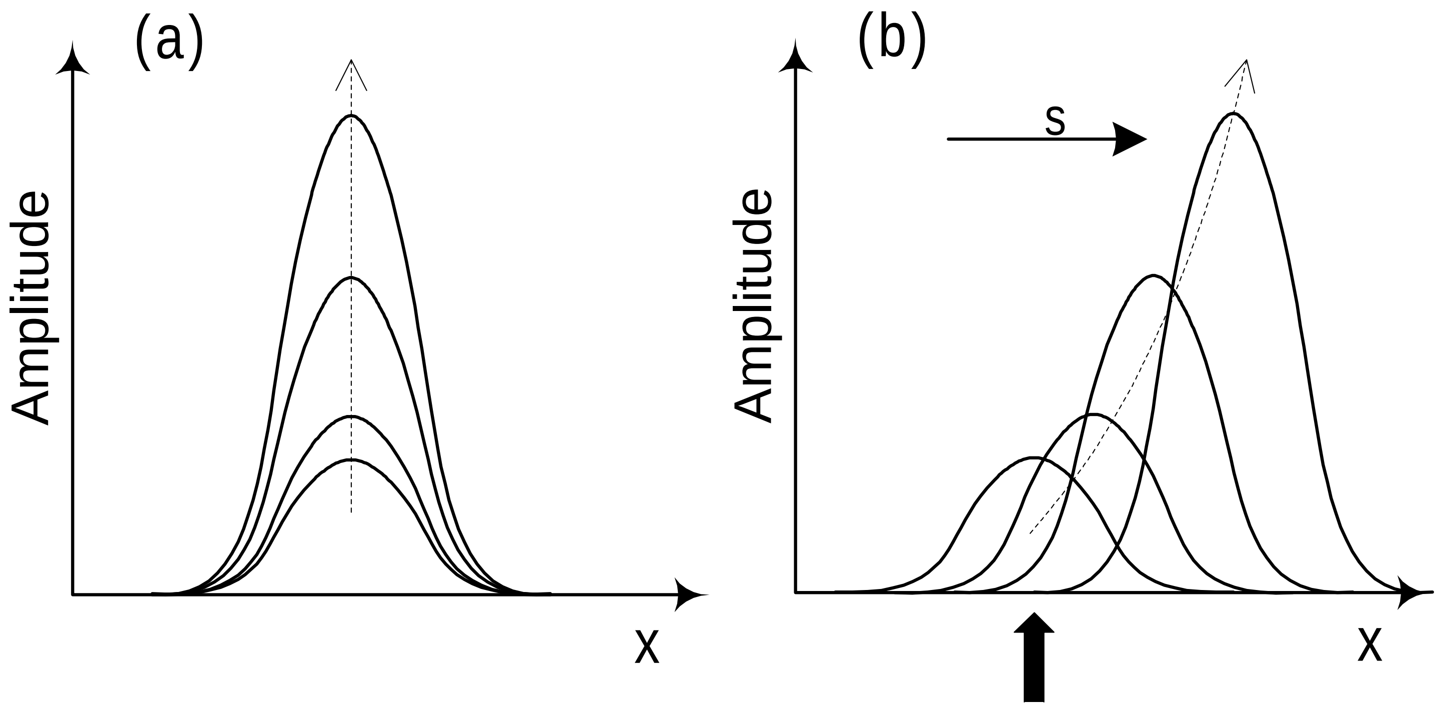
Important new concept: absolute v. convective instability

Absolute and convective instability



Conditions for convective and absolute instability

- **Convective instability:** same as condition for instability to Fourier mode
- **Absolute instability:** for a growth rate spectrum σ_q, the system is absolutely unstable if

$$\mathrm{Re}\,\sigma(\mathbf{q}_s) = 0$$

where $\mathbf{q}_s$ is a *complex* wave vector given by the solution of the stationary phase condition

$$\frac{d\sigma_{\mathbf{q}}}{d\mathbf{q}} = 0$$

Derivation of condition for absolute instability

In the linear regime the disturbance growing from any given initial condition $u_p(\mathbf{x}, t = 0)$ can be expressed as (for simplicity restricting attention to one dimension)

$$u_p(x, t) = \int_{-\infty}^{\infty} dq\, e^{iqx+\sigma_q t} \int_{-\infty}^{\infty} dx'\, u_p(x', 0) e^{-iqx'}$$

