Chapter 9

Dimensions

9.1 Capacity and Hausdorff dimension

9.1.1 Capacity

The structure observed in the demonstrations of chaotic systems (e.g. the 2d maps of chapter 5)—the "chaotic attractor"—seems to be neither space filling (an area for the 2d maps) or a simple curve (a line). This complex geometry can be characterized by a non-integral dimension, and the structure is then called a fractal.

The *capacity* or *box counting dimension* is a simple way of defining a nonintegral dimension. It is related to the Hausdorff dimension, and is usually equal to this (and often assumed to be so in the context of dynamical systems), although there are counterexamples. The construction is as follows. Suppose we have a set in an m-dimensional space. Imagine covering the space with equal size m-cubes of side ε, and count how many m-cubes contain points in the set, say $N(\varepsilon)$. The capacity is defined as

$$D_C = \lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{\log\left(\varepsilon^{-1}\right)} \tag{9.1}$$

i.e. $N(\varepsilon) \sim \varepsilon^{-D_C}$. Often a sequence of box sizes $\varepsilon_n = b^{-n}$ is used, and other shapes, and boxes not arranged on the regular mesh may also be used to find, for example, the least number of boxes of size ε to cover the set.

This definition gives sensible results for standard situations. For example, for a straight line of length L we need $N(\varepsilon) = L/\varepsilon$ boxes to cover the line, so

$$D_C = \lim_{\varepsilon \to 0} \frac{\log L + \log \varepsilon^{-1}}{\log \varepsilon^{-1}} = 1 \quad . \tag{9.2}$$

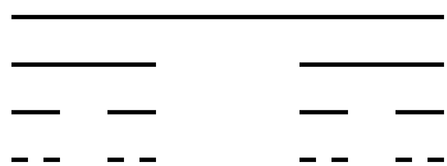

Figure 9.1: The construction of the $\frac{1}{3}$ Cantor set

A famous example of a set with non-integral dimension is the "middle third Cantor set". This set is produced from the unit interval by successively removing the middle one third of each line element remaining (see figure 9.1). Iterating this process indefinitely leaves a set of measure zero (the measure of the remaining line elements at the nth level is $\left(\frac{2}{3}\right)^n$). The capacity may be found choosing boxes of size $\varepsilon_m = \left(\frac{1}{3}\right)^m$. Then $N\left(\varepsilon_m\right) = (2)^m$ so that

$$D_C = \lim_{m \to \infty} \frac{m \log(2)}{m \log(3)} = \frac{\log 2}{\log 3} \simeq 0.63 \quad . \tag{9.3}$$

For sets generated by iterating dynamical systems there are, of course, limitations in applying this idea. For example, due to the finite data set $N(\varepsilon)$ will clearly saturate when there is one box per data point. On the other hand if the data is perturbed by measurement error or experimental noise, the distribution of points at fine scales will be disturbed, and again there will be deviations from the expected behavior. Thus in practice $\log N(\varepsilon)$ will not be proportional to $\log \varepsilon^{-1}$ for all ε. Instead, plotting $\log N(\varepsilon)$ against $\log \varepsilon^{-1}$ will yield a straight line only over some intermediate range of ε where the asymptotic dependence is a reasonable approximation, but the imperfections of the data at small scales is not yet a problem: the slope of this intermediate range is used to estimate D_C. This produces severe limitations on the ability to measure dimensions accurately from real data, particularly when the dimension becomes large. Even for numerically generated sets such as for the Hénon map, accurate estimates are hard to get: due to the phenomenon called "lacunarity" [1] the estimate of the dimension oscillates as the length scales over which the fit is made varies—for example, the variation is at about the 5% level for fits over ranges from $2^{-4\pm 2}$ to $2^{-10\pm 2}$.

The box counting scheme for estimating dimensions of fractal sets of chaotic attractors is implemented for the Hénon map in demonstration 1. The limitations of the approach can be tested by varying the number of data points, range of fits etc.

9.1.2 Hausdorff dimension

The Hausdorff dimension D_H is often equal to the capacity, but gives more reasonable answers in some special cases. Since analogous formulations will be useful later when we consider generalized dimensions, it is convenient to introduce the definition here, although it is rarely implemented on experimental or numerical data sets.

Cover the set with m-cubes of variable edge length $l_i \le \varepsilon$. Define a "partition function"

$$\Gamma(d, \varepsilon) = \inf \sum_i l_i^d \tag{9.4}$$

(inf means "the smallest"). Then it is found there exists a D_H such that

$$\Gamma(d) = \lim_{\varepsilon \to 0} \Gamma(d, \varepsilon) = \begin{cases} 0 & \text{for } d > D_H \\ \infty & \text{for } d < D_H \end{cases}, \tag{9.5}$$

thereby defining D_H.

It is easy enough to see that $D_C \ge D_H$. To do this cover the set with the equal size boxes as in the definition of D_C. Then because of the inf in the definition of Γ we have

$$\Gamma(d, \varepsilon) \le N(\varepsilon)\varepsilon^d \sim \varepsilon^{-D_C}\varepsilon^d. \tag{9.6}$$

Thus

$$\Gamma(d) \le \lim_{\varepsilon \to 0} \varepsilon^{-(D_C - d)} \to 0 \quad \text{for all} \quad d > D_C \tag{9.7}$$

and so D_C cannot be less than D_H.

