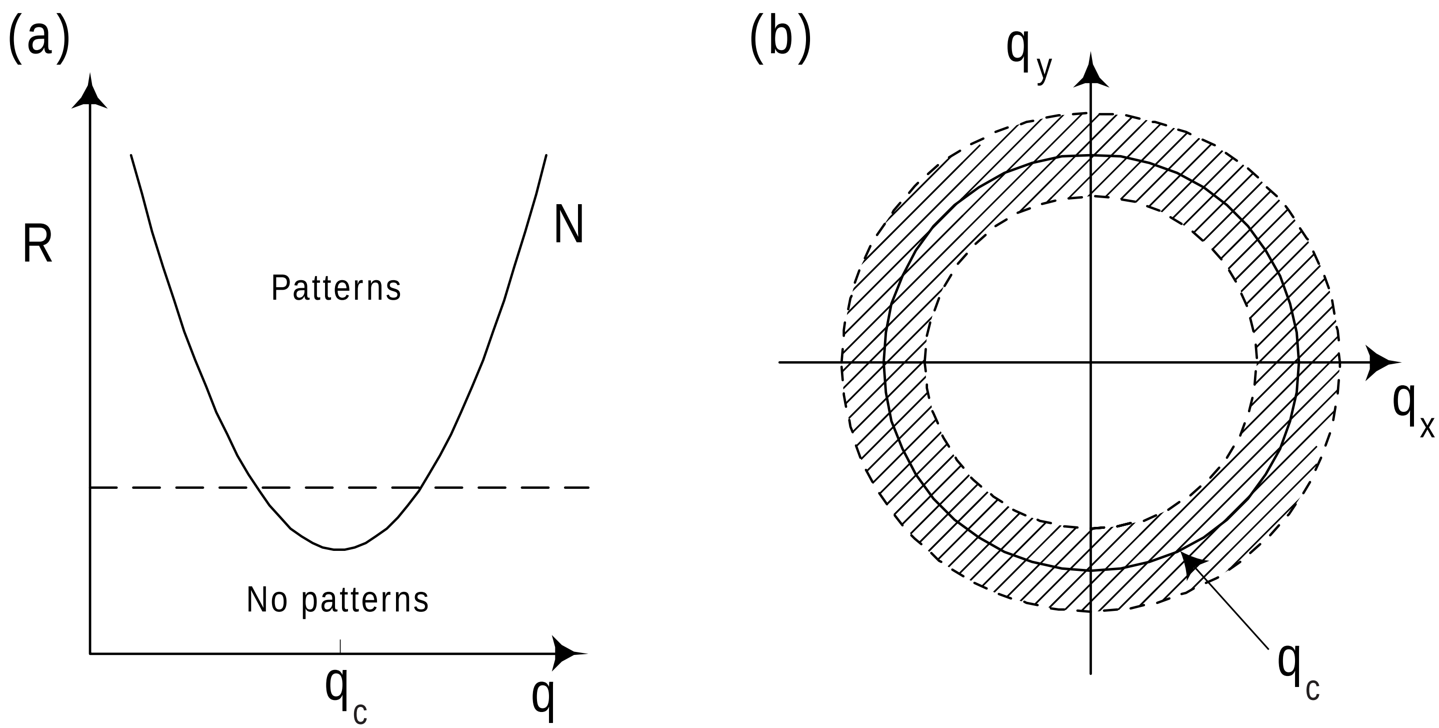
Pattern Formation in Spatially Extended Systems

Lecture 2: Symmetry

- Rotational invariance near threshold
  - ◇ Amplitude equation
  - ◇ Swift-Hohenberg equation
- Translational invariance: the phase equation
  - ◇ Near threshold
  - ◇ Far from threshold
- Defects

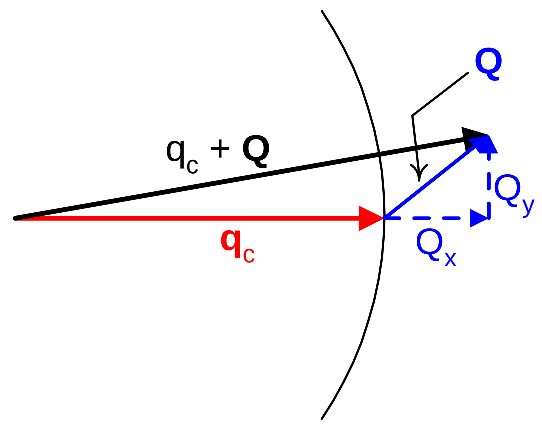
Rotational Symmetry: Linear Instability



Rotational Symmetry: Amplitude Equation

For a 2d, rotationally invariant system the gradient term is more complicated

$$\tau_0 \partial_t A = \varepsilon A + \xi_0^2 \left(\partial_x - \frac{i}{2q_c} \partial_y^2 \right)^2 A - g_0 |A|^2 A$$



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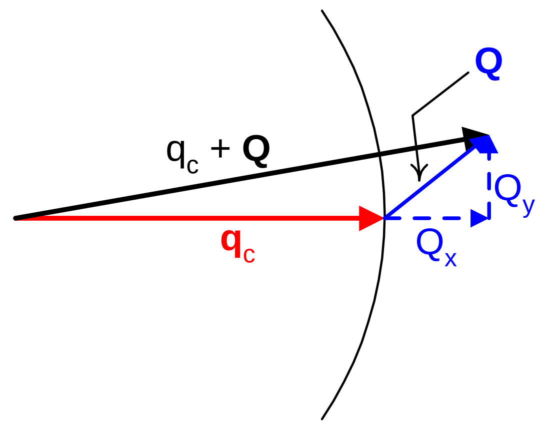

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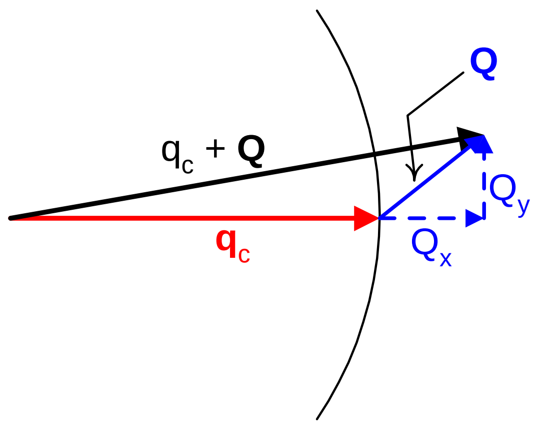