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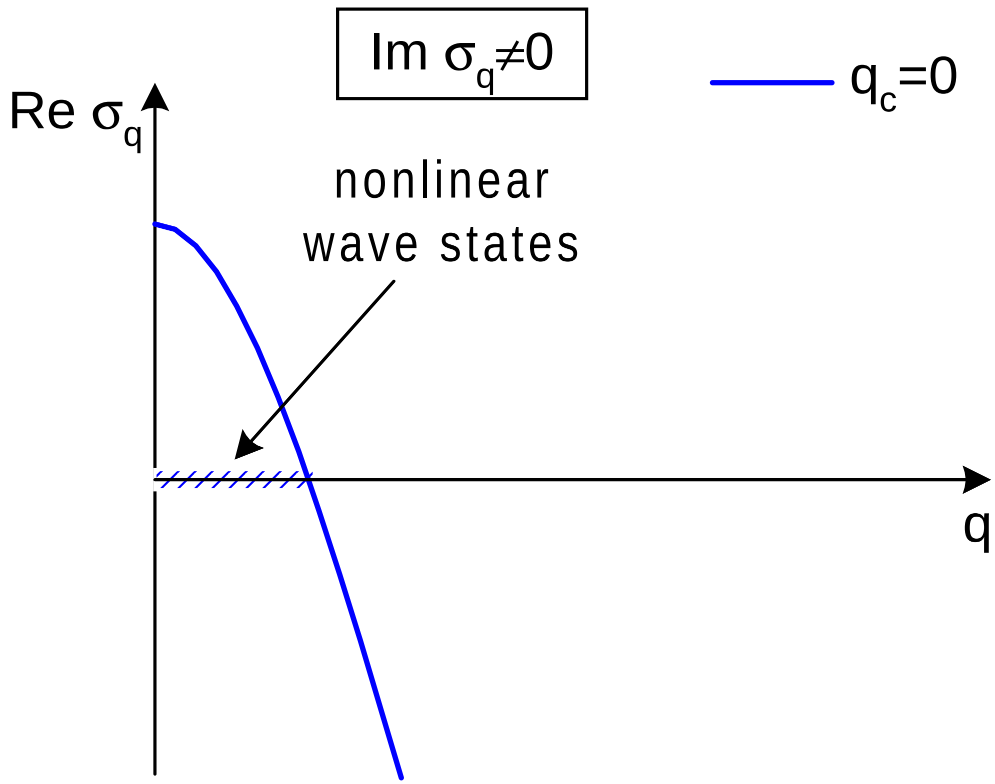
Thus the system will be absolutely unstable for $\mathrm{Re}\,\sigma_{q_s} > 0$.

Nonlinear oscillators and waves

Insights from amplitude and phase equations

- Oscillatory instability $q_c = 0$
- Wave instability $q_c \neq 0$

Oscillatory intability: Complex Ginzburg-Landau



1d: $$\partial_T \bar{A} = (1 + ic_0)\bar{A} + (1 + ic_1)\partial_X^2 \bar{A} - (1 - ic_3)\left|\bar{A}\right|^2 \bar{A}$$

