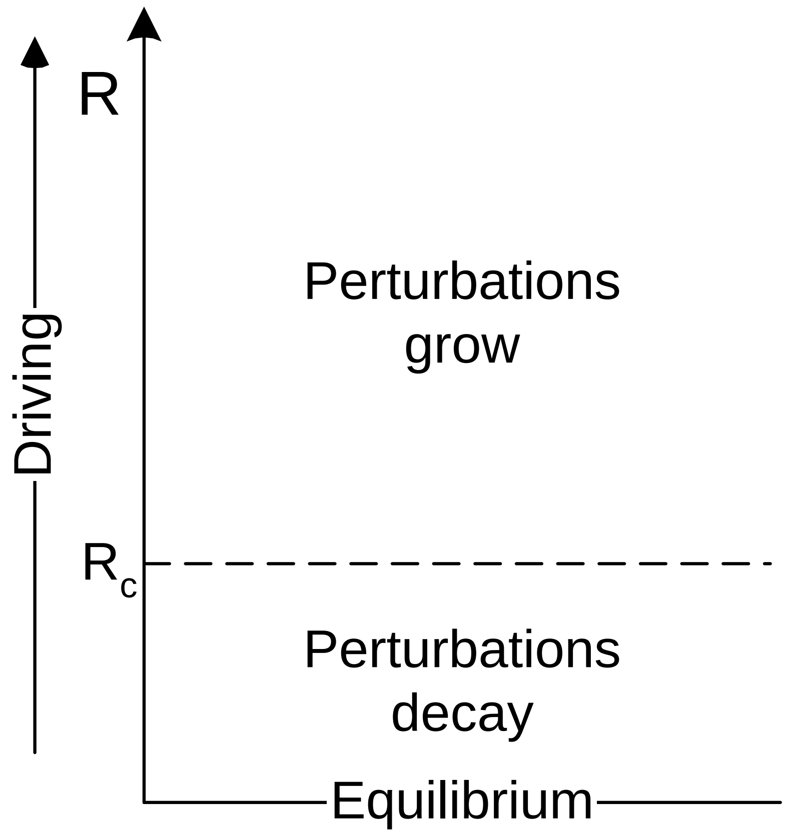
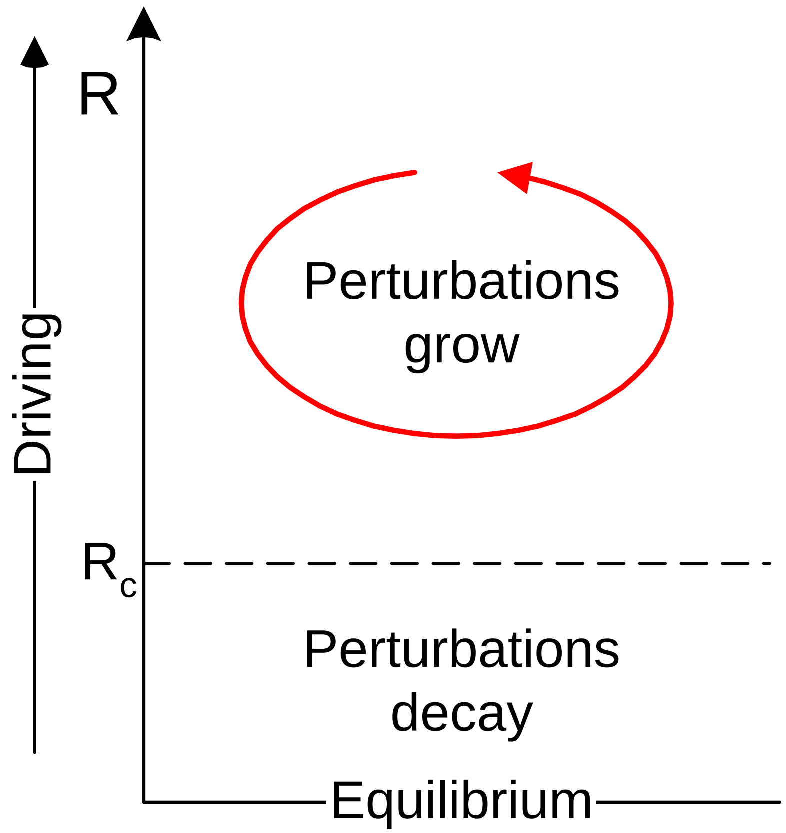
Pattern Formation in Spatially Extended Systems

Lecture 1

- linear instability
- nonlinear saturation
- stability balloon
- amplitude equation


R
Driving
Perturbations
grow
R_c
Perturbations
decay
Equilibrium


R
Driving
Perturbations
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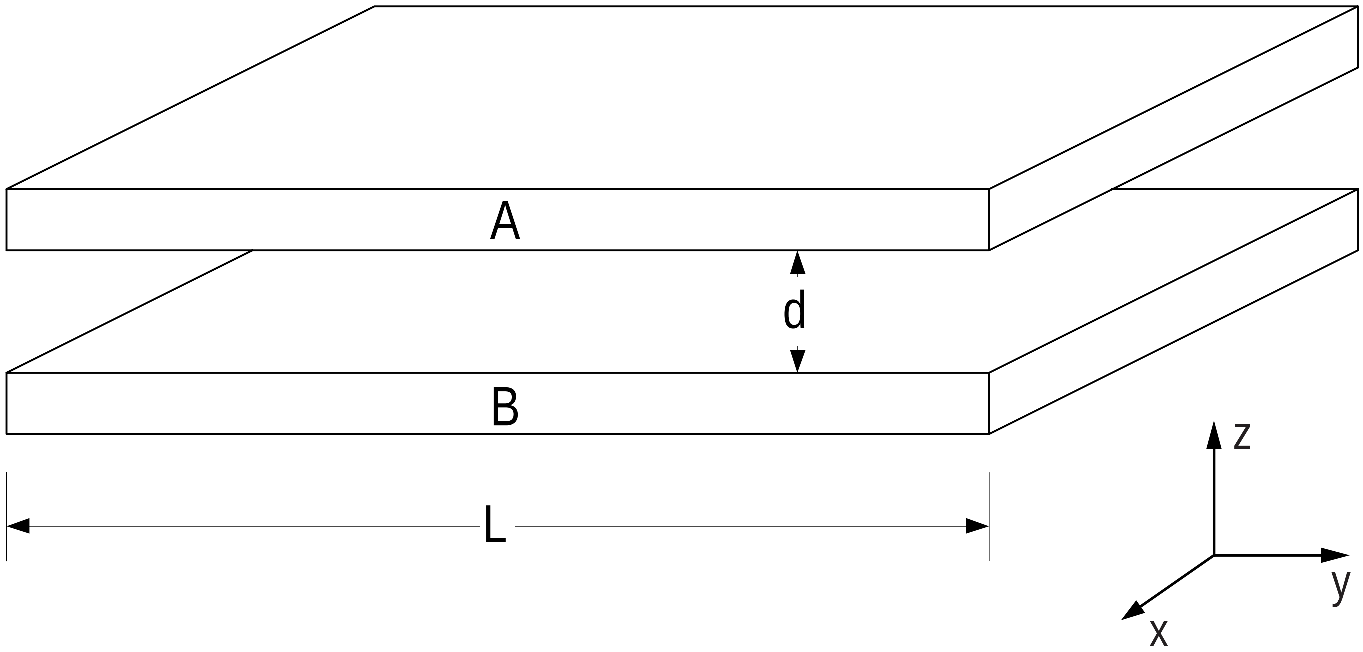
Equilibrium

Pattern formation occurs in a spatially extended system when the growing perturbation about the spatially uniform state has spatial structure (a mode with nonzero wave vector).


A
d
B
L
z
y
x

A first approach to patterns: linear stability analysis

1. Find equations of motion of the physical variables $\mathbf{u}(x, y, z, t)$

2. Find the *uniform* base solution $\mathbf{u}_b(z)$ *independent* of x, y, t

3. Focus on deviation from $\mathbf{u}_b$

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{u}_b(z) + \delta\mathbf{u}(\mathbf{x}, t)$$

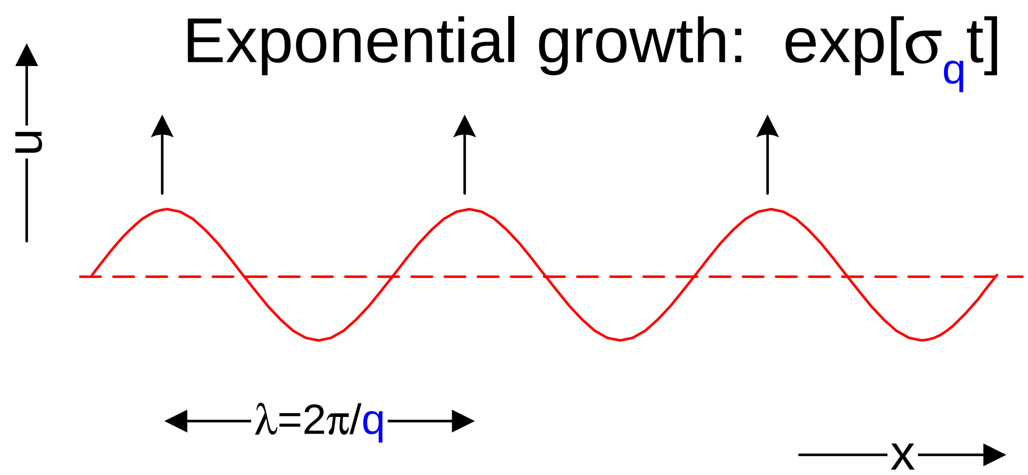
4. Linearize equations about $\mathbf{u}_b$, i.e. substitute into equations of part (1) and keep all terms with just one power of $\delta\mathbf{u}$. This will give an equation of the form

$$\partial_t \delta\mathbf{u} = \hat{\mathbf{L}}\, \delta\mathbf{u}$$

   where $\hat{\mathbf{L}}$ may involve $\mathbf{u}_b$ and include spatial derivatives acting on $\delta\mathbf{u}$

5. Since $\hat{\mathbf{L}}$ is independent of x, y, t we can find solutions

$$\delta\mathbf{u}_{\mathbf{q}}(\mathbf{x}_\perp, z, t) = \mathbf{u}_{\mathbf{q}}(z)\, e^{i\mathbf{q}\cdot\mathbf{x}_\perp}\, e^{\sigma_{\mathbf{q}} t}$$



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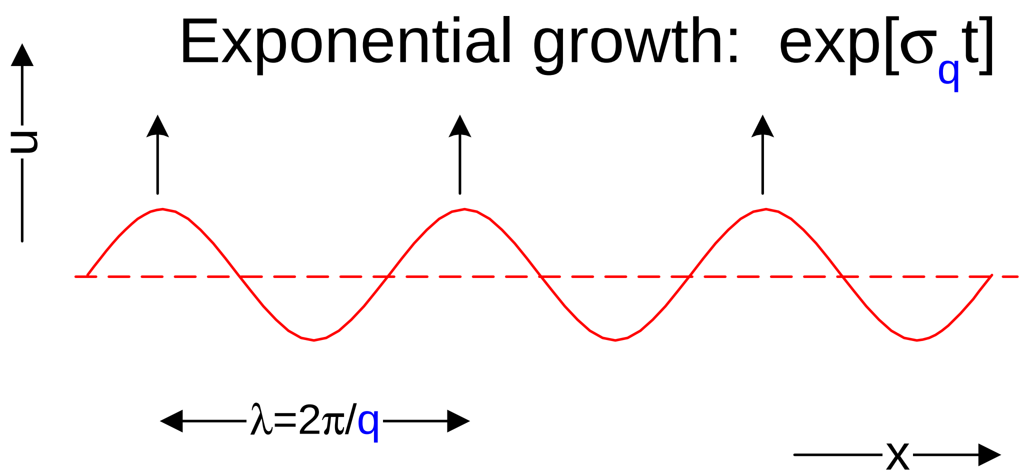
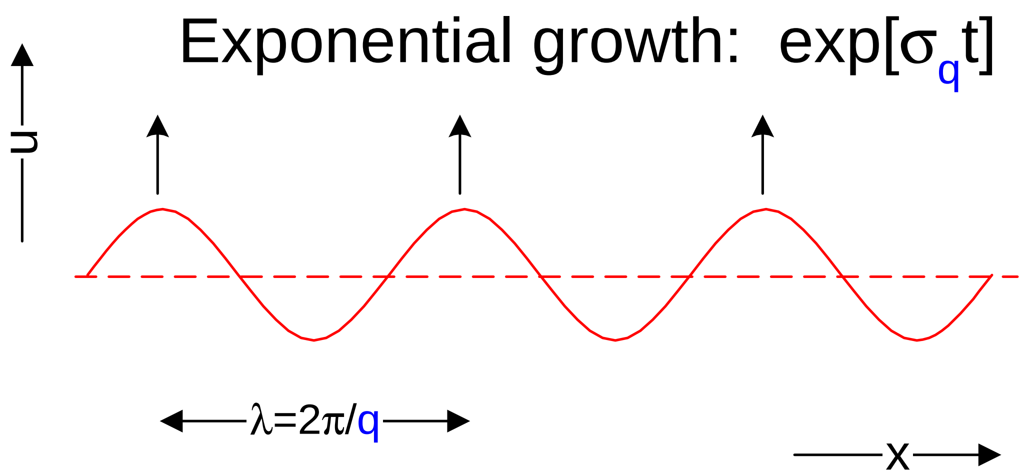


$$\delta \mathbf{u_q}(\mathbf{x}_\perp, z, t) = \mathbf{u_q}(z)\, e^{i\,\mathbf{q}\cdot\mathbf{x}_\perp}\, e^{\sigma_\mathbf{q} t}$$