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Isotropic system gives anisotropic scaling: $x = \varepsilon^{-1/2} \xi_0 \; X; \; y = \varepsilon^{-1/4} (\xi_0 / q_c)^{1/2} \; Y$

Swift-Hohenberg equation

Simple equation for an *order parameter* $\psi(x, y, t)$ that is rotationally invariant in the plane and captures the same physics as the amplitude equation

$$\partial_t \psi = \left[r - (\nabla_\perp^2 + 1)^2\right] \psi - \psi^3$$

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- equation is relaxational

$$\partial_t \psi = -\frac{\delta V}{\delta \psi}, \qquad V = \iint dx dy \left\{-\tfrac{1}{2} r \psi^2 + \tfrac{1}{2}\left[(\nabla^2 + 1)\psi\right]^2 + \tfrac{1}{4}\psi^4\right\}$$

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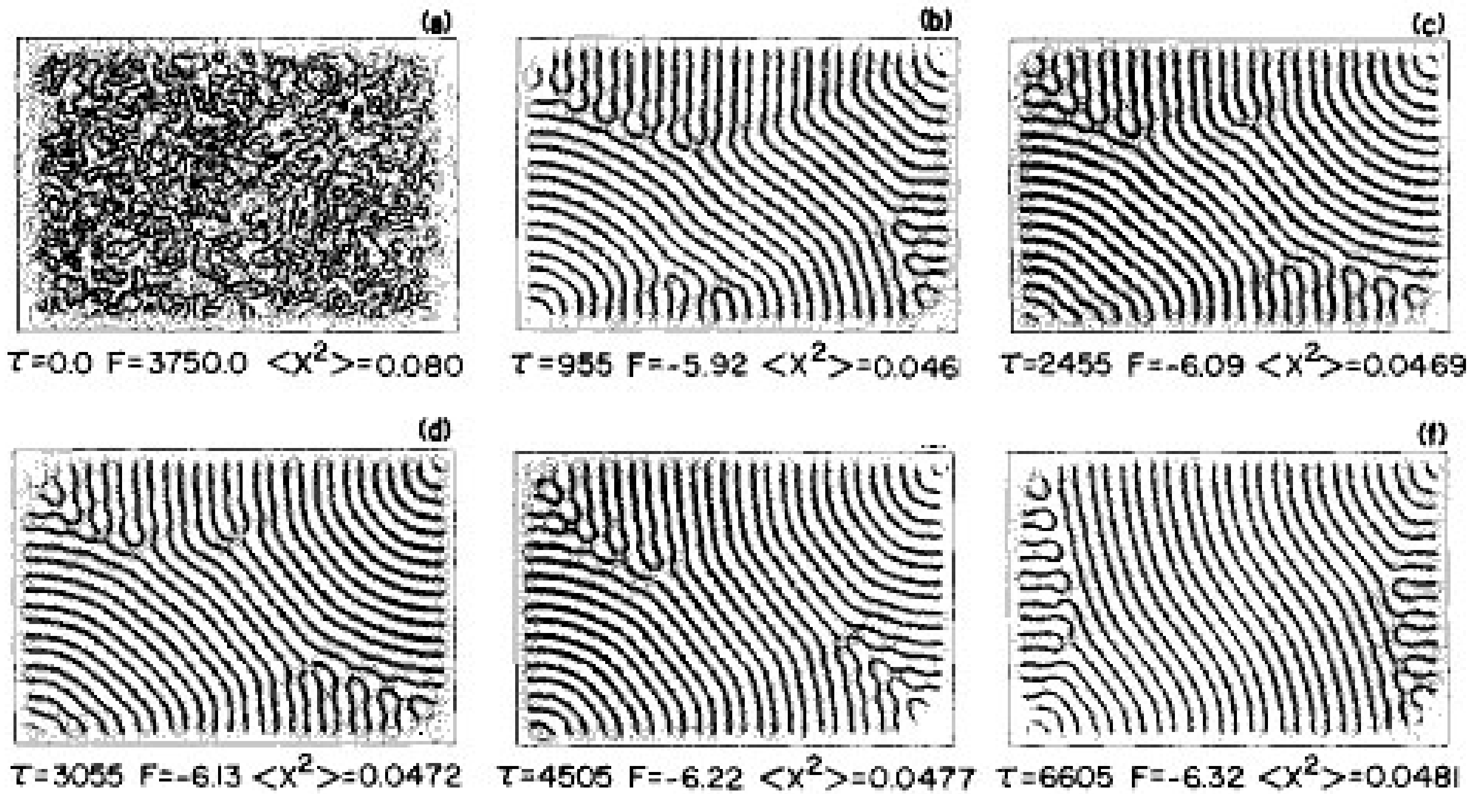
- Alternatively can think