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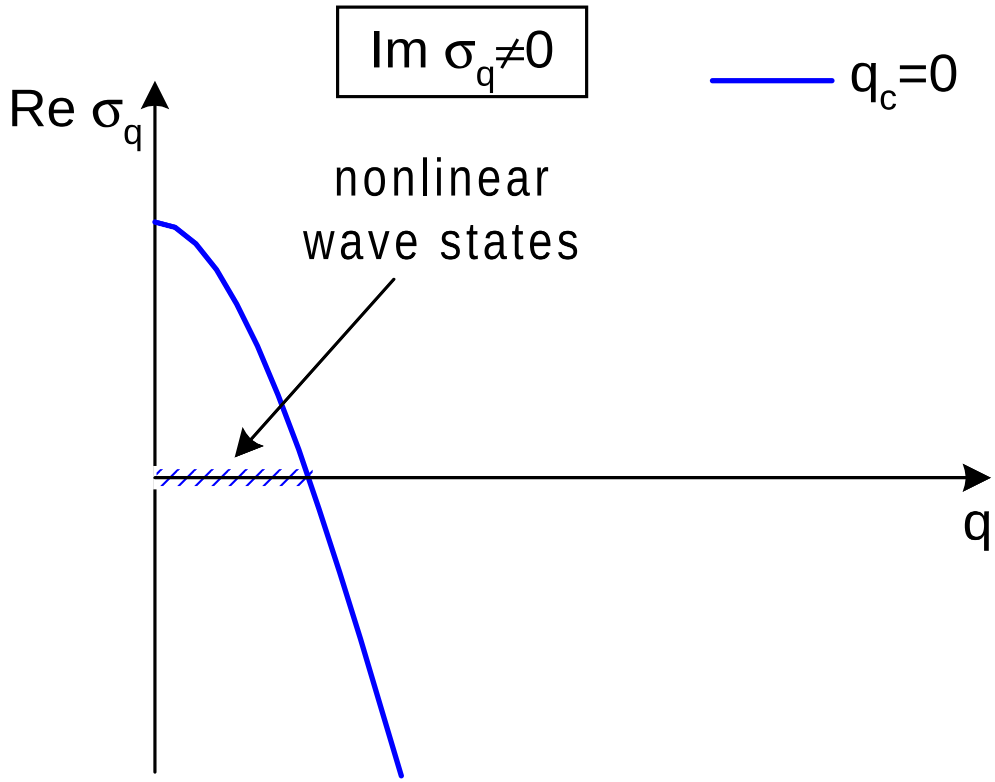

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2d: $$\partial_T \bar{A} = (1 + ic_0)\bar{A} + (1 + ic_1)\nabla_\perp^2 \bar{A} - (1 - ic_3)\left|\bar{A}\right|^2 \bar{A}$$

Simulations of the CGL equation

General equation (2d)

$$\partial_T \bar{A} = (1 + ic_0)\bar{A} + (1 + ic_1)\nabla_\perp^2 \bar{A} - (1 - ic_3) \left|\bar{A}\right|^2 \bar{A}$$

Case simulated: $c_1 = 0$ (choice of parameter), $c_0 = -c_3$ (for simplicity of plots)

$$\partial_T \bar{A} = (1 - ic_3)\bar{A} + \nabla_\perp^2 \bar{A} - (1 - ic_3) \left|\bar{A}\right|^2 \bar{A}$$

Simulations…

Nonlinear wave patterns

As well as uniform oscillations, CGL equation supports travelling wave solutions, but with properties that are strange to those of us brought up on linear waves:

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As well as uniform oscillations, CGL equation supports travelling wave solutions, but with properties that are strange to those of us brought up on linear waves:

- Waves annhilate at shocks rather than superimpose
- Waves disappear at boundaries rather than reflect (not shown)
- Defects: importance as persistent sources
- Spiral defects play a conspicuous role, because they are topologically defined persistent sources.
- Instabilities can lead to spatiotemporal chaos

Wave solutions

$$\partial_T \bar{A} = (1 + ic_0)\bar{A} + (1 + ic_1)\nabla_\perp^2 \bar{A} - (1 - ic_3)\left|\bar{A}\right|^2 \bar{A}$$

Travelling wave solutions

$$\bar{A}_K(\mathbf{X}, T) = a_K e^{i(\mathbf{K}\cdot\mathbf{X} - \Omega_K T)}$$

$$a_K^2 = 1 - K^2 \qquad \Omega_K = -(c_0 + c_3) + (c_1 + c_3)K^2$$

Group speed

$$S = d\Omega_K/dK = 2(c_1 + c_3)K$$

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Standing waves, based on the addition of waves at $\mathbf{K}$ and $-\mathbf{K}$ can be constructed, but they are unstable towards travelling waves