Re $\sigma_\mathbf{q}$ gives exponential growth or decay



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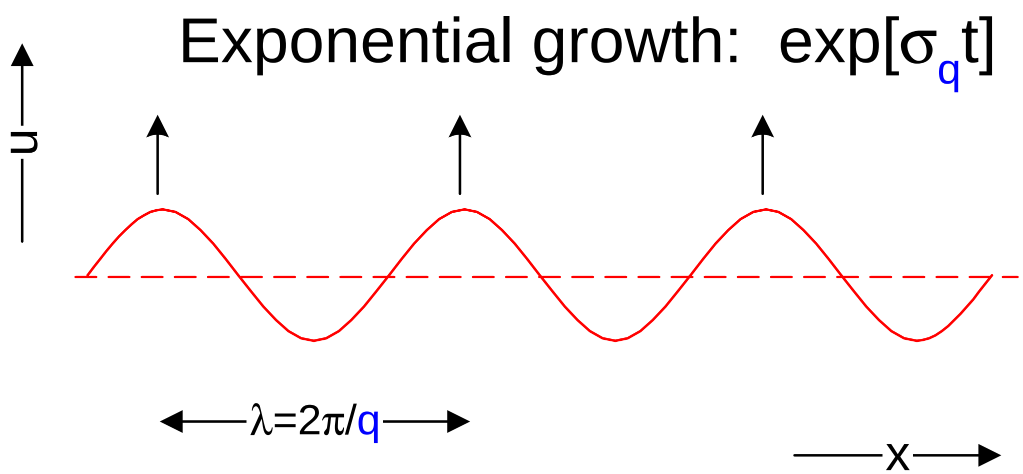
Im $\sigma_\mathbf{q} = -\omega_\mathbf{q}$ gives oscillations, waves $e^{i(\mathbf{q}\cdot\mathbf{x}_\perp - \omega_\mathbf{q} t)}$

$$\begin{aligned} \text{Im } \sigma_\mathbf{q} = 0 \quad &\Longrightarrow \quad \text{Stationary instability} \\ \text{Im } \sigma_\mathbf{q} \neq 0 \quad &\Longrightarrow \quad \text{Oscillatory instability} \end{aligned}$$



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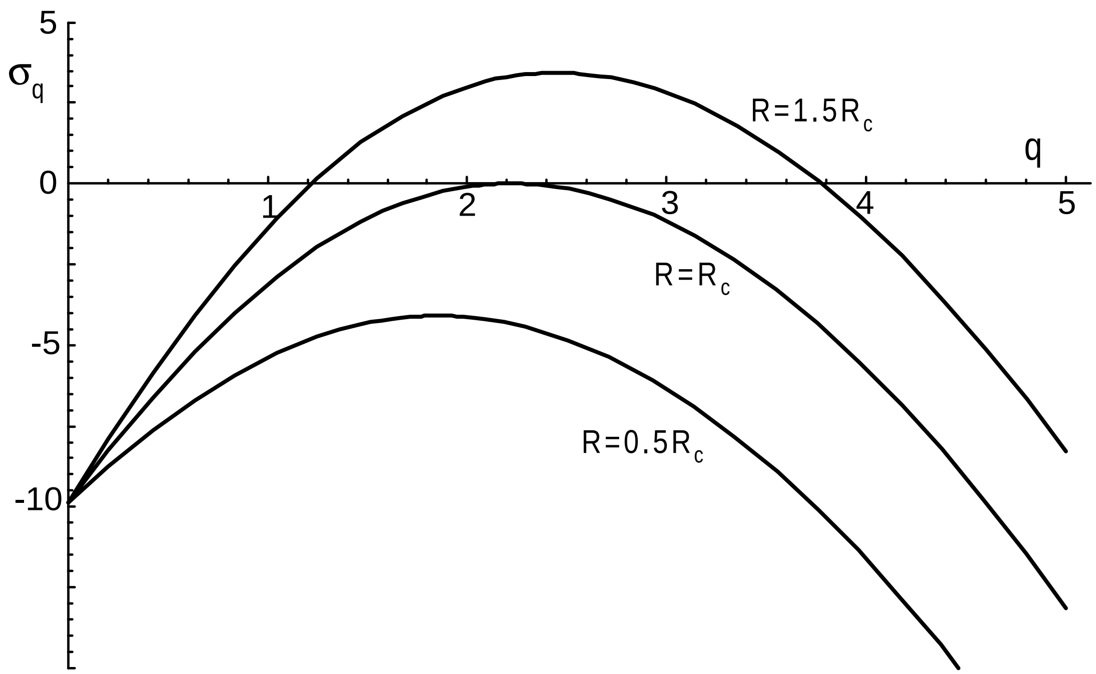
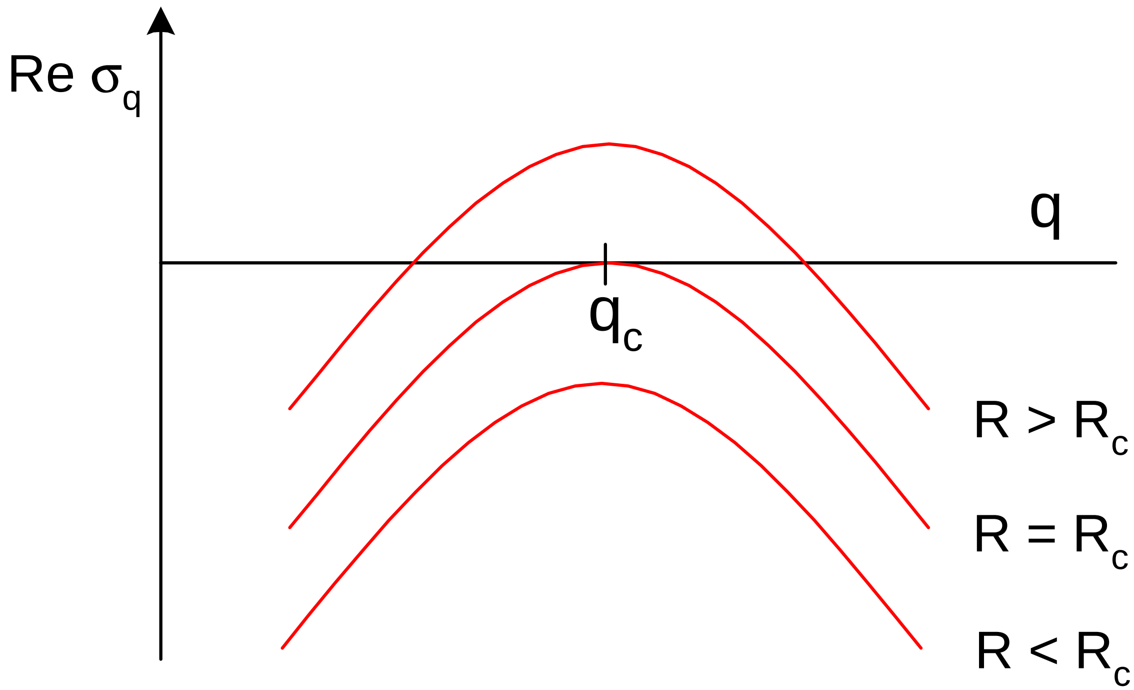
For this lecture I will look at the case of stationary instability

Rayleigh's calculation



$$(\sigma^{-1}\sigma_q + \pi^2 + q^2)(\sigma_q + \pi^2 + q^2) - Rq^2/(\pi^2 + q^2) = 0$$

Parabolic approximation near maximum



For R near R_c and q near q_c

$$\mathrm{Re}\ \sigma_{\mathbf{q}} = \tau_0{}^{-1}[\varepsilon - \xi_0{}^2\,(q - q_c)^2] \qquad \text{with} \qquad \varepsilon = \frac{R - R_c}{R_c}$$

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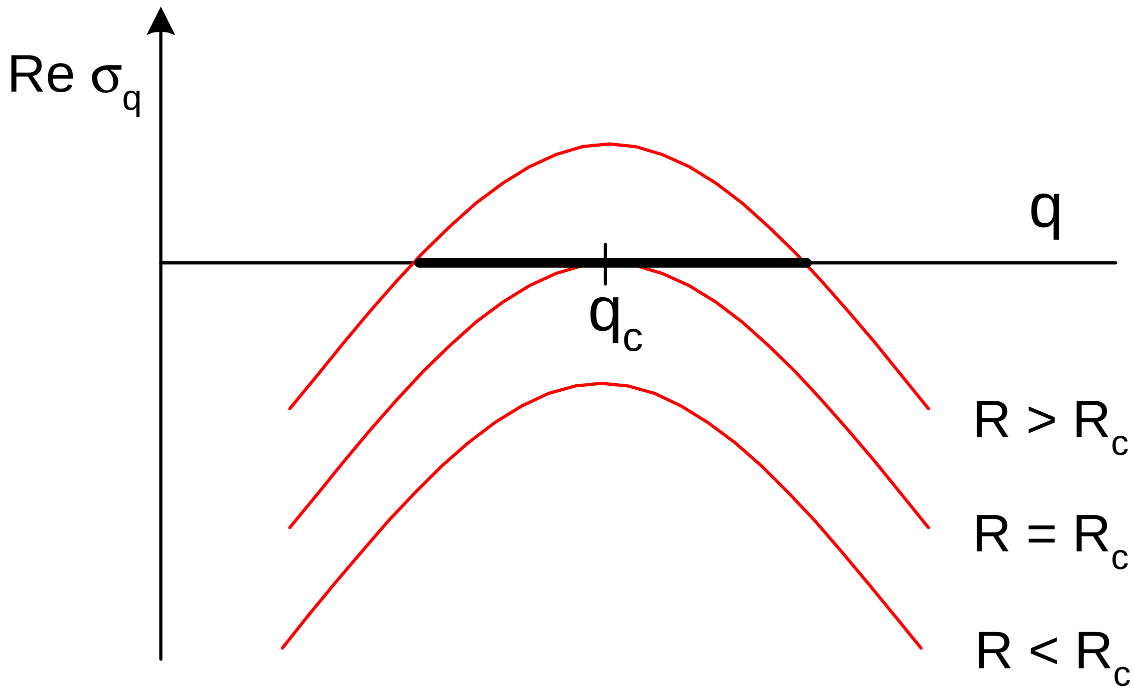

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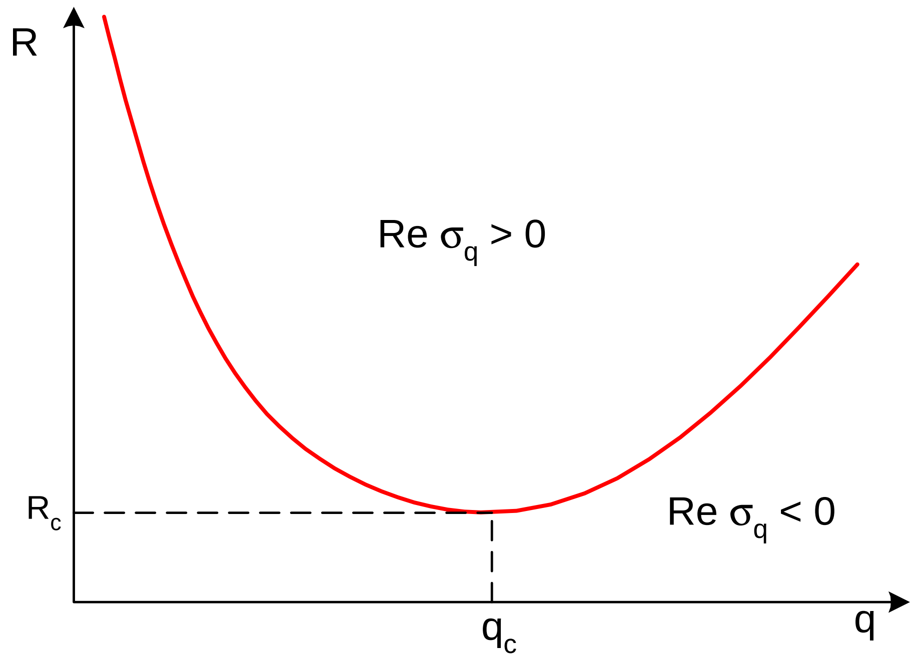
Neutral stability curve



Setting Re $\sigma_{\mathbf{q}} = 0$ defines the neutral stability curve $R = R_c(q)$

$$\text{Rayleigh :} \qquad R_c(q) = \frac{(q^2 + \pi^2)^3}{q^2} \Rightarrow \qquad R_c = \frac{27\pi^4}{4},\ q_c = \frac{\pi}{\sqrt{2}}$$

Linear stability theory is often a useful first step in understanding pattern formation:

- Often is quite easy to do either analytically or numerically
- Displays the important physical processes
- Gives the length scale of the pattern formation $1/q_c$