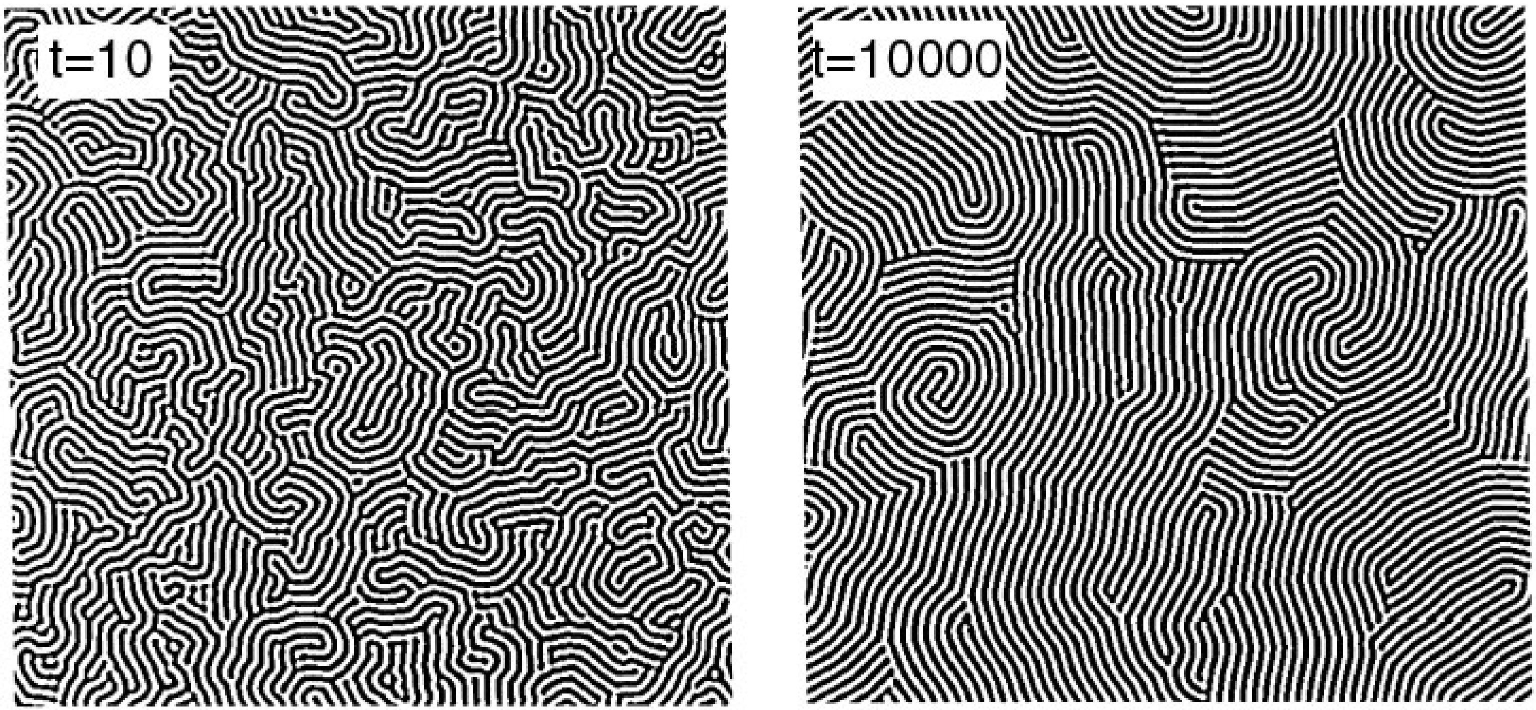
$$A(x, y)e^{i\mathbf{q}_c x} \Rightarrow \psi(x, y)$$

Relaxation to steady state



(from Greenside and Coughran, 1984)

Coarsening in a periodic geometry



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$$\partial_t \psi = \left[r - (\nabla_\perp^2 + 1)^2\right]\psi + (\nabla\psi)^2\nabla^2\psi$$

- model effects of rotation

$$\partial_t \psi = \left[r - (\nabla_\perp^2 + 1)^2\right]\psi - \psi^3 + g_2\hat{\mathbf{z}}\cdot\nabla\times[(\nabla\psi)^2\nabla\psi] + g_3\nabla\cdot[(\nabla\psi)^2\nabla\psi]$$

Order parameter equation

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This approach is rotationally invariant, and removes the limitations of the Swift-Hohenberg equation, but seems only easy to formulate in Fourier representation. It is not known how to treat real boundaries properly.

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- Near threshold θ is simply the phase of the complex amplitude, and an equation for the phase dynamics can be derived from the amplitude equation for $\eta \ll \varepsilon$ (Pomeau and Manneville, 1979)

Equation for small phase distortions near threshold

For a phase variation $\theta = kx + \delta\theta$

$$\partial_t \delta\theta = D_\parallel \partial_x^2 \delta\theta + D_\perp \partial_y^2 \delta\theta$$

with diffusion constants for the state with wave number $q = q_c + k$

$$D_\parallel = (\xi_0^2 \tau_0^{-1}) \frac{\varepsilon - 3\xi_0^2 k^2}{\varepsilon - \xi_0^2 k^2}$$

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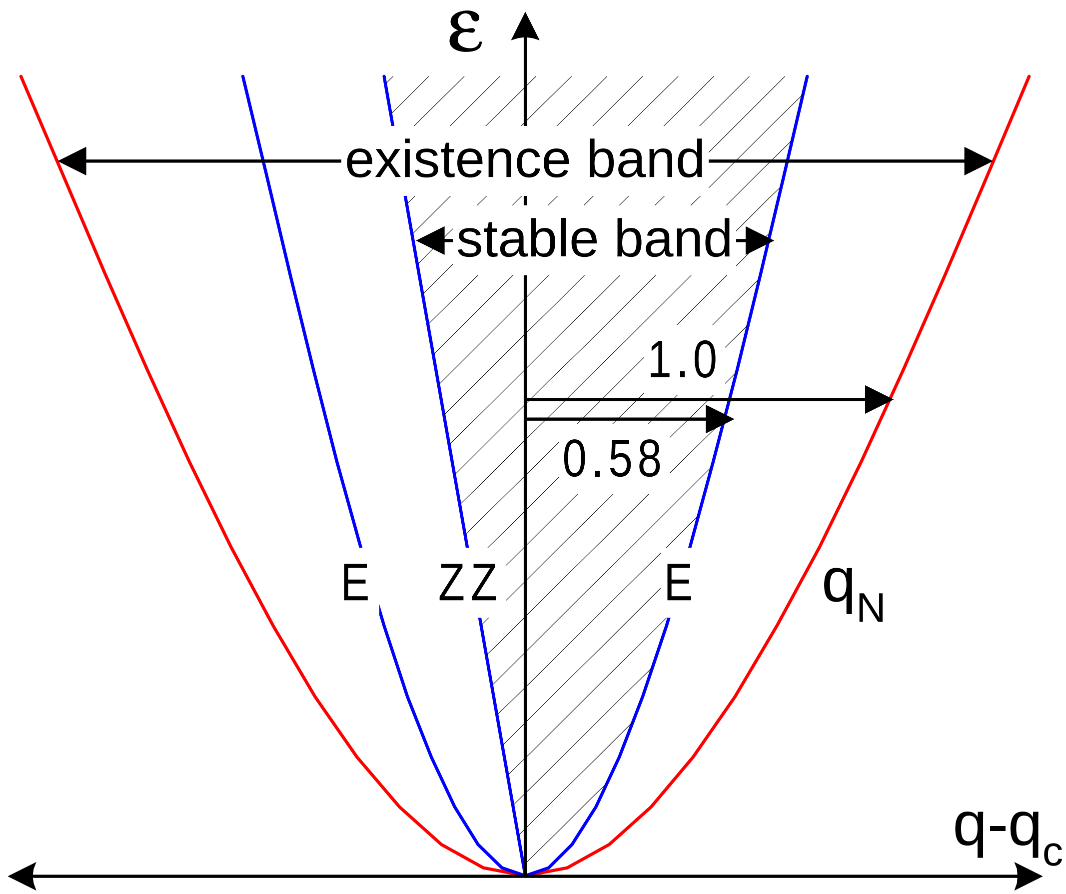
A negative diffusion constant leads to exponentially growing solutions, i.e. the state with wave number $q_c + k$ is unstable to long wavelength phase perturbations for