Stability analysis

$$\bar{A}_K(\mathbf{X}, T) = (a_K + \delta a)e^{i(\mathbf{K}\cdot\mathbf{X} - \Omega_K T + \delta\theta)}$$

For *small*, *slowly varying* phase perturbations

$$\partial_T \delta\theta + S\partial_X \delta\theta = D_\parallel(K)\partial_X^2 \delta\theta + D_\perp(K)\partial_Y^2 \delta\theta$$

with longitudinal and transverse diffusion with constants

$$D_\parallel(K) = (1 - c_1 c_3)\frac{1 - \nu K^2}{1 - K^2} \qquad D_\perp(K) = (1 - c_1 c_3)$$

with

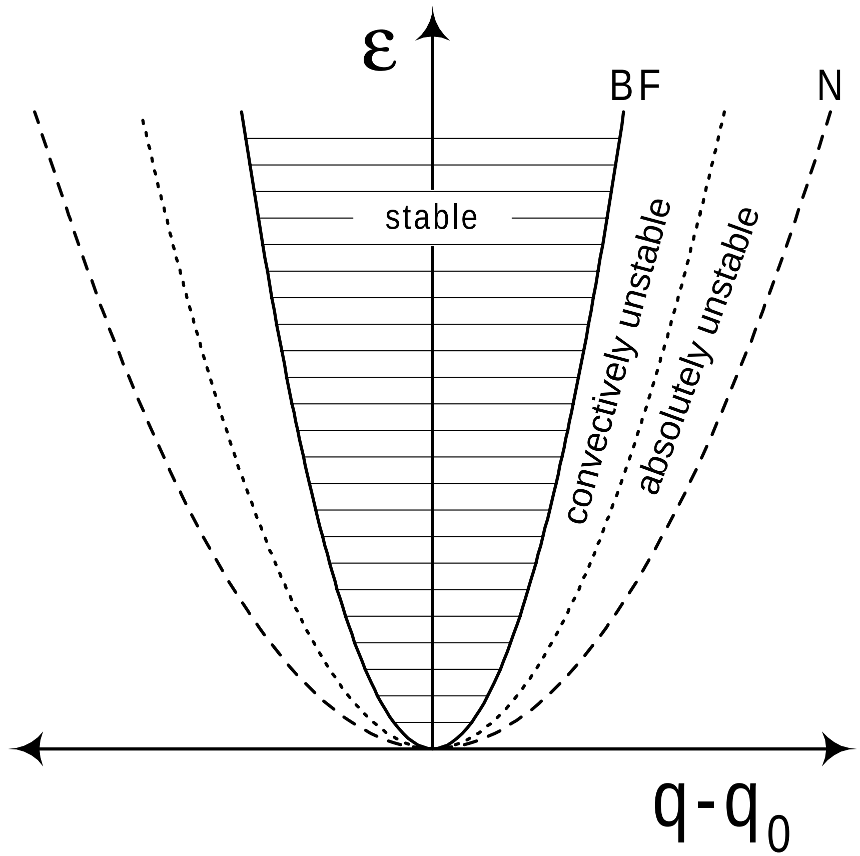
$$\nu = \frac{3 - c_1 c_3 + 2c_3^2}{1 - c_1 c_3}$$

- $D_\parallel = 0 \Rightarrow$ Benjamin-Feir instability (longitudingal sideband instability analogous to Eckhaus) for

  $$|K| \geq \Lambda_B = \nu^{-1}$$

  leaving a stable band of wave numbers with width a fraction ν^{-1} of the existence band.
- For $1 - c_1 c_3 < 0$ *all* wave states are unstable (Newell)

Stability balloon



Shocks: the nonlinear phase equation

For slow phase variations about spatially uniform oscillations (now keeping all terms up to second order in derivatives)

$$\partial_T \theta = \Omega + \alpha \nabla_\perp^2 \theta - \beta (\vec{\nabla}_\perp \theta)^2$$

with

$$\alpha = 1 - c_1 c_3$$
$$\beta = c_1 + c_3$$
$$\Omega = c_0 + c_3$$

Cole-Hopf transformation

The Cole-Hopf transformation

$$\chi(X, Y, T) = \exp[-\beta\theta(X, Y, T)/\alpha]$$

transforms the nonlinear phase equation into the *linear* equation for χ

$$\partial_T \chi = \alpha \nabla_X^2 \chi$$

Plane wave solutions

$$\chi = \exp\left[(\mp\beta K X + \beta^2 K^2 T)/\alpha\right]$$

correspond to the phase variations

$$\theta = \pm K X - \beta K^2 T$$

Cole-Hopf transformation (cont)