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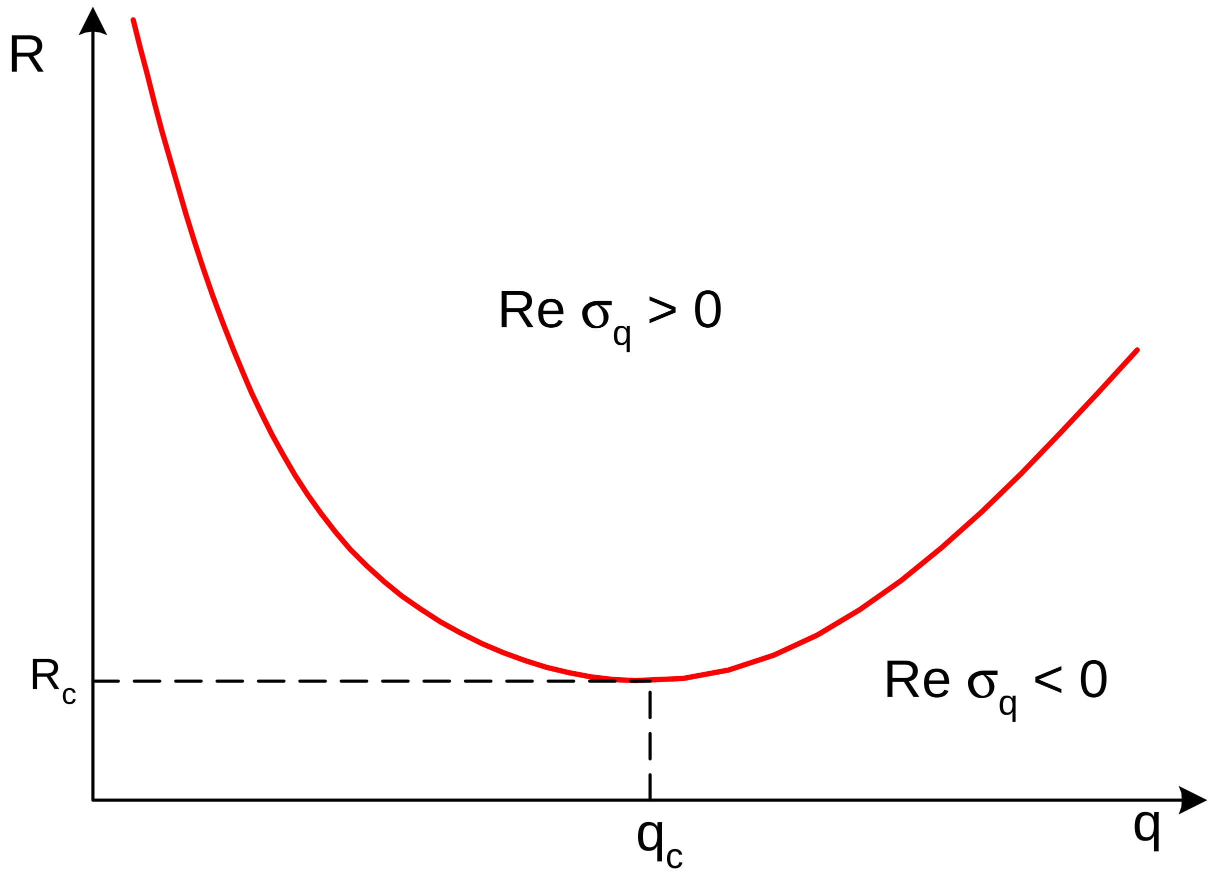
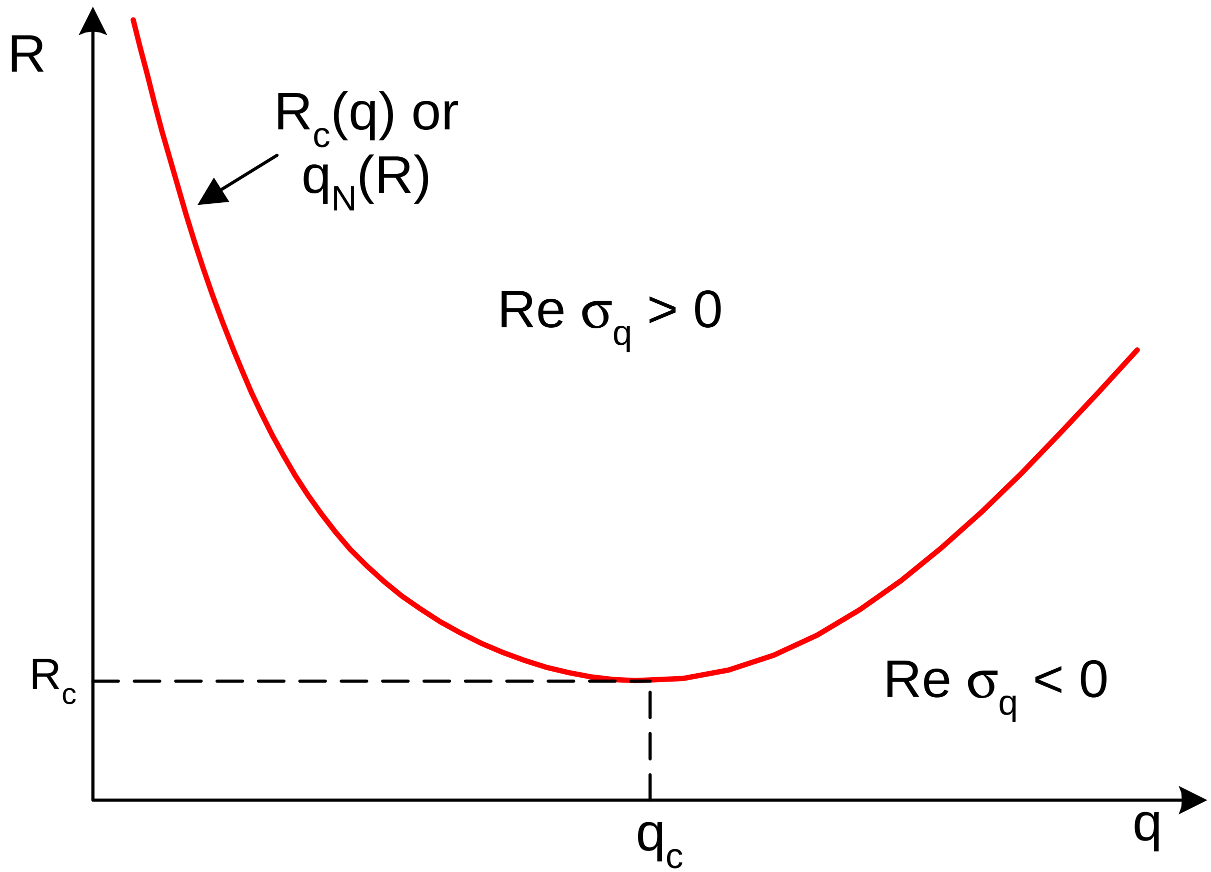
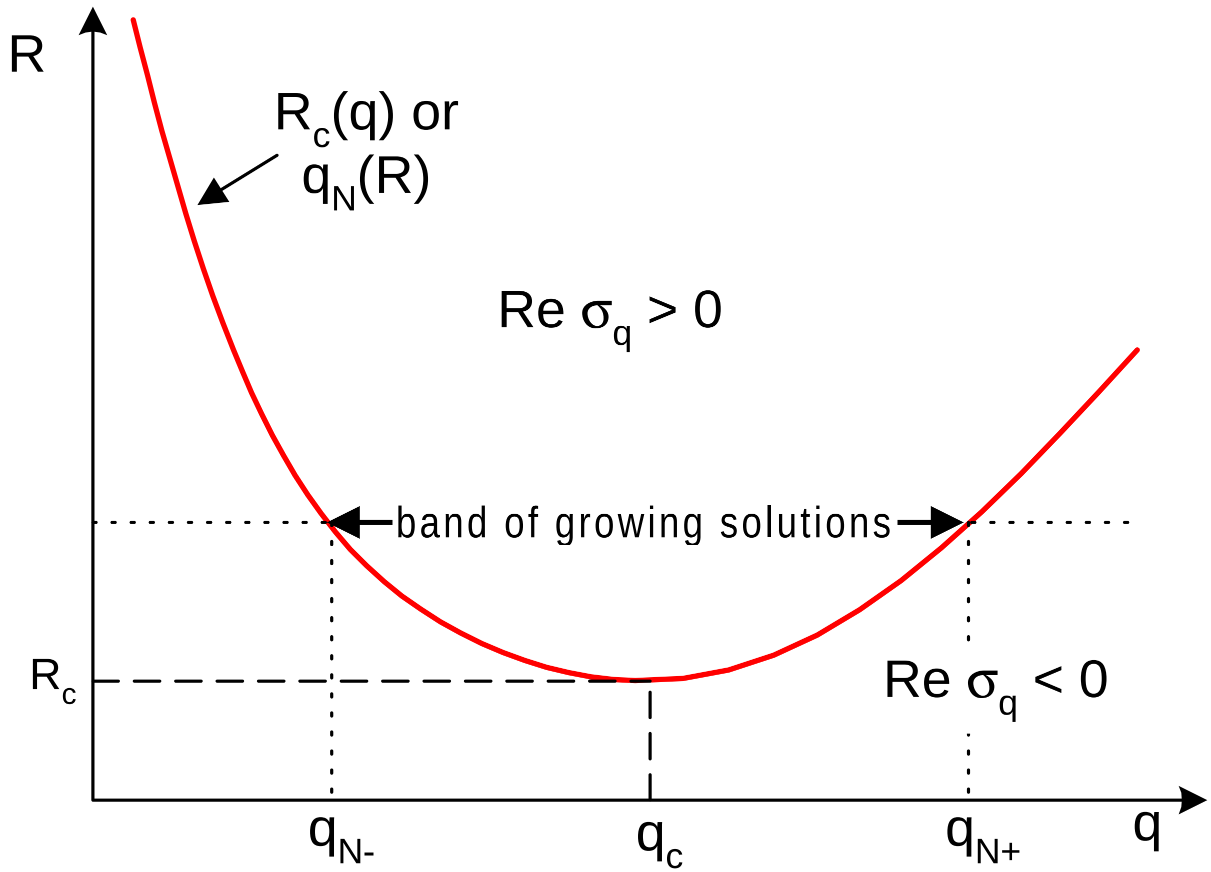
But:

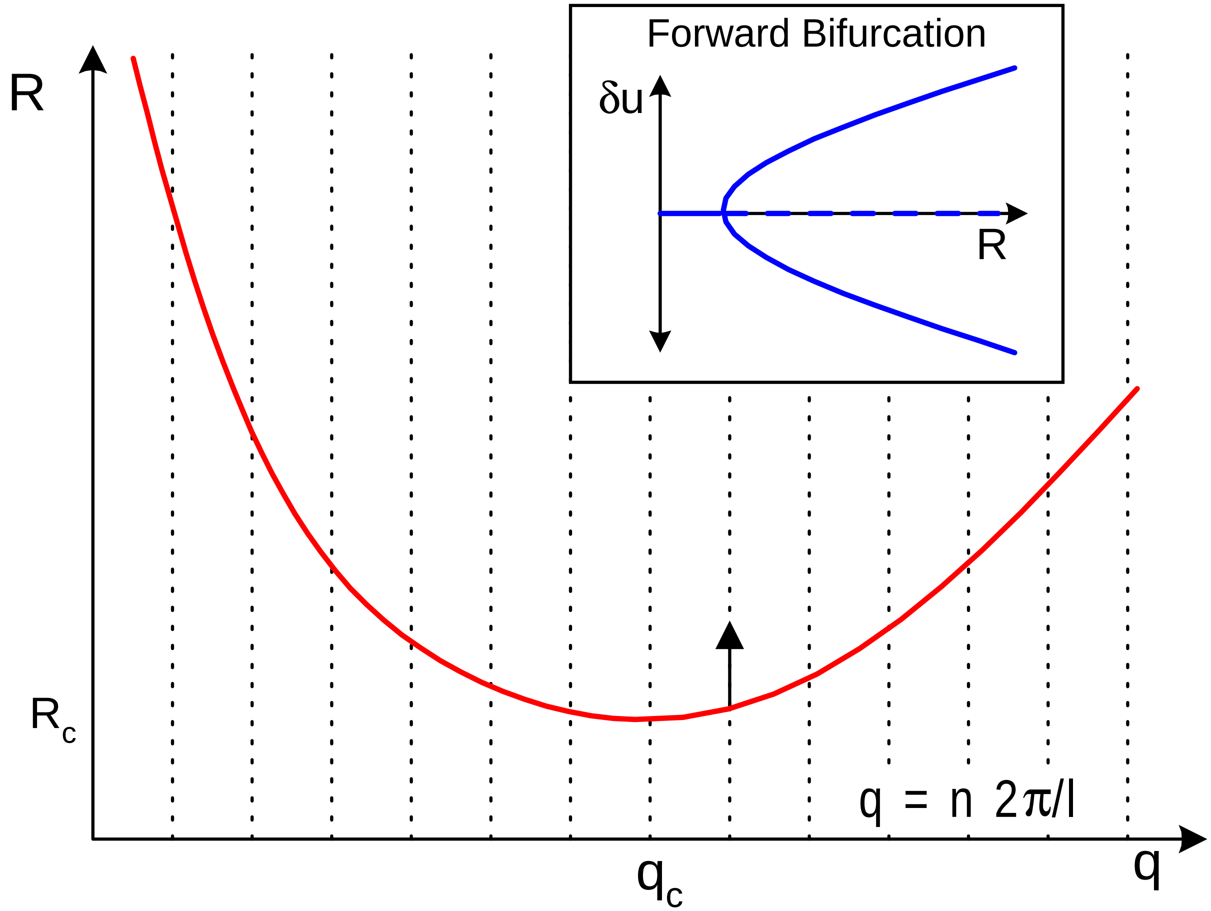
- Leaves us with unphysical exponentially growing solutions