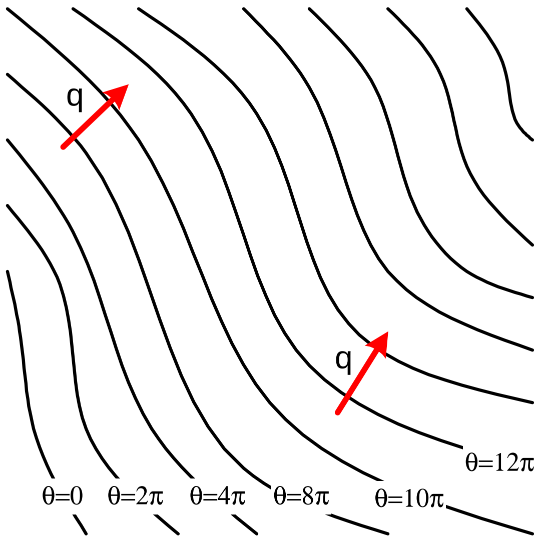
$\lvert\xi_0 k\rvert > \varepsilon^{1/2}/\sqrt{3}$	longitudinal (Eckhaus)
$k < 0$	transverse (ZigZag)

Stability balloon near threshold



Phase dynamics away from threshold (MCC and Newell, 1984)

Away from threshold the other degrees of freedom relax even more quickly, and so idea of a slow phase equation remains.



- pattern is given by the lines of constant phase θ of a local stripe solution;
- wave vector $\mathbf{q}$ is the gradient of this phase $\mathbf{q} = \nabla\theta$.

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The form of the equation derives from symmetry and smoothness arguments, and expanding up to second order derivatives of the phase.

The parameters $\tau(q)$, $B(q)$ are system dependent functions depending on the equations of motion, $\mathbf{u}_q$, etc.

Small deviations from stripes

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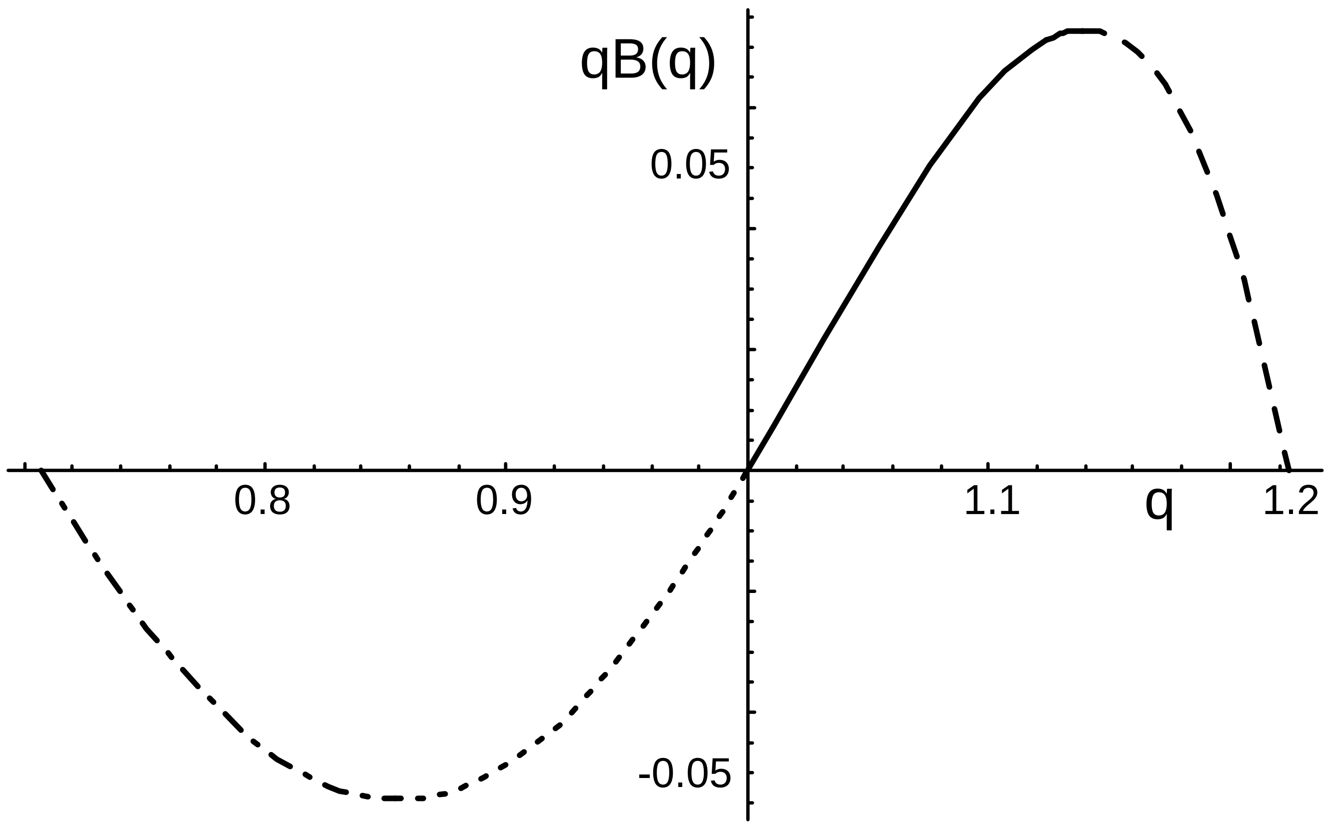
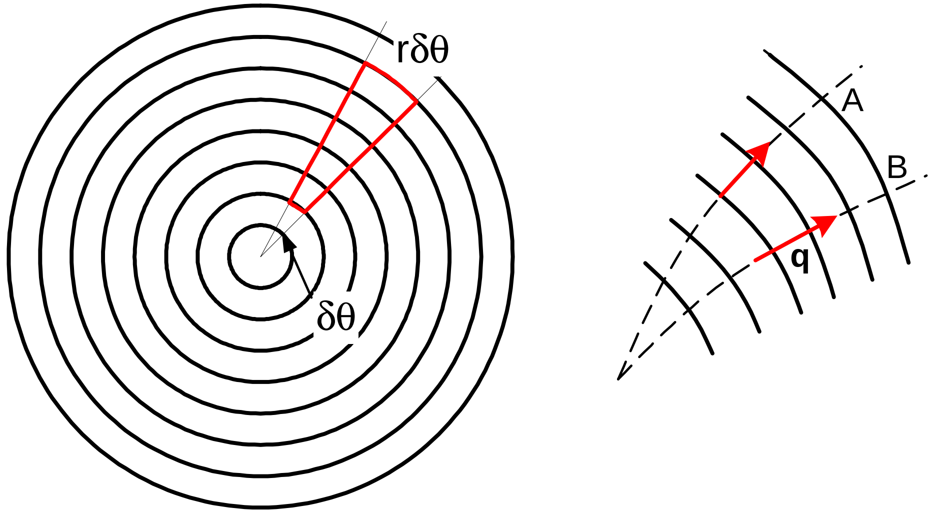
A negative diffusion constant signals instability:

- $[qB(q)]' < 0$: Eckhaus instability
- $B(q) < 0$: zigzag instability

Phase parameters for the Swift-Hohenberg equation



Application: wave number selection by a focus



$$\nabla \cdot (\mathbf{q} B(q)) = 0 \qquad \Rightarrow \qquad \oint B(q)\mathbf{q} \cdot \hat{\mathbf{n}}\, dl = \mathbf{0}$$

$$q B(q) = \frac{C}{r} \underset{r \to \infty}{\longrightarrow} 0$$

i.e. $q \to q_f$ with $B(q_f) = 0$, the wave number of the zigzag instability!

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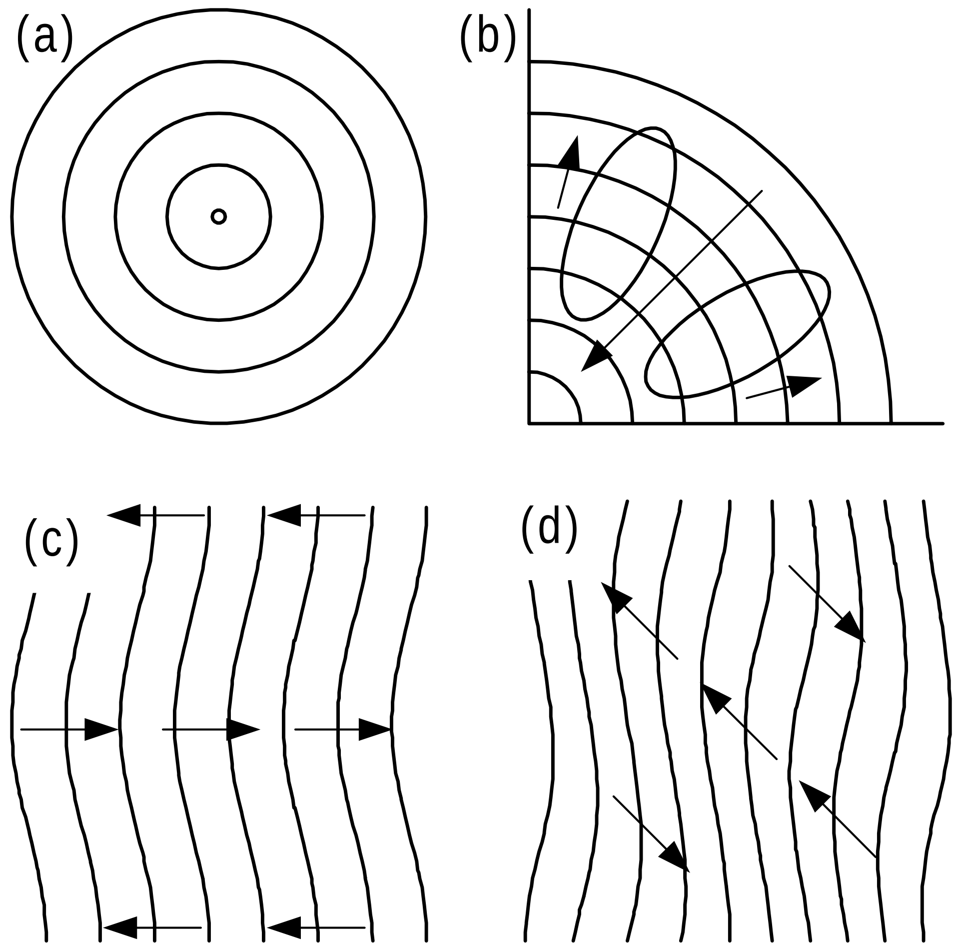
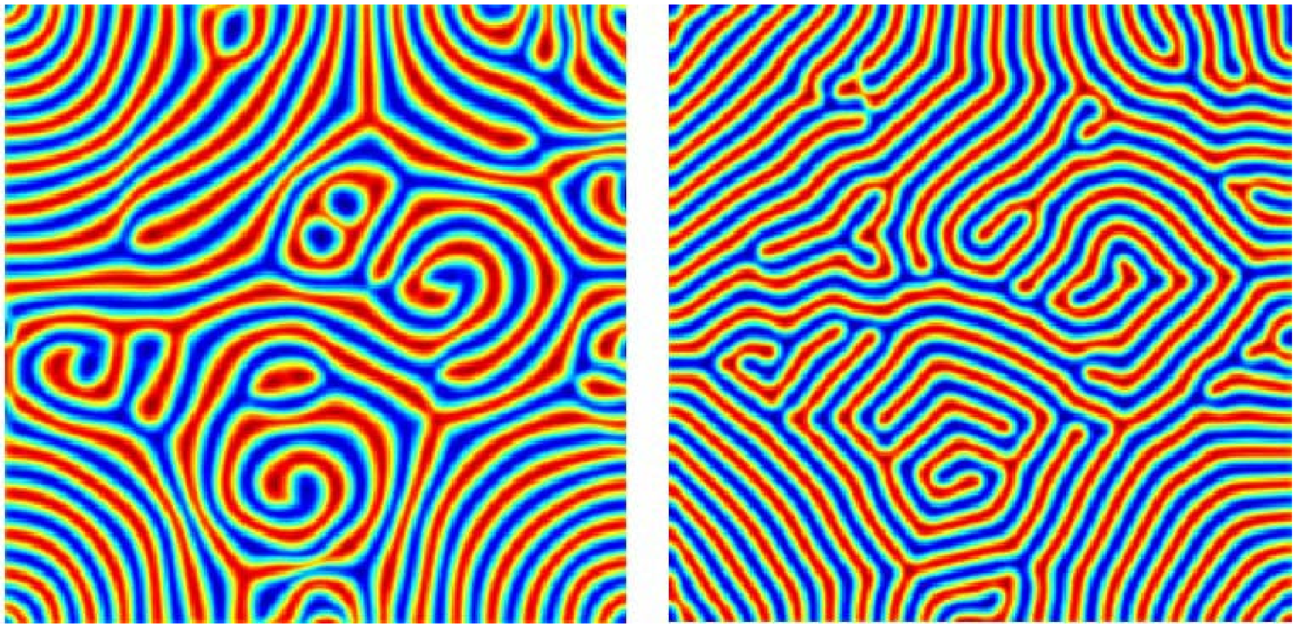
$$\partial_t \theta \rightarrow \partial_t \theta + \mathbf{V} \cdot \nabla \theta .$$