Since the χ equation is *linear*, we can superimpose a pair of these solutions

$$\chi = \exp\left[(-\beta K X + \beta^2 K^2 T)/\alpha\right] + \exp\left[(+\beta K X + \beta^2 K^2 T)/\alpha\right]$$

The phase is

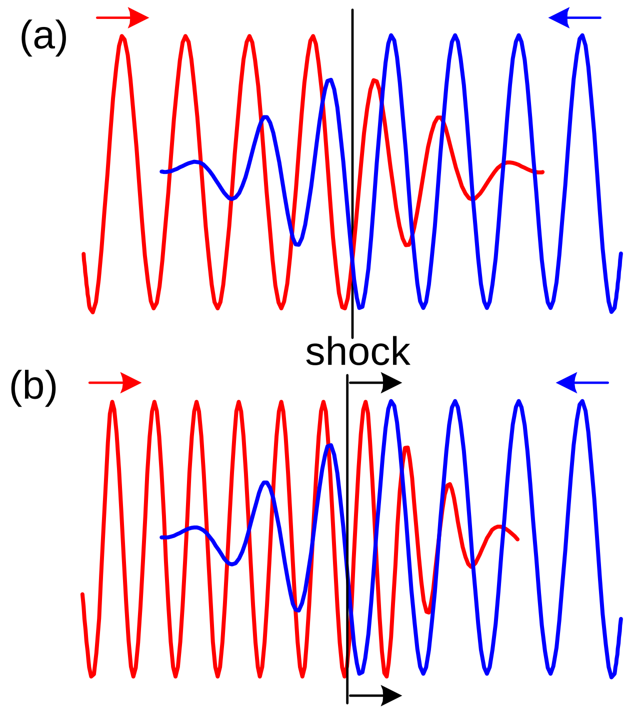
$$\theta = -\beta K^2 T - \frac{\alpha}{\beta} \ln[2\cosh(\beta K X/\alpha)].$$

For large $|X|$ the phase is given by (assuming βK positive)

$$\theta \to -K X - \beta K^2 T - \tfrac{\alpha}{\beta} \exp(-2\beta K X/\alpha) \quad \text{for} \quad X \to +\infty \tag{5}$$

i.e. left moving waves plus exponentially decaying right moving waves with the decay length $\alpha/2\beta K$. Similarly for $X \to -\infty$ get left moving waves with exponentially small right moving waves.

Shocks



- Shocks are sinks, not sources
- For positive group speed shocks between waves of different freuqency move so that the higher frequency region expands

Spiral Defects

m-armed spiral: $$\oint \nabla\theta \cdot \mathbf{dl} = m \times 2\pi$$

$$\bar{A} = a(R)e^{i(K(R)R + m\theta - \Omega_s T)}$$

with for $R \to \infty$

$$a(R) \to a_K \qquad K(R) \to K_s \qquad \text{with} \qquad \Omega_K(K_s) = \Omega_s$$

Uniqueness

A key question is whether there is a family of spirals giving a continuous range of possible frequencies Ω_s or whether there is a discrete set of possible spiral structures or even a unique one with a prescribed frequency that selects a particular wave number.