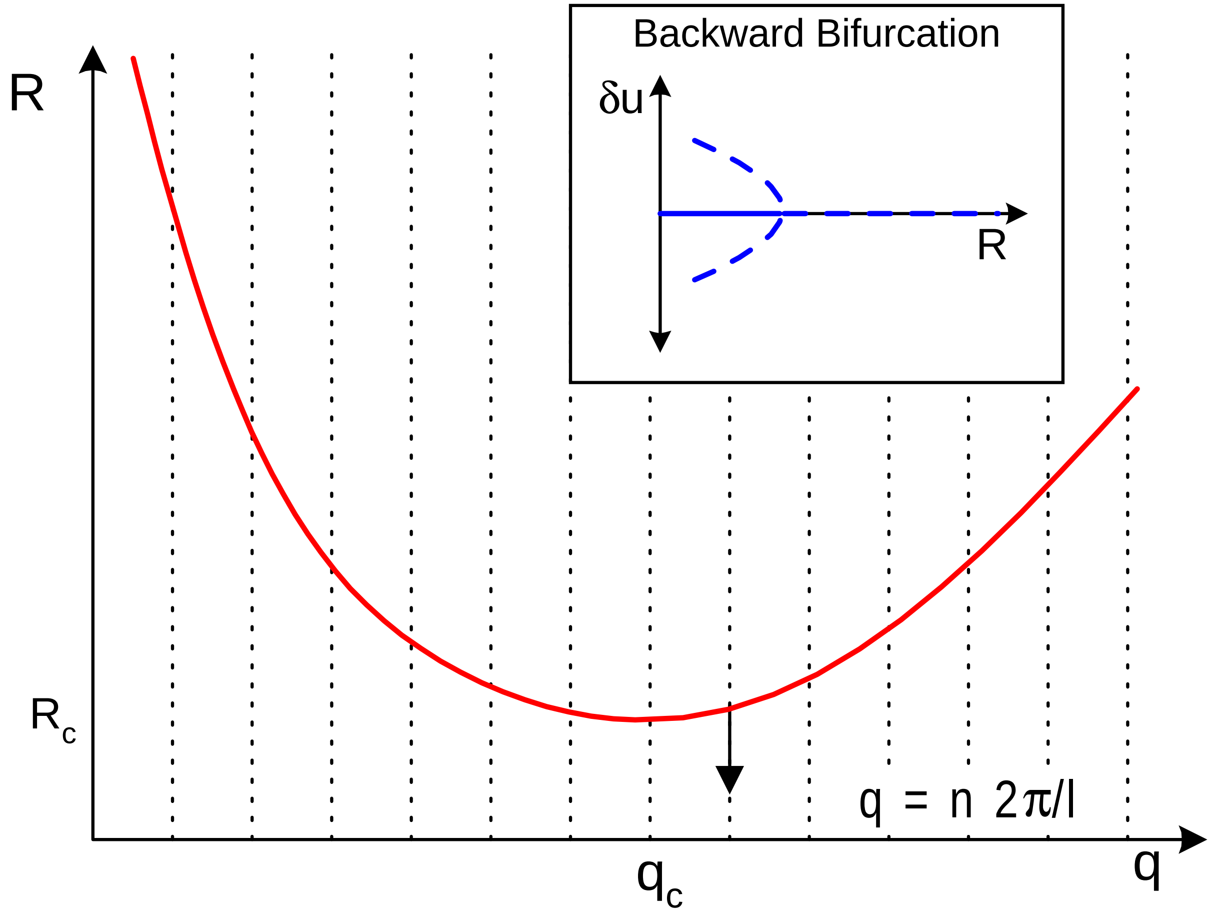
Nonlinearity

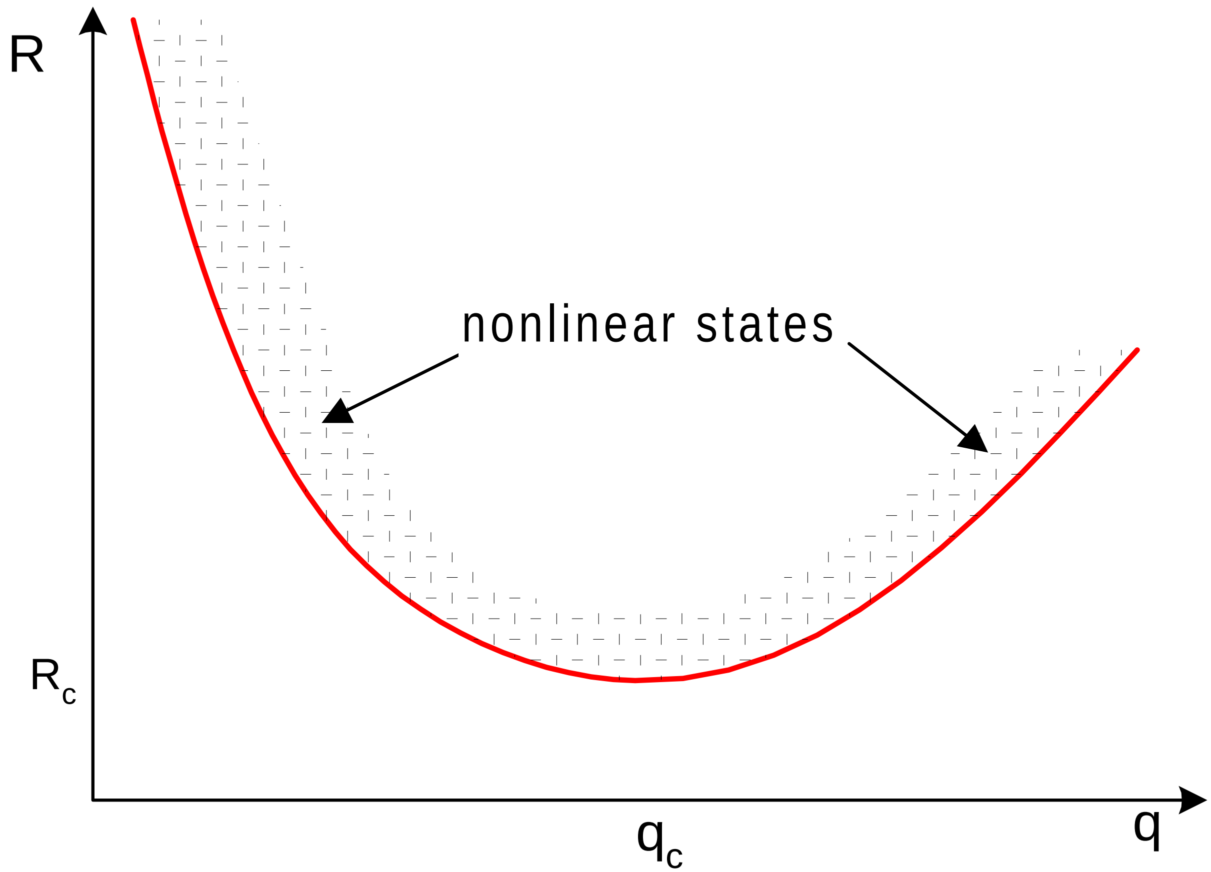

R
Re σq > 0
Rc
Re σq < 0
qc
q

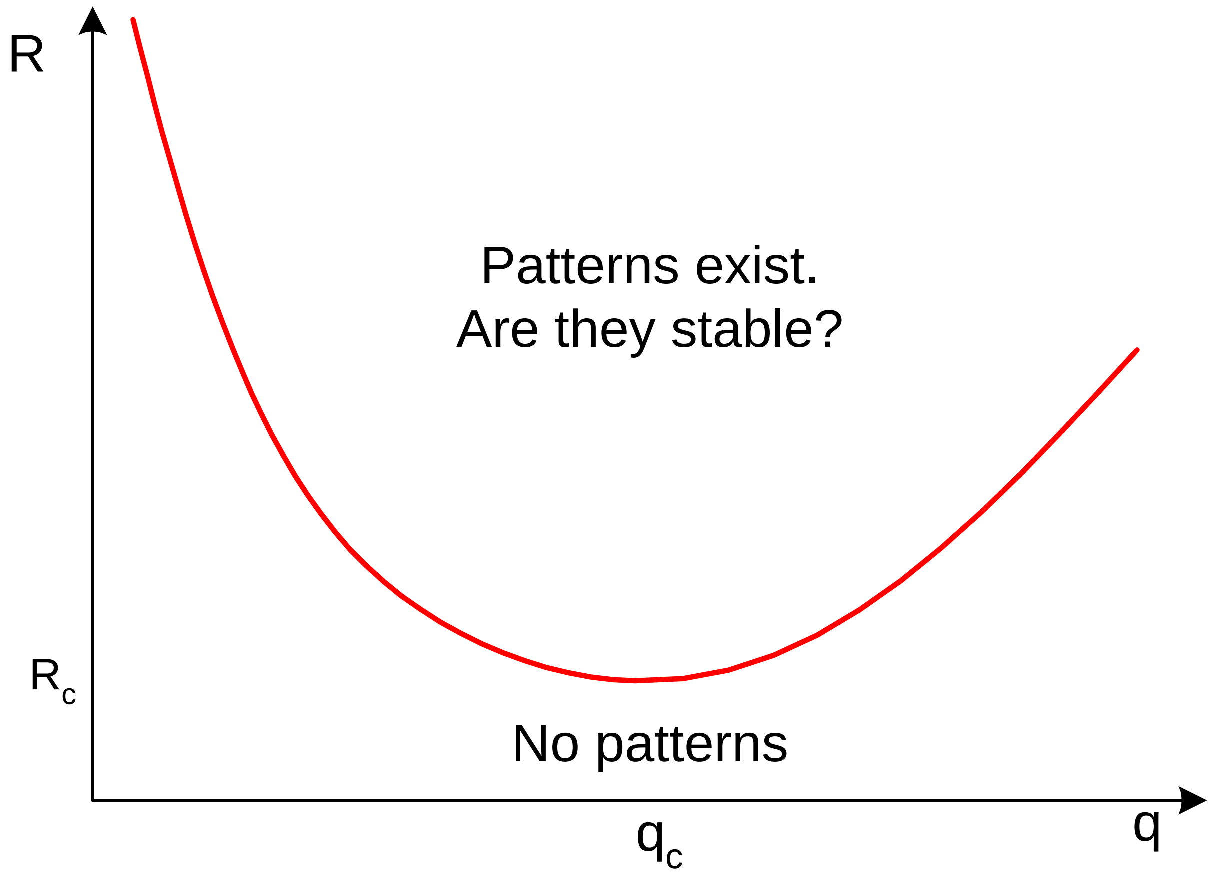

R
$R_c(q)$ or
$q_N(R)$
Re $\sigma_q > 0$
R_c
Re $\sigma_q < 0$
q_c
q

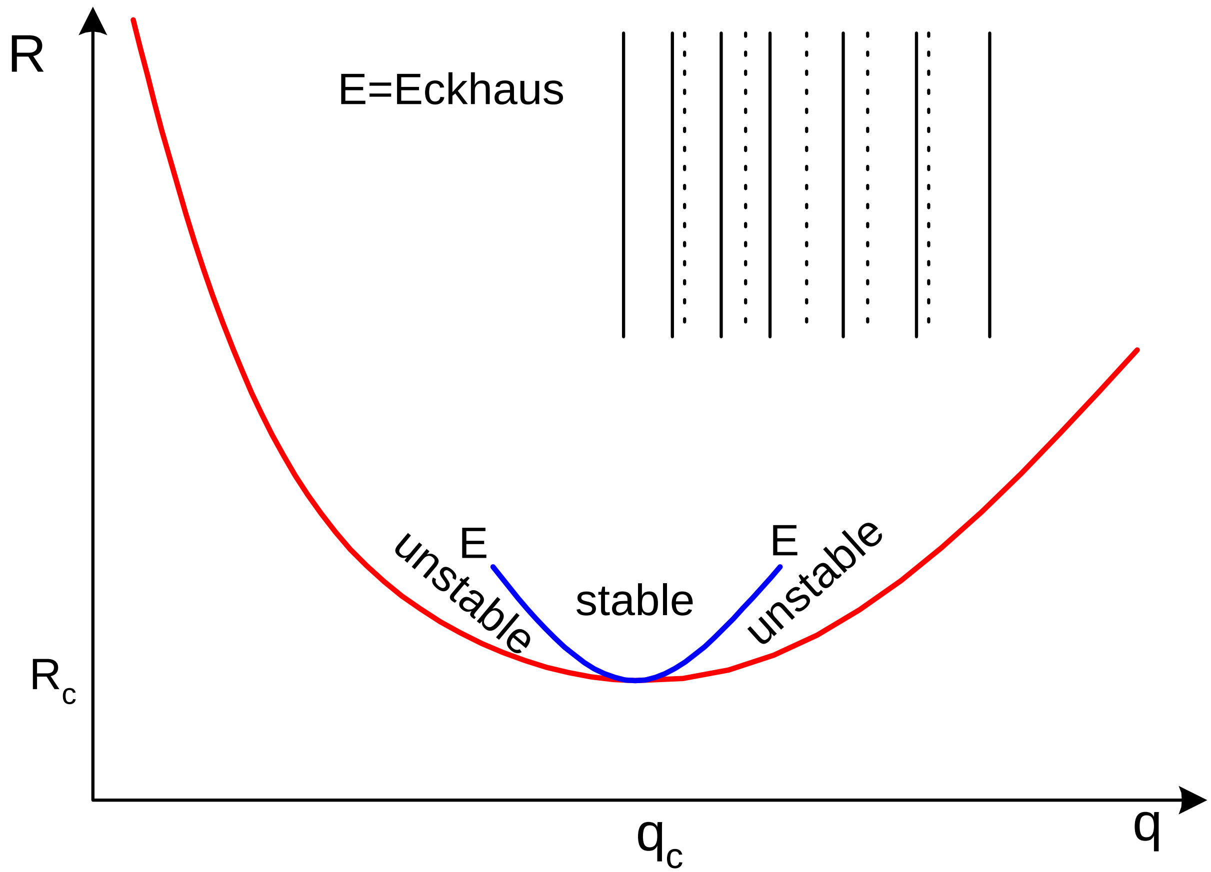

R
Rc(q) or
qN(R)
Re σq > 0
band of growing solutions
Rc
Re σq < 0
qN-
qc
qN+
q

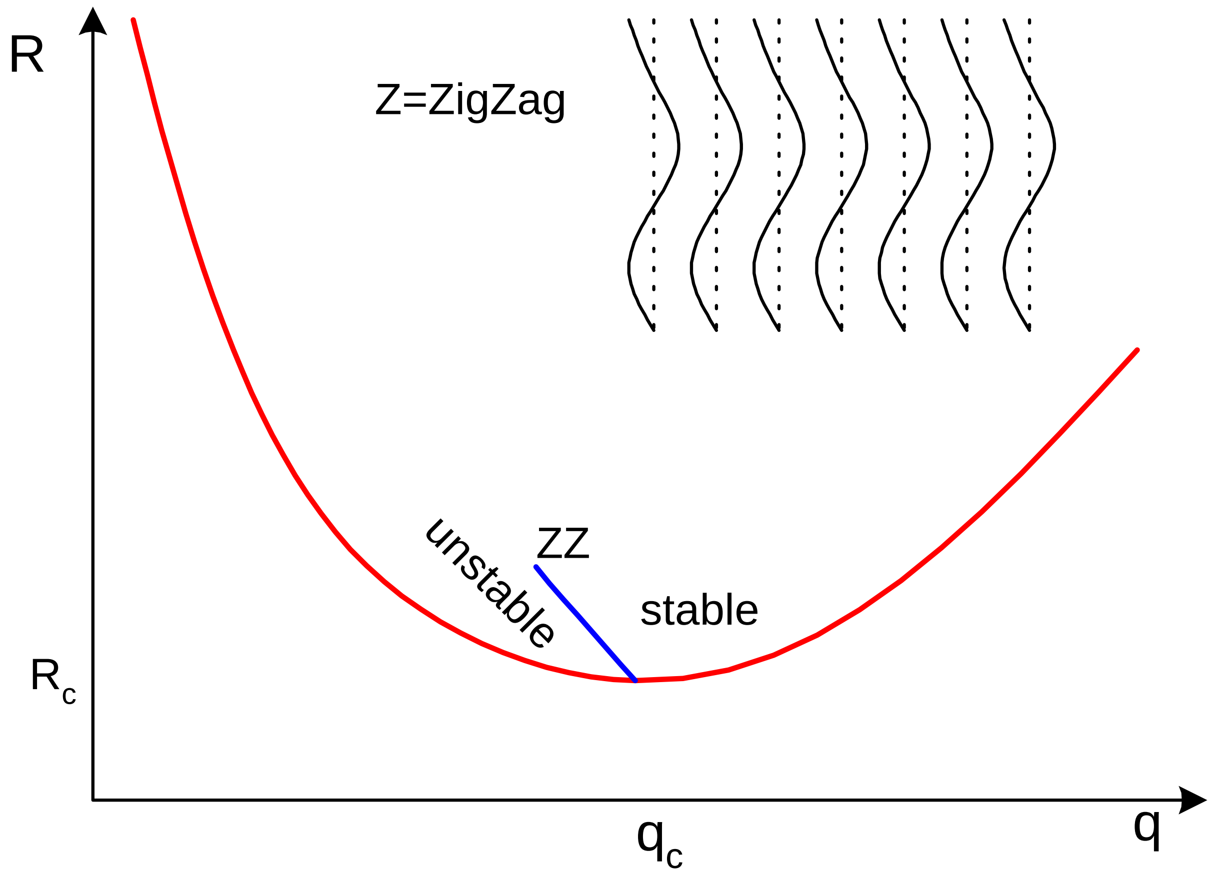

Forward Bifurcation
δu
R
R
R_c
q = n 2π/l
q_c
q

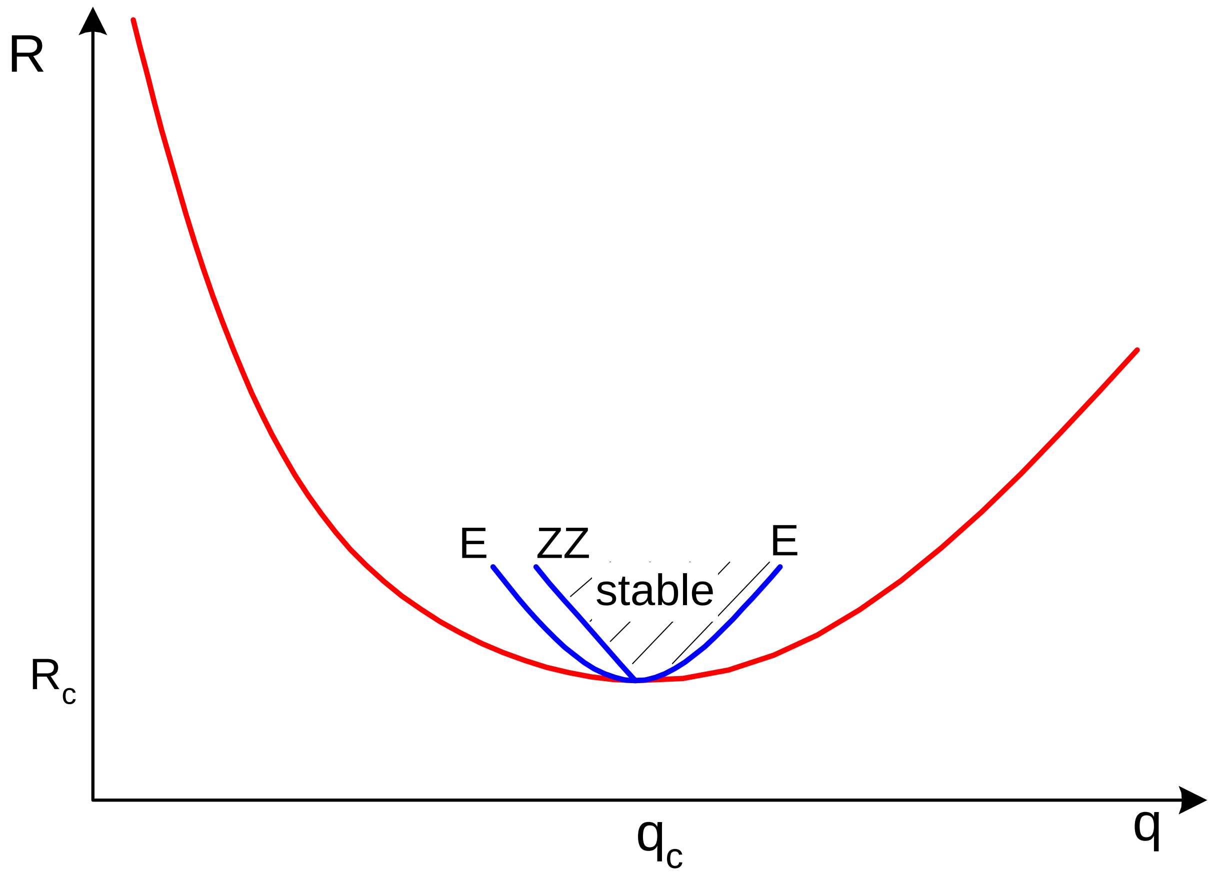

Backward Bifurcation
δu
R
R
R_c
$q = n\ 2\pi/l$
q_c
q

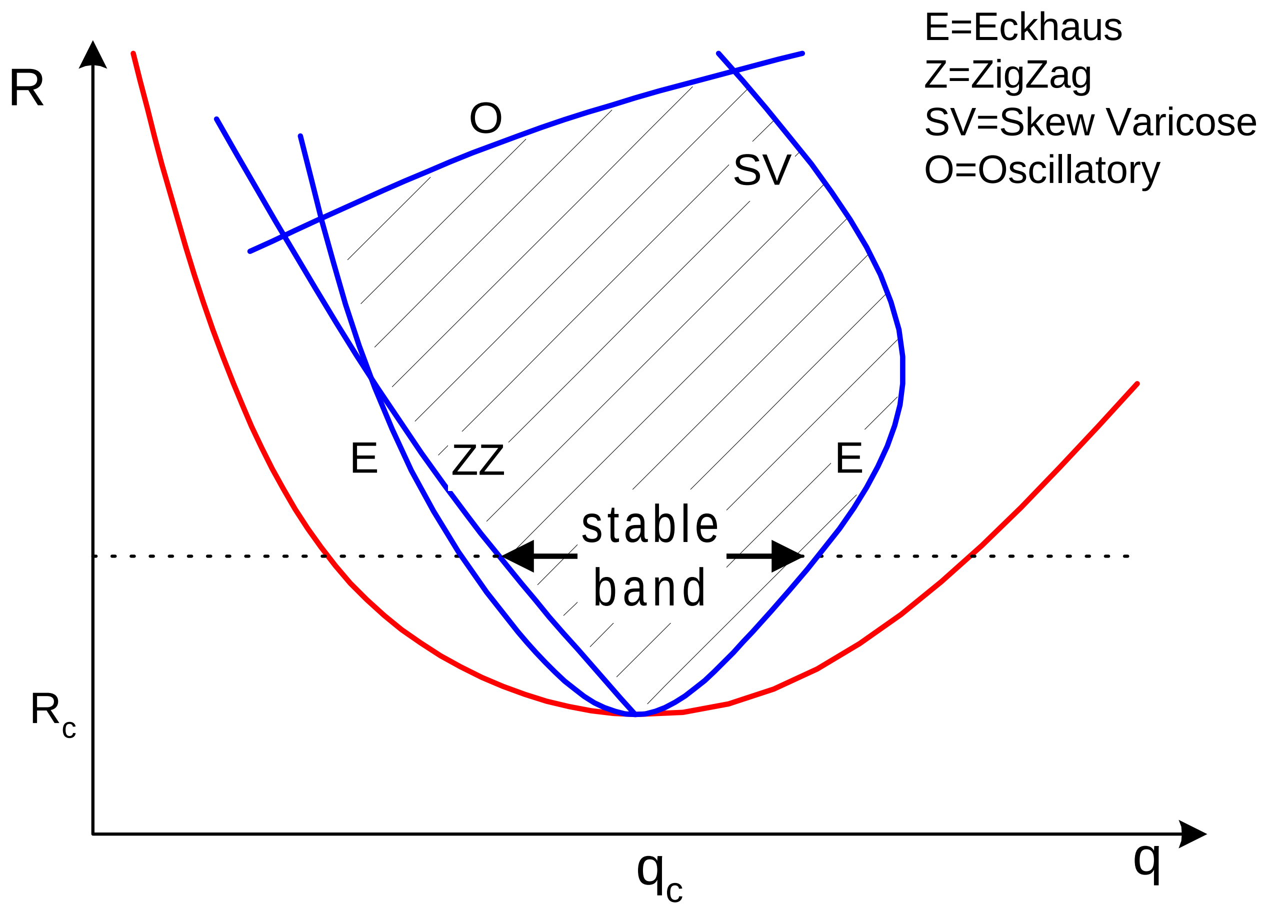

R
nonlinear states
R_c
q_c
q

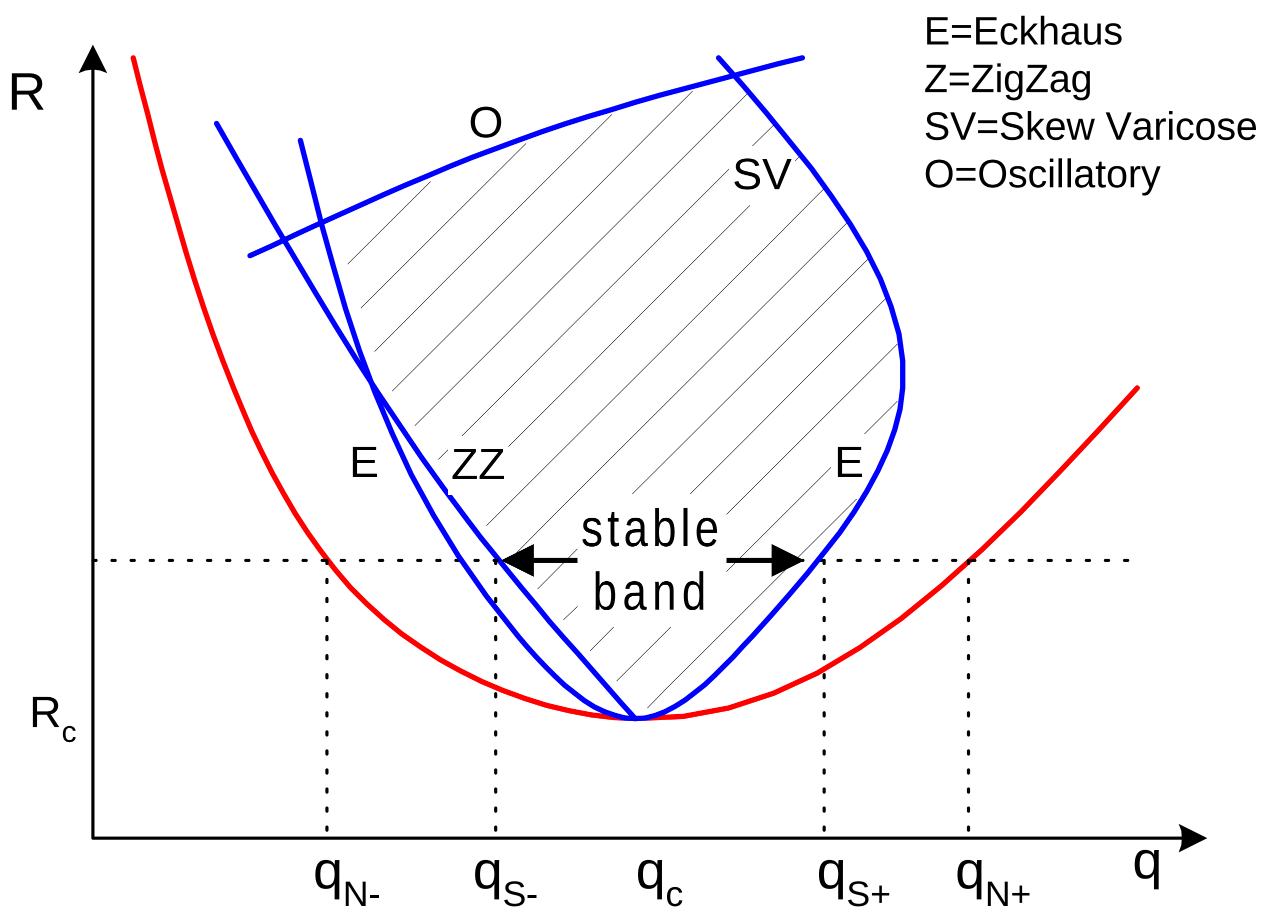

R
Patterns exist.
Are they stable?
R_c
No patterns
q_c
q


R
E=Eckhaus
E
E
unstable
stable
unstable
R_c
q_c
q


R
Z=ZigZag