The advection horizontal velocity **V** is in turn driven by the pattern. Writing **V** in terms of a stream function ζ so that $\mathbf{V} = (-\partial_y \zeta, \partial_x \zeta)$

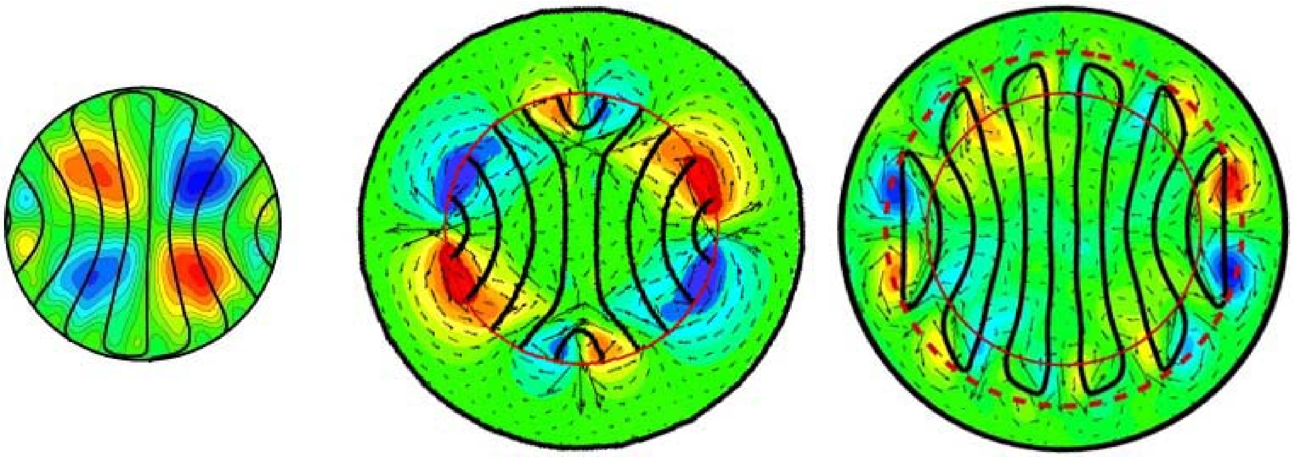
$$\nabla_\perp^2 \zeta = \hat{\mathbf{z}} \cdot \nabla_\perp \times \mathbf{V} = \gamma \hat{\mathbf{z}} \cdot \nabla_\perp \times [\mathbf{k} \nabla \cdot (\mathbf{k} A^2)]$$


(a)
(b)
(c)
(d)



(a) with mean flow (b) mean flow quenched

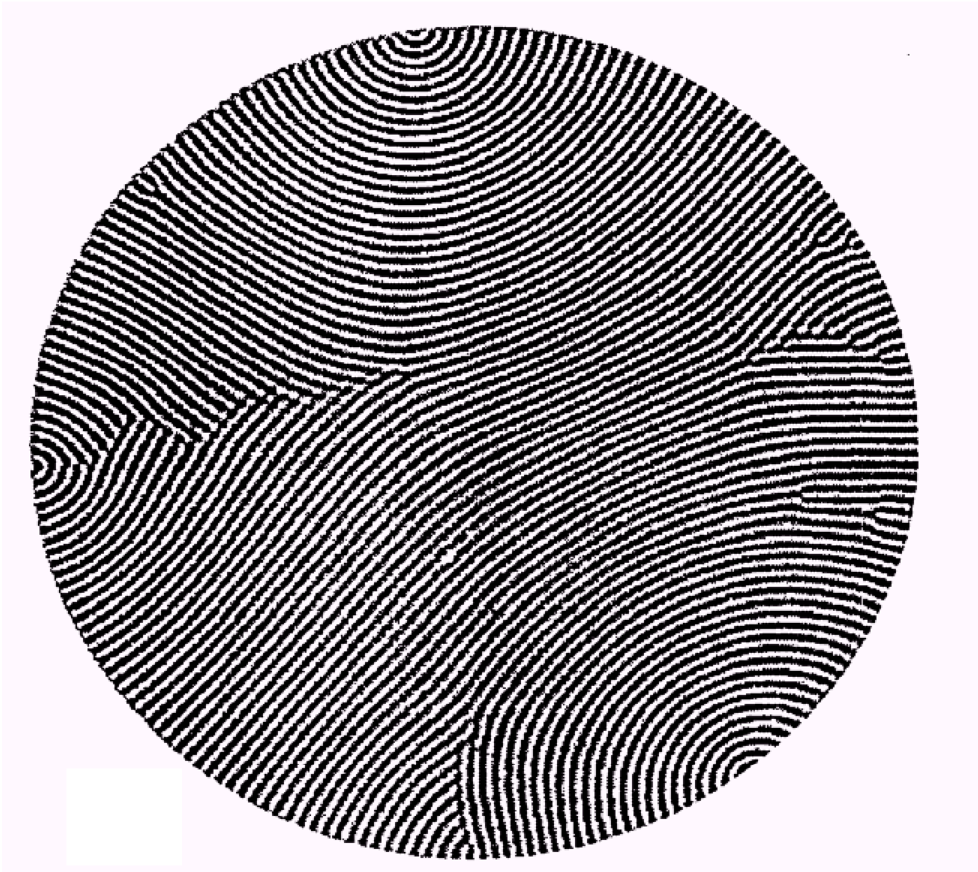
[Chiam, Paul, MCC, and Greenside (2003)]