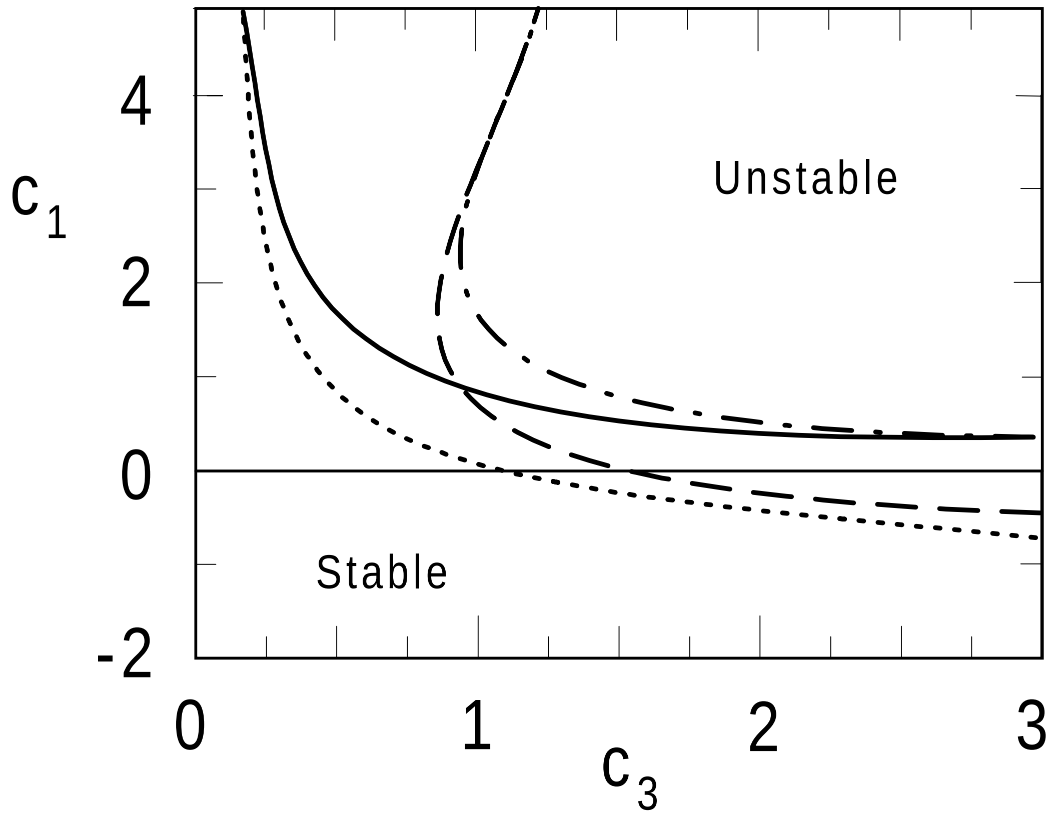
A perturbative treatments of the CGLE for small $c_1 + c_3$ about the real amplitude equation predicts a unique stable spiral structure, with a wave number K_s that varies as

$$K_s \rightarrow \frac{1.018}{|c_1 + c_3|} \exp[-\frac{\pi}{2\,|c_1 + c_3|}].$$

(Hagan, 1982)

Stability revisited

- Wave number of nonlinear waves determined by spirals
- Only BF stability of waves at K_s relevant to stability of periodic state
- Convective instability may not lead to breakdown
- Core instabilities may intervene