unstable
ZZ
stable
R_c
q_c
q


R
E
ZZ
E
stable
R_c
q_c
q


R
O
SV
E
ZZ
E
stable
band
R_c
q_c
q
E=Eckhaus
Z=ZigZag
SV=Skew Varicose
O=Oscillatory


R
R_c
q
O
SV
E
ZZ
E
stable
band
q_N-
q_S-
q_c
q_S+
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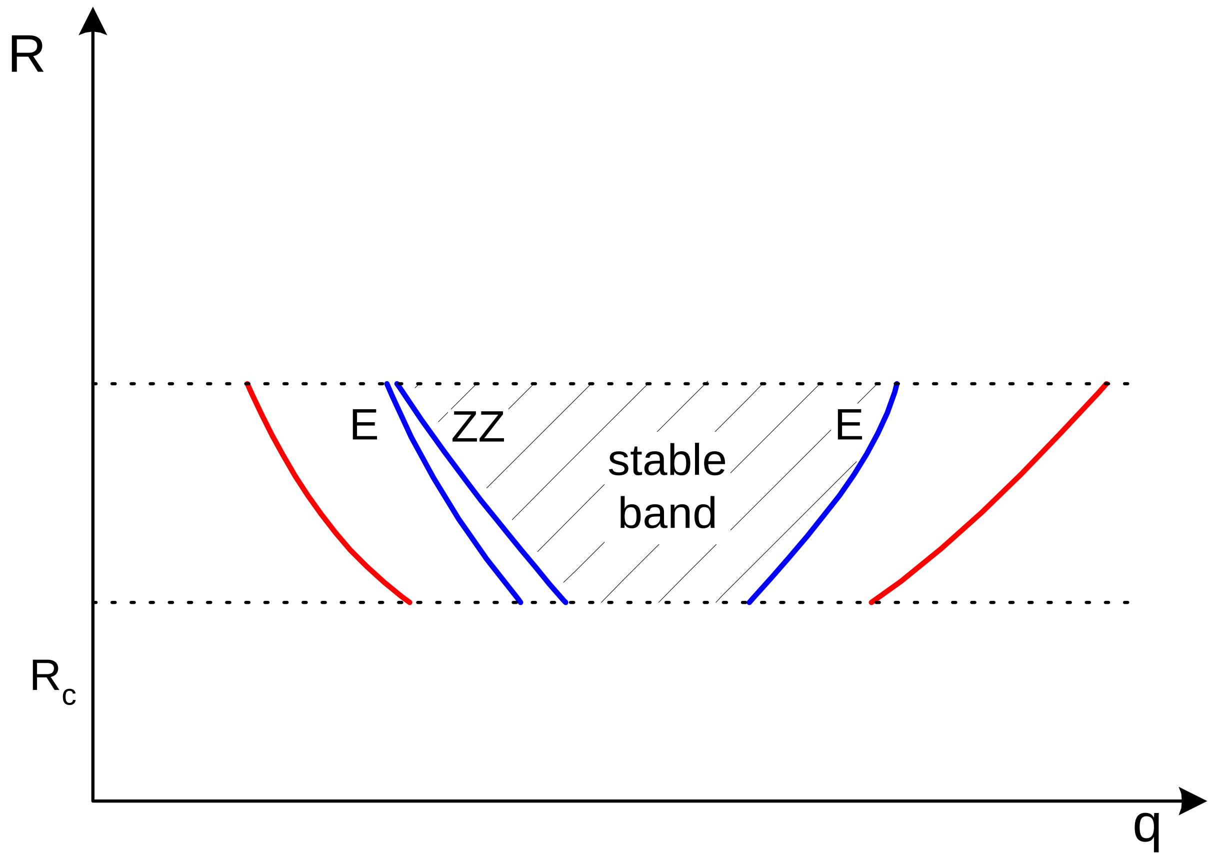
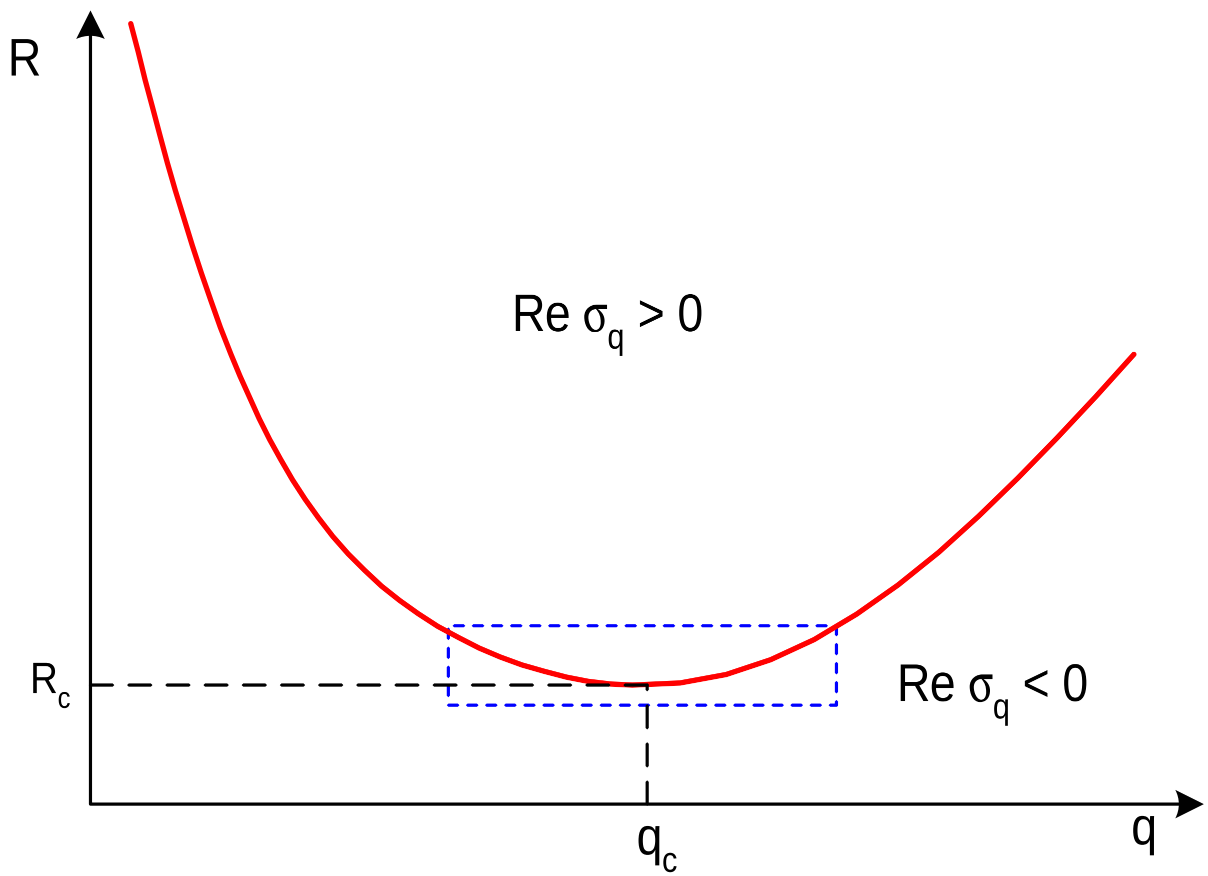

R
E
ZZ
stable
band
E
R
c
q

Tools for the Nonlinear Problem

Amplitude Equations

Systematic approach for describing weakly nonlinear solutions near onset



Linear onset solution

$$\delta \mathbf{u_q}(\mathbf{x}_\perp, z, t) \quad = \quad \left[a_0 e^{i(\mathbf{q}-\mathbf{q}_c)\cdot\mathbf{x}_\perp}\, e^{\mathrm{Re}\,\sigma_\mathbf{q} t}\right] \quad \times \quad \left[\mathbf{u_q}(z)\, e^{i\mathbf{q}_c\cdot\mathbf{x}_\perp}\right] \quad + \quad c.c.$$

Small terms near onset   Onset solution

Linear onset solution

$$\delta \mathbf{u}_{\mathbf{q}}(\mathbf{x}_\perp, z, t) \quad = \quad \underbrace{\left[a_0 e^{i(\mathbf{q}-\mathbf{q}_c)\cdot\mathbf{x}_\perp}\, e^{\mathrm{Re}\,\sigma_{\mathbf{q}} t}\right]}_{\text{Small terms near onset}} \quad \times \quad \underbrace{\left[\mathbf{u}_{\mathbf{q}}(z)\, e^{i\mathbf{q}_c\cdot\mathbf{x}_\perp}\right]}_{\text{Onset solution}} \quad + \quad c.c.$$

Weakly nonlinear, slowly modulated, solution

$$\delta \mathbf{u}(\mathbf{x}_\perp, z, t) \quad \approx \quad \underbrace{A(\mathbf{x}_\perp, t)}_{\text{Complex amplitude}} \quad \times \quad \underbrace{\left[\mathbf{u}_{\mathbf{q}_c}(z)\, e^{i\mathbf{q}_c\cdot\mathbf{x}_\perp}\right]}_{\text{Onset solution}} \quad + \quad c.c.$$