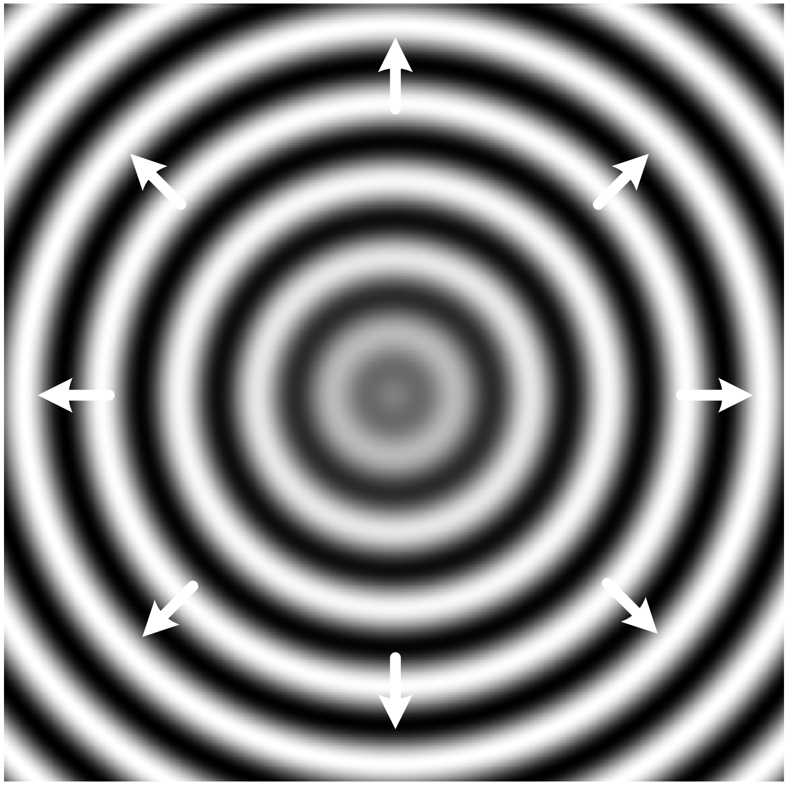
3 convection cells with different side wall conditions: (a) rigid; (b) finned; and (c) ramped. Case (a) is dynamic, the others static.

[Paul, MCC, and Fischer (2002)]