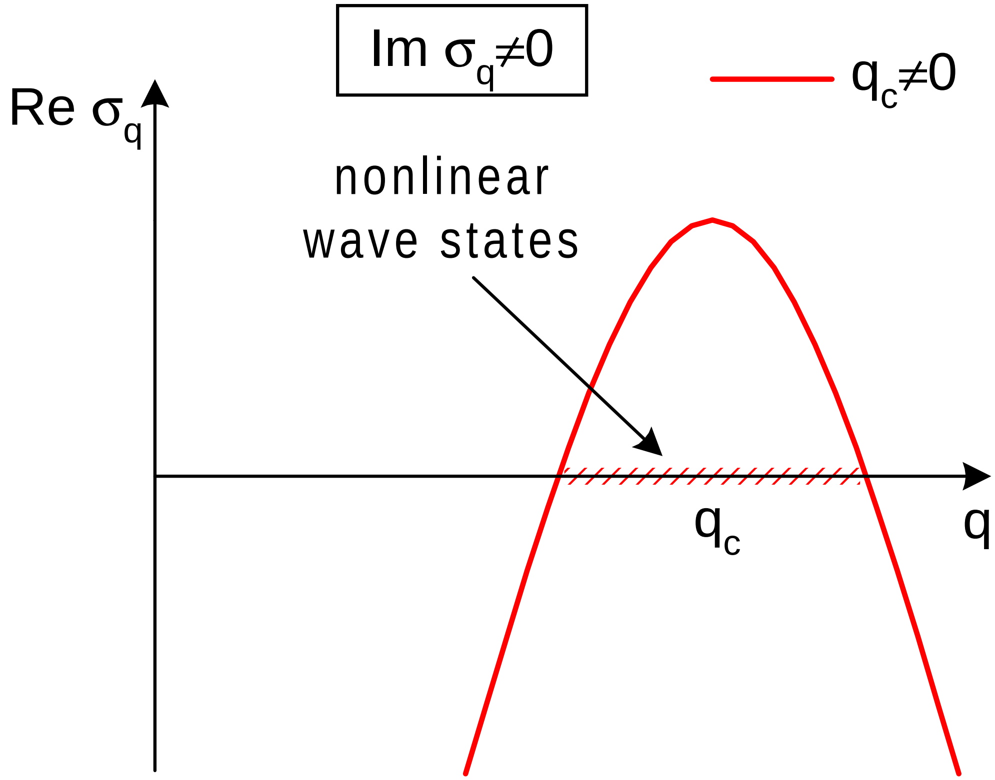
Stability lines of the CGLE. Solid line: Newell criterion $c_1 c_3 = 1$; dotted line: (convective) Benjamin-Feir instability of spiral-selected wavenumber; dashed: absolute instability of spiral selected wavenumber; dashed-dotted: abolute instability of whole wavenumber band. Unstable states are towards larger postive $c_1 c_3$.

Waves in excitable media

Waves in reaction-diffusion systems such as chemicals or heart tissue show similar properties

[From Winfree and Strogatz (1983) and the website of G. Bub, McGill]

Wave instabilities



$$\tau_0(\partial_t + s\partial_x)A = \varepsilon(1+ic_0)A + \xi_0^2(1+ic_1)\partial_x^2 A - g_0(1-ic_3)\left|A\right|^2 A$$

No *single* scaling of x, t with ε eliminates the small parameter ε from equation.

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- Lowest order equation is just the propagation

$$\partial_{T_p} \bar{A} + s \partial_X \bar{A} = 0.$$

Solution is that $\bar{A}$ is a function of the reduced coordinate $\xi = X - sT_p$, i.e. $\bar{A}(X, T_p, T) = \bar{A}(\xi, T)$. Physically this corresponds to transforming to a frame moving at the group speed s.

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- Then use

$$\partial_\xi \to \varepsilon^{-1/2} \partial_x, \qquad \partial_T \to \varepsilon^{-1}(\partial_t + s\partial_x).$$

Solutions

$$(\partial_t + s\partial_x)A = \varepsilon(1 + ic_0)A + \xi_0^2(1 + ic_1)\partial_x^2 A - g_0(1 - ic_3)\left|A\right|^2 A$$

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Also can treat counterpropagating waves

$$(\partial_T + S\partial_X)\bar{A}_R = (1 + ic_0)\bar{A}_R + (1 + ic_1)\partial_X^2 \bar{A}_R - (1 - ic_3)\left|\bar{A}_R\right|^2 \bar{A} - g_1(1 - ic_2)\left|\bar{A}_L\right|^2 \bar{A}_R$$

$$(\partial_T - S\partial_X)\bar{A}_L = (1 + ic_0)\bar{A}_L + (1 + ic_1)\partial_X^2 \bar{A}_L - (1 - ic_3)\left|\bar{A}_L\right|^2 \bar{A} - g_1(1 - ic_2)\left|\bar{A}_R\right|^2 \bar{A}_L$$