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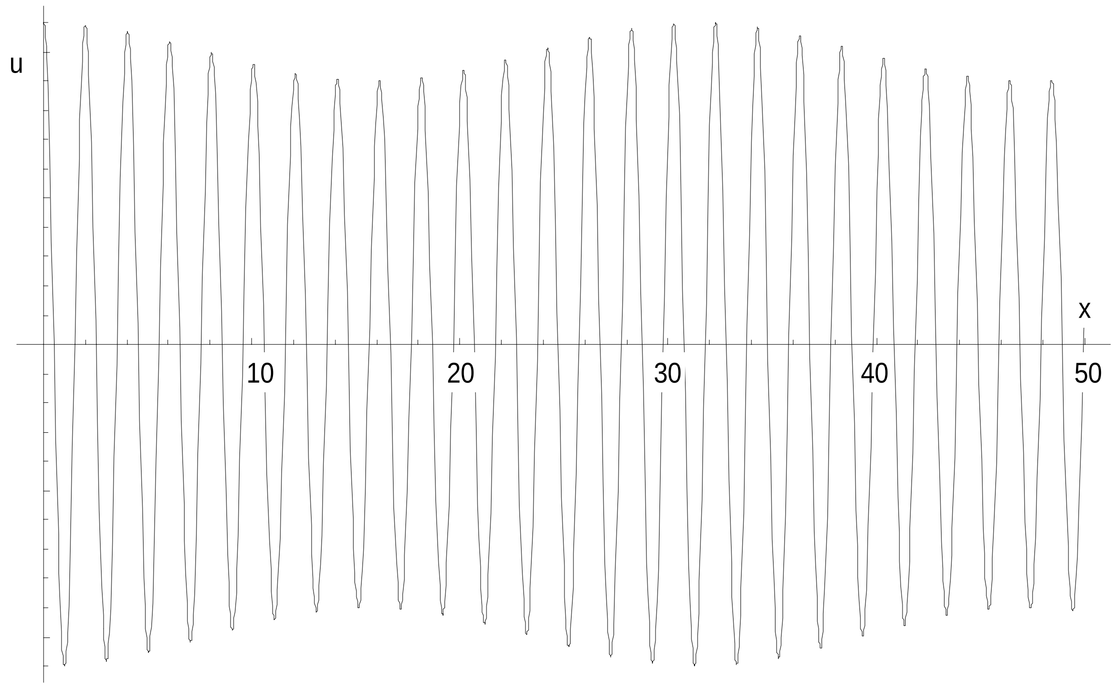
Substituting into the dynamical equations gives the amplitude equation, which in 1d [$\mathbf{q}_c = q_c\hat{\mathbf{x}}$, $A = A(x, t)$] is

$$\tau_0 \partial_t A = \varepsilon A + \xi_0^2 \partial_x^2 A - g_0 |A|^2 A, \qquad \varepsilon = \frac{R - R_c}{R_c}$$

Pictorially

A convection pattern that varies gradually in space



$$u \propto \mathrm{Re}[A(x)e^{iq_cx}]$$

$$q_c = 3.117; \qquad A(x) = 1 + 0.1\cos(0.2x)$$

Complex Amplitude

Magnitude and phase of A play very different roles

$$A(x, y, t) = a(x, y, t)e^{i\theta(x,y,t)}$$

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  $A = ae^{ikx}$ corresponds to $q = q_c + k$

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- y-gradient $\partial_y\theta$ gives rotation of wave vector through angle $\partial_y\theta/q_c$
  (plus $O[(\partial_y\theta)^2]$ change in wave number)

The amplitude equation describes

$$\tau_0 \partial_t A = \varepsilon A + \xi_0^2 \partial_x^2 A - g_0 |A|^2 A$$

growth dispersion/diffusion saturation

Parameters

$$\tau_0 \partial_t A = \varepsilon A + \xi_0^2 \partial_x^2 A - g_0 |A|^2 A,$$

- control parameter $\varepsilon = (R - R_c)/R_c$

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  - ◇ τ_0, ξ_0 fixed by matching to linear growth rate
    $A = a\, e^{i\mathbf{k}\cdot\mathbf{x}_\perp} e^{\sigma_\mathbf{q} t}$ gives pattern at $\mathbf{q} = \mathbf{q}_c \hat{x} + \mathbf{k}$)

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$$\sigma_\mathbf{q} = {\tau_0}^{-1}[\varepsilon - {\xi_0}^2 (q - q_c)^2]$$

  - ◇ g_0 by calculating nonlinear state at small ε and $q = q_c$.

Scaling

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Introduce scaled variables

$$x = \varepsilon^{-1/2} \xi_0 \, X$$

$$t = \varepsilon^{-1} \tau_0 \, T$$

$$A = (\varepsilon / g_0)^{1/2} \, \bar{A}$$

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This reduces the amplitude equation to a *universal* form

$$\partial_T \bar{A} = \bar{A} + \partial_X^2 \bar{A} - \left| \bar{A} \right|^2 \bar{A}$$

Scaling

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$$\partial_T \bar{A} = \bar{A} + \partial_X^2 \bar{A} - \left|\bar{A}\right|^2 \bar{A}$$

Since solutions to this equation will develop on scales $X, Y, T, \bar{A} = O(1)$ this gives us scaling results for the physical length scales.

Derivation

$$\tau_0 \partial_t A = \varepsilon A + \xi_0^2 \partial_x^2 A - g_0 |A|^2 A, \qquad \varepsilon = \frac{R - R_c}{R_c}$$

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- Symmetry arguments: equation invariant under:
  - ◇ $A(\mathbf{x}_\perp) \to A(\mathbf{x}_\perp) e^{i\Delta}$ with Δ a constant, corresponding to a physical translation;
  - ◇ $A(\mathbf{x}_\perp) \to A^*(-\mathbf{x}_\perp)$, corresponding to inversion of the horizontal coordinates (parity symmetry);

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  - ◇ $A(\mathbf{x}_\perp) \rightarrow A^*(-\mathbf{x}_\perp)$, corresponding to inversion of the horizontal coordinates (parity symmetry);
- Multiple scales perturbation theory (Newell and Whitehead, Segel 1969)

Derivation

$$\tau_0 \partial_t A = \varepsilon A + \xi_0^2 \partial_x^2 A - g_0 |A|^2 A, \qquad \varepsilon = \frac{R - R_c}{R_c}$$

- Symmetry arguments: equation invariant under:
  - ◇ $A(\mathbf{x}_\perp) \to A(\mathbf{x}_\perp)e^{i\Delta}$ with Δ a constant, corresponding to a physical translation;
  - ◇ $A(\mathbf{x}_\perp) \to A^*(-\mathbf{x}_\perp)$, corresponding to inversion of the horizontal coordinates (parity symmetry);
- Multiple scales perturbation theory (Newell and Whitehead, Segel 1969)
- Mode projection (MCC 1980)

Amplitude Equation = Ginzburg Landau equation

$$\tau_0 \partial_t A = \varepsilon A + \xi_0^2 \partial_x^2 A - g_0 |A|^2 A,$$

Familiar from other branches of physics:

- Good: take intuition from there
- Bad: no *really* new effects

e.g. equation is relaxational (potential, Lyapunov)

$$\tau_0 \partial_t A = -\frac{\delta V}{\delta A^*}, \qquad V = \int dx \left[-\varepsilon |A|^2 + \tfrac{1}{2} g_0 |A|^4 + \xi_0^2 |\partial_x A|^2\right]$$

This leads to

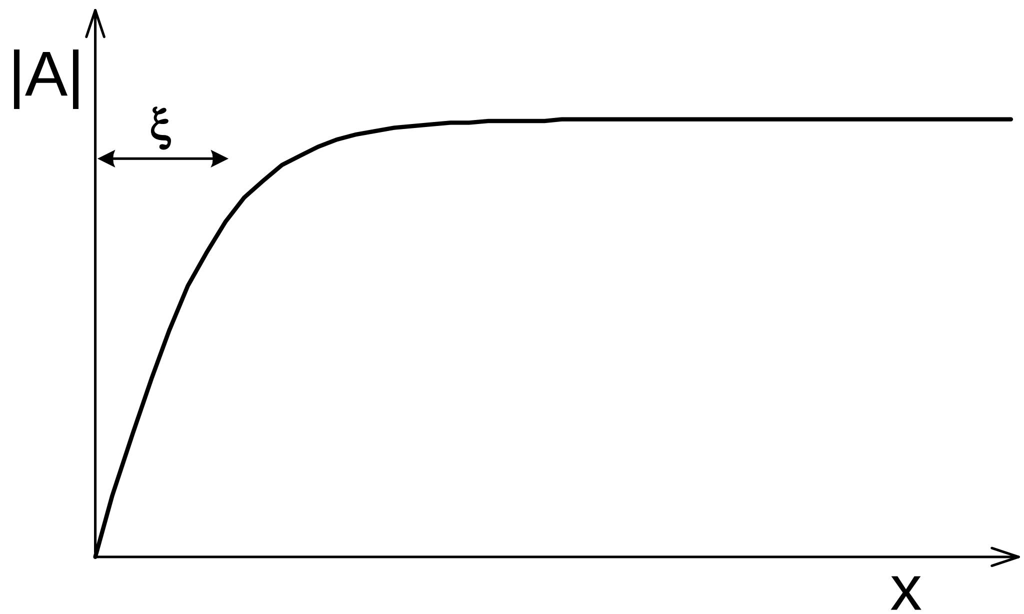
$$\frac{dV}{dt} = -\tau_0^{-1} \int dx \, |\partial_t A|^2 \leq 0$$

and dynamics runs “down hill” to a minimum of V — no chaos!

Example: one dimensional geometry with boundaries that suppress the pattern (e.g. rigid walls in a convection system)

First consider a single wall

$$\partial_T \bar{A} = \bar{A} + \partial_X^2 \bar{A} - \left|\bar{A}\right|^2 \bar{A} \qquad \bar{A}(0) = 0$$



$$\bar{A} = e^{i\theta} \tanh(X/\sqrt{2})$$

$$A = e^{i\theta}(\varepsilon/g_0)^{1/2} \tanh(x/\xi) \qquad \text{with} \qquad \xi = \sqrt{2}\varepsilon^{-1/2}\xi_0$$

$$A = e^{i\theta}(\varepsilon/g_0)^{1/2}\tanh(x/\xi)$$

- arbitrary position of rolls
- asymptotic wave number is $k = 0$, giving $q = q_c$: no band of existence

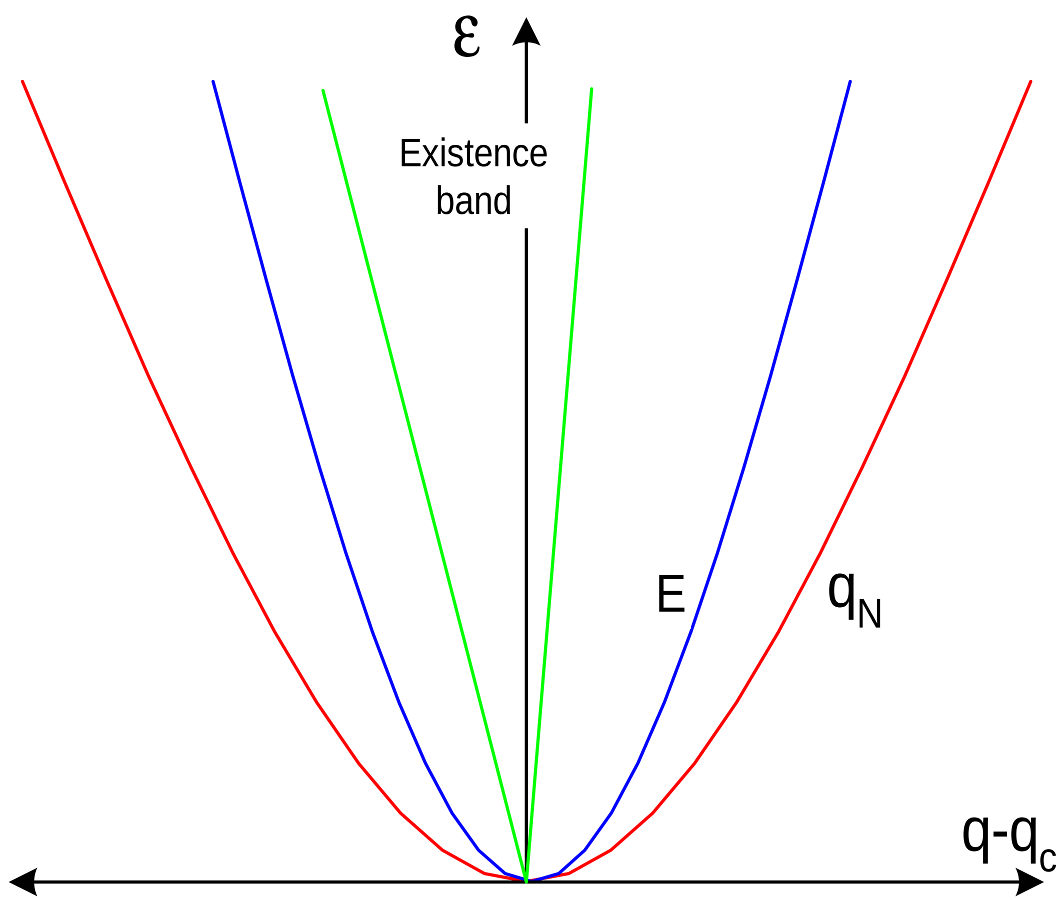
$$A = e^{i\theta}(\varepsilon/g_0)^{1/2}\tanh(x/\xi)$$