Defects

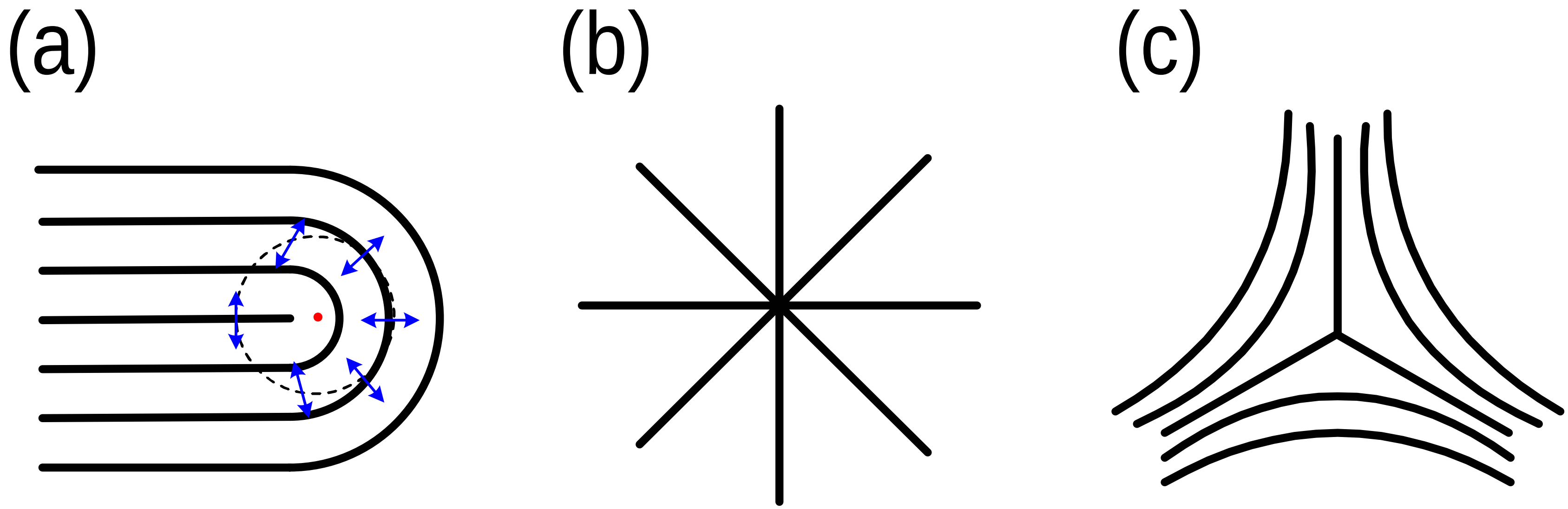


Focus/target defect



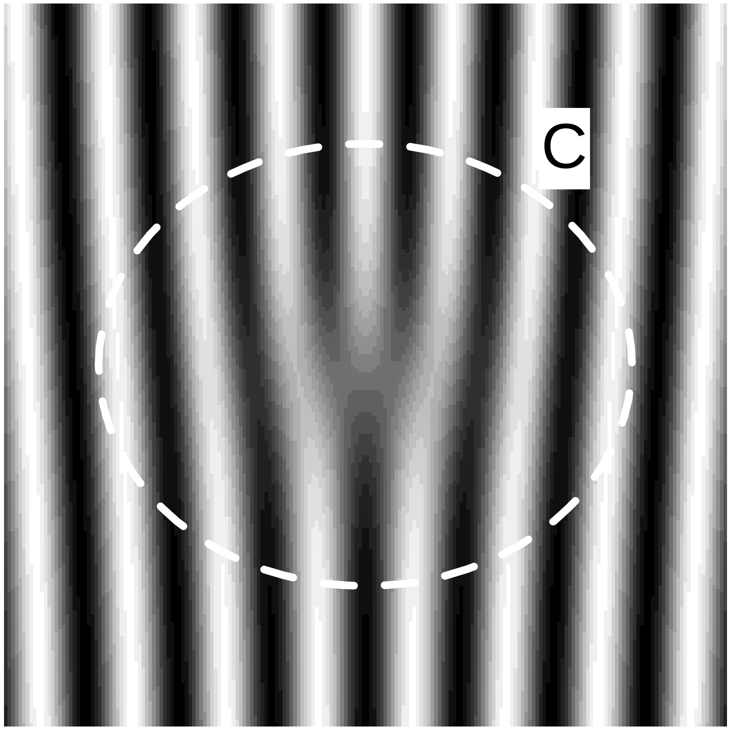
Wavevector winding number = 1

Disclinations



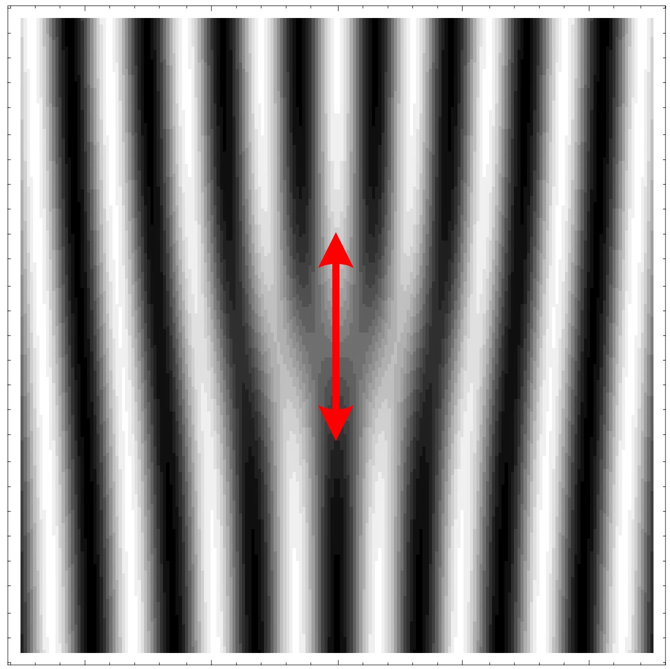
Winding numbers: (a) $\frac{1}{2}$; (b) 1; (c) -1

Dislocation



$$\text{Phase winding number } = \frac{1}{2\pi}\oint \nabla\theta \cdot \mathbf{dl} = 1$$

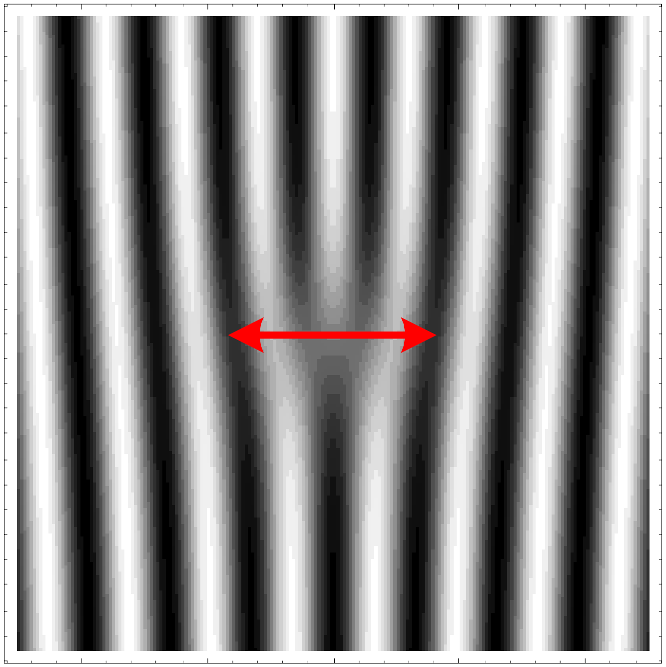
Dislocation climb



Smooth motion through symmetry related states

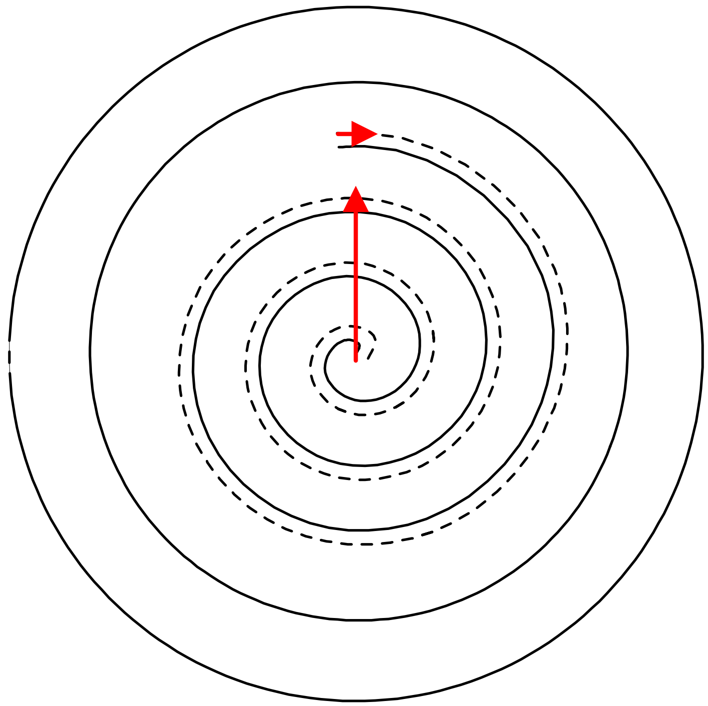
$$v_d \approx \beta(q - q_d)$$

Dislocation glide



Motion involves stripe pinch off, and is pinned to the periodic structure

Spiral Dynamics: experiments of Plapp et al. (1998)



Dislocation motion

$$v_d = \omega r_d = \beta(q(r_d) - q_d) \qquad (*)$$

Spiral motion from phase equation

$$\omega = -\tau_q^{-1} \frac{1}{r} \frac{\partial}{\partial r} (r q B(q))$$

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$$q(r) - q_f = -\omega r/2\alpha + Cr^{-1}.$$

Evaluating at r_d and combining with Eq. (*) gives ω.

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Is this relevant to spiral defect chaos?

Conclusions

In today's lectures I have described the implications of symmetry on the theoretical methods for stationary patterns:

- amplitude equation in 2d
- Swift-Hohenberg equation and generalizations
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The methods have various advantages and disadvantages, and have given great insights, but none is a complete approach even near threshold.

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Next lecture: oscillatory instabilities.