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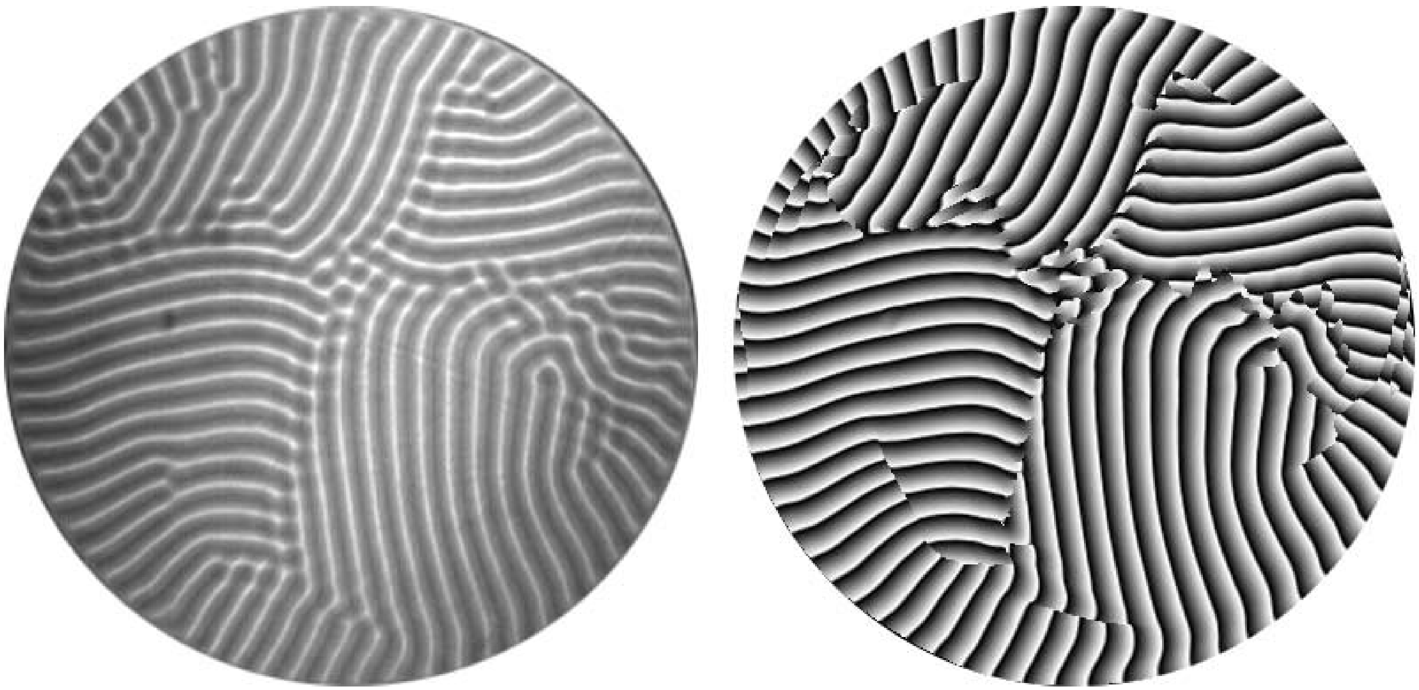
- (Knobloch and de Luca 1990) For $s = O(1)$ and ε small, interaction with inhomogeneity in medium or counterpropagating wave is *nonlocal* e.g.
$\left|\bar{A}_L\right|^2\bar{A}_R \to \left(L^{-1}\int\,\left|\bar{A}_L\right|^2 dX\right)\bar{A}_R$

Wave instability in 2d



[From La Porta and Surko (1998)]

Other issues for wave instabilities

- Noise sustained structures in convectively unstable domain
- Global modes (e.g. for $\varepsilon(x)$): local absolutely unstable region sustains disturbance in convectively unstable region (Chomaz et al., 1988)
- Complex dynamics of counterpropagating waves in finite geometry (e.g. blinking states)

Conclusions

In this lecture I discussed pattern formation near oscillatory instabilities.

Some of the key concepts were:

- **Convective v. Absolute Instability**
- **Oscillatory Instability**
  - ◇ CGL equation
  - ◇ Benjamin-Feir instability
  - ◇ Properties of nonlinear waves
  - ◇ Importance of spiral sources in 2d
- **Wave Instability**
  - ◇ Importance of propagation term