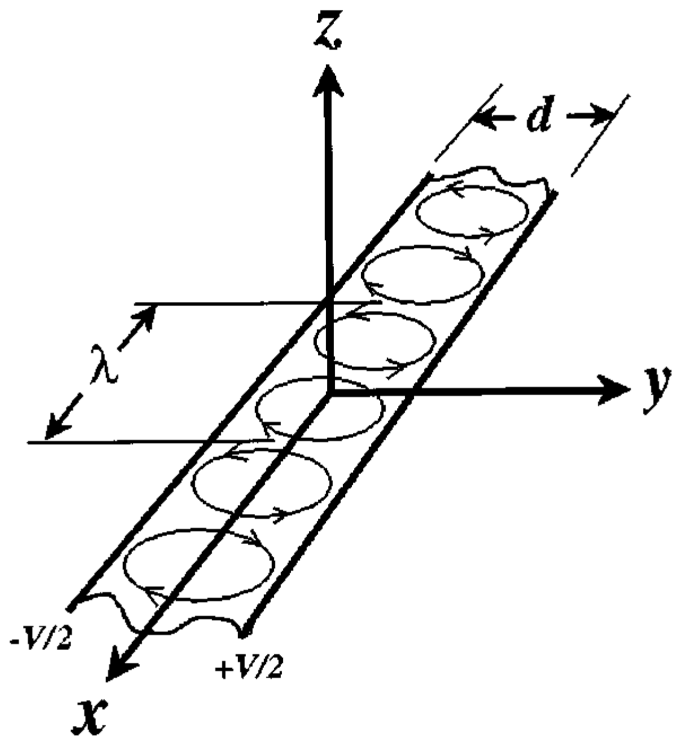
- arbitrary position of rolls
- asymptotic wave number is $k = 0$, giving $q = q_c$: no band of existence

Extended amplitude equation to next order in ε (MCC, Daniels, Hohenberg, and Siggia 1980) shows

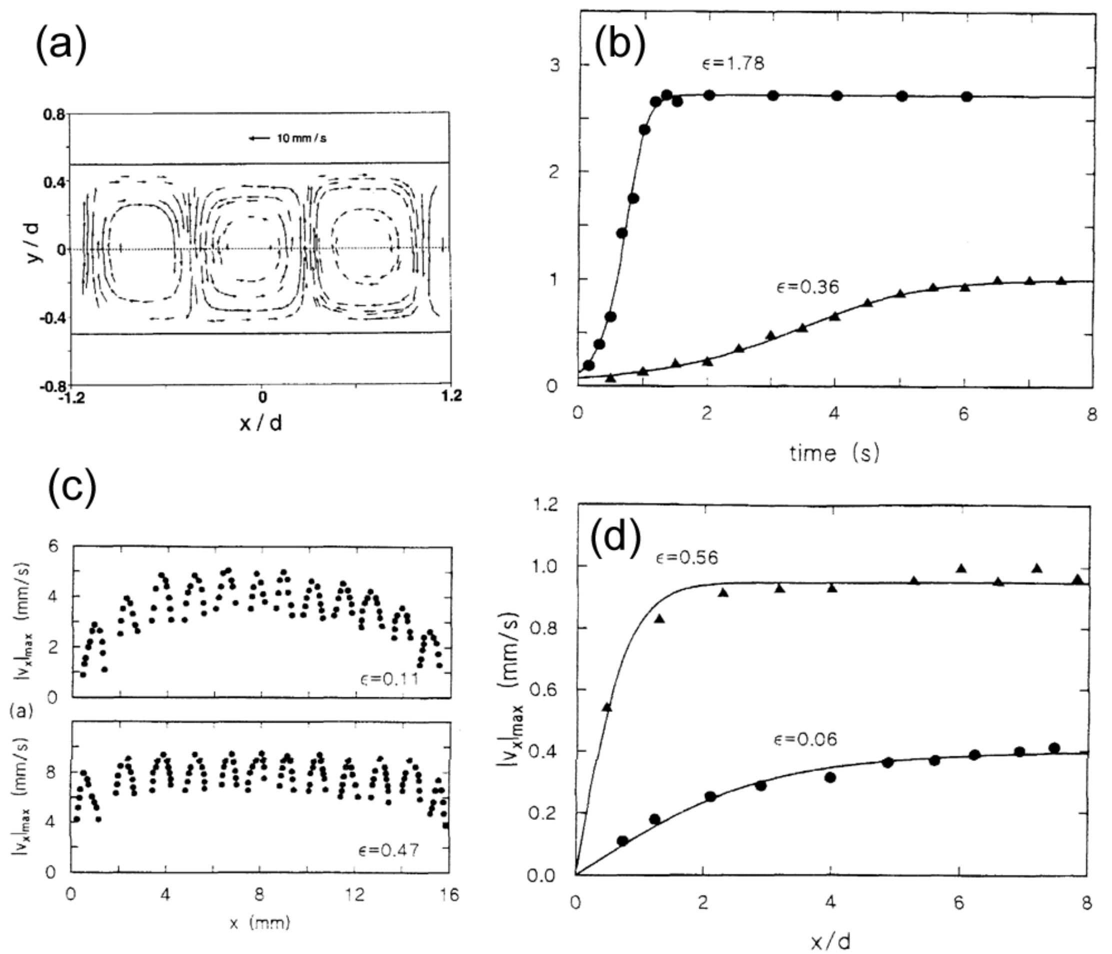
- discrete set of roll positions
- solutions restricted to a narrow $O(\varepsilon^1)$ wave number band with wave number far from the wall

$$\alpha_-\varepsilon < q - q_c < \alpha_+\varepsilon$$

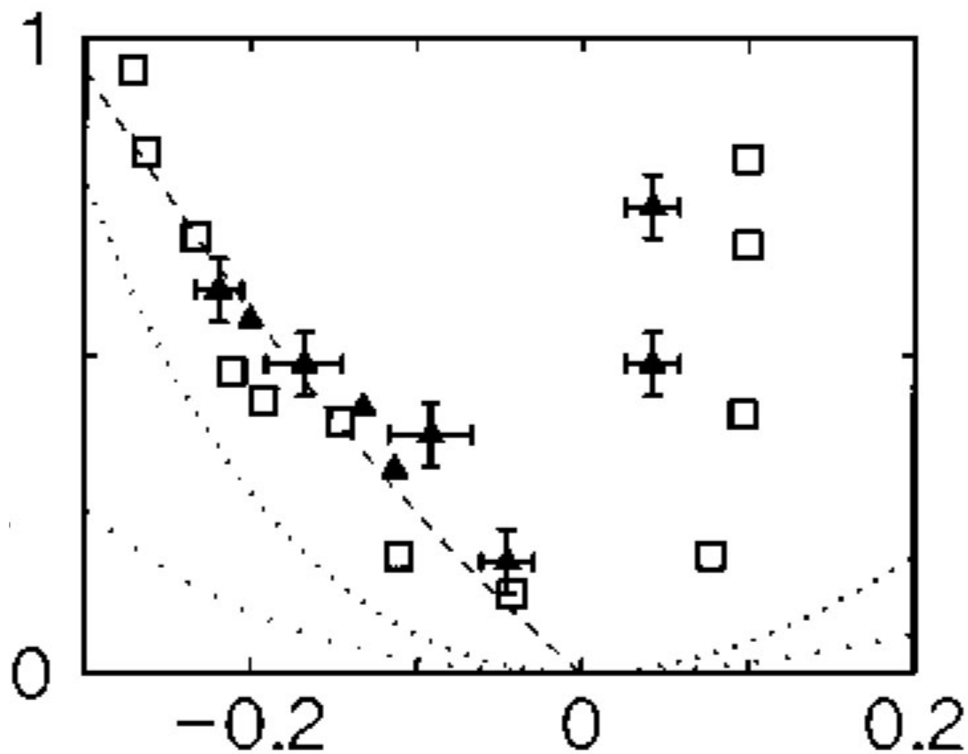

ε
Existence
band
E
q_N
$q-q_c$



V. B. Deyirmenjian, Z. A. Daya, and S. W. Morris (1997)

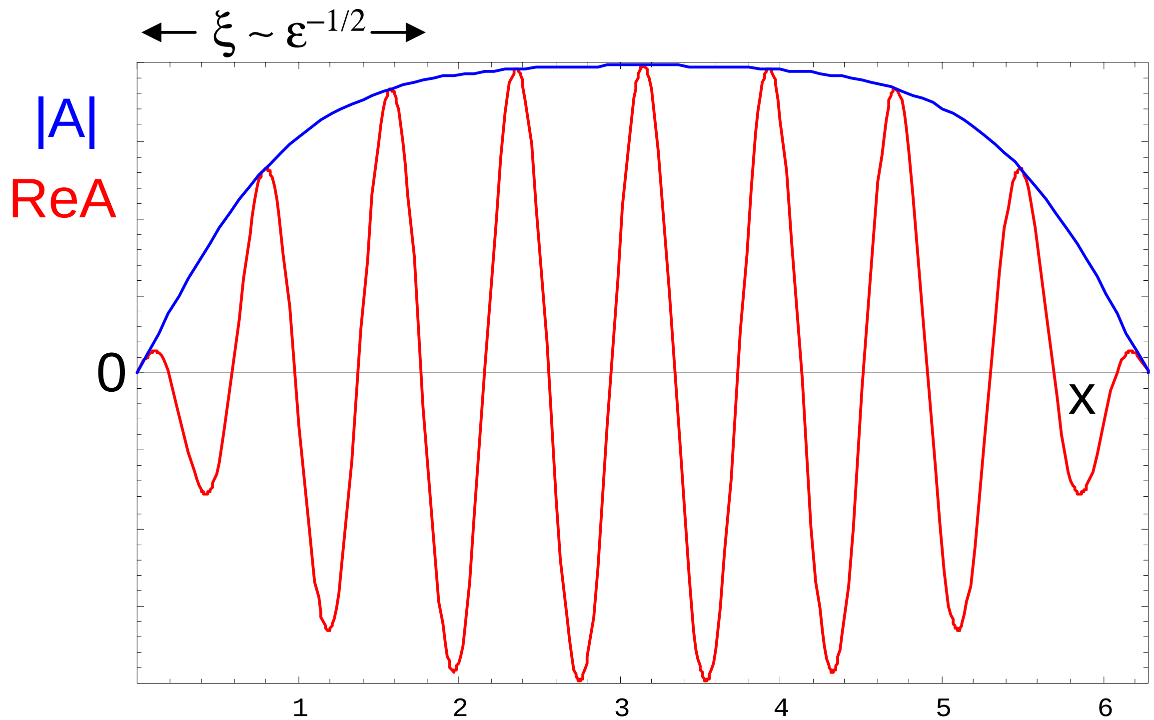


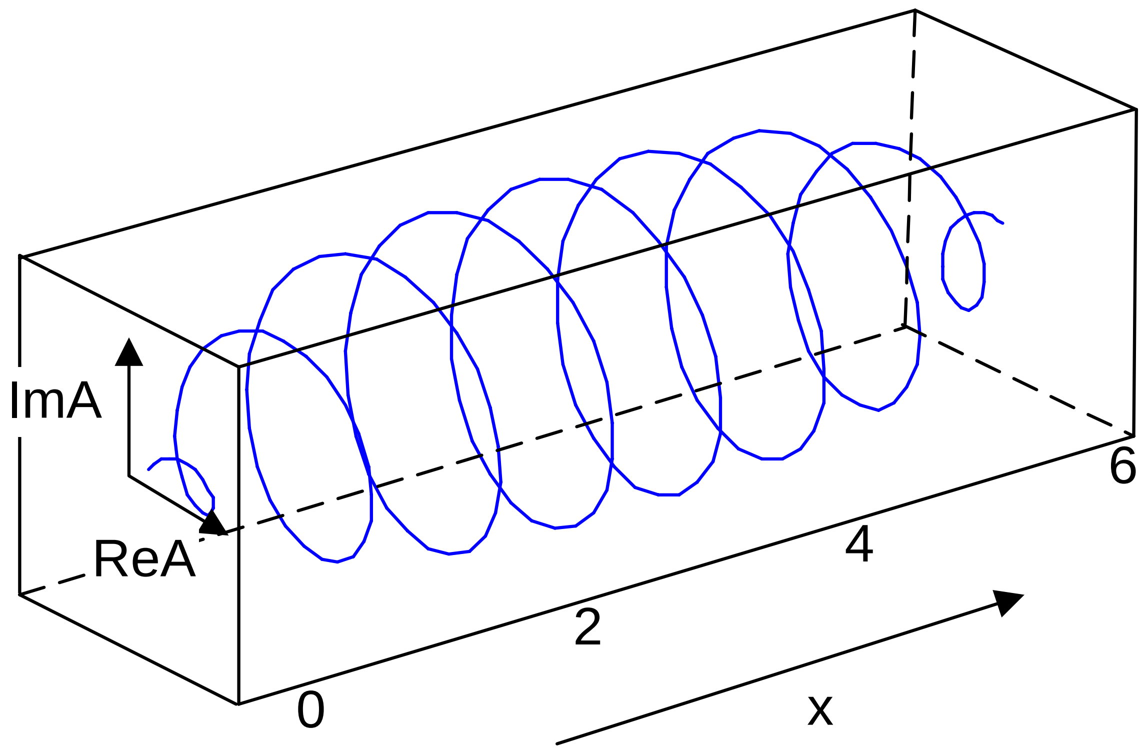
From Morris et al. (1991) and Mao et al. (1996)



Mao et al. (1996)

Two sidewalls




ImA
ReA
0
2
4
6
x

Conclusions

In today's lectures I introduced some of the basic ideas of pattern formation:

- linear instability at nonzero wave number;
- nonlinear saturation;
- stability balloons.

I then introduced the amplitude equation which is the simplest theoretical approach that captures the key effects in pattern formation (growth, saturation, and dispersion).

I focussed on the equation in one dimension, and on a phenomenological derivation. You can find more technical aspects in the supplementary notes.

Next lecture: the role of continuous symmetries — rotation and translation