9.2 Generalized Dimensions

The capacity and Hausdorff dimensions are purely geometric, making no mention of the measure of the attractor, i.e. the number of times the dynamics visits different regions of the phase space. This may make them hard to calculate, since rarely visited regions may contribute significantly to the dimension. In addition a single number is certainly an incomplete characterization of the sets encountered in dynamical systems, which are typically not self similar as in the one-third Cantor set. The *generalized dimensions* are an attempt to address these issues. There are two formulations, analogous to the formulations of the capacity and Hausdorff dimensions, that are usually taken as giving the same result. The former is most useful for actual implementation on real data sets, the later for theoretical manipulations.

9.2.1 Box counting approach

Cover the attractor with boxes of size ε and define the probability of finding a point in the ith box $p_i = N_i/N$ with N_i the number out of a total N points in the ith box. The p_i are estimates of the measure associated with the box $\int_{V_i} \rho(x)\, dV$. The qth generalized dimension D_q is defined as

$$D_q = \lim_{\varepsilon \to 0} \frac{1}{q-1} \frac{\log \sum_i p_i^q}{\log \varepsilon}. \tag{9.8}$$

The generalized dimensions of the Hénon attractor are investigated suing the box counting algorithm in demonstration 2.

9.2.2 Partition function approach

The approach of the Hausdorff dimension is generalized. Cover the set with m-boxes of size $l_i \leq \varepsilon$. Define the partition function

$$\Gamma(q, \tau, \varepsilon) = \begin{cases} \inf \sum_i \frac{p_i^q}{l_i^\tau} & q \leq 1, \tau \leq 0 \\ \sup \sum_i \frac{p_i^q}{l_i^\tau} & q \geq 1, \tau \geq 0 \end{cases} \tag{9.9}$$

and then there exists a $\tau(q)$ that

$$\Gamma(q, \tau) = \lim_{\varepsilon \to 0} \Gamma(q, \tau, \varepsilon) = \begin{cases} 0 & \text{for } \tau < \tau(q) \\ \infty & \text{for } \tau > \tau(q) \end{cases}. \tag{9.10}$$

The dimension D_q^H is then defined as

$$D_q^H = \frac{\tau(q)}{q-1}. \tag{9.11}$$

9.2.3 Properties

It can be shown $D_{q'} \leq D_q$ for $q^{'} > q$. Sets for which D_q is not a constant are known as multifractals. Certain D_q have simple interpretations or are particularly easy to calculate.

Capacity, Hausdorff Dimension, $q = 0$

Clearly from the definitions the $q = 0$ generalized dimension reduces to the capacity or Hausdorff dimension, e.g. for the box counting algorithm $D_0 = D_C$.

Information Dimension, $q = 1$

For D_1 we calculate D_q for $q \to 1$ (since $\sum_i p_i$ is 1 and the log is zero):

$$D_1 = \lim_{\varepsilon \to 0} \frac{1}{q-1} \frac{\log \sum_i p_i p_i^{q-1}}{\log \varepsilon} \simeq \lim_{\varepsilon \to 0} \frac{1}{q-1} \frac{\log \sum_i p_i \left[1 + (q-1) \log p_i\right]}{\log \varepsilon} \tag{9.12}$$

so that

$$D_1 = \lim_{\varepsilon \to 0} \frac{-\sum_i p_i \log p_i}{-\log \varepsilon}. \tag{9.13}$$

This expression tells us how the information scales with the box size, and so D_1 is called the information dimension.

Since to calculate the dimension D_1 each box is weighted with the measure, this is the dimension that characterizes most directly the measure of the attractor. The following constructions also typically lead to a value equal to D_1:

- The θ-Capacity is defined in terms of the number of boxes $N(\varepsilon; \theta)$ of size ε needed to cover the *smallest* set containing a fraction θ of the total measure

$$D_C(\theta) = \lim_{\varepsilon \to 0} \frac{\log N(\varepsilon; \theta)}{\log \left(\varepsilon^{-1}\right)}. \tag{9.14}$$

Typically $D_C(\theta) = D_1$ for $\theta < 1$. Of course $D_C(1) = D_C$. Since usually $D_1 < D_0 = D_C$, and the number of boxes of size ε needed to cover a set scales as $(1/\varepsilon)^{D_C}$, this means that almost all of the measure of the set is contained in a tiny fraction of the boxes needed to cover the set, and the fraction goes to zero in the limit ε goes to zero. A consequence of this is that it is hard to calculate D_C which depends on numerous points that contribute little to the measure i.e. are visited rarely in an evolution.

- The pointwise dimension $D_p(x)$ is defined in terms of how the measure $p_\varepsilon(x)$ associated with a box of size ε at the point x on the attractor scales with ε i.e. $p_\varepsilon \sim \varepsilon^{D_p}$or more precisely

$$D_p(x) = \lim_{\varepsilon \to 0} \frac{\log p_\varepsilon(x)}{\log(\varepsilon)}. \tag{9.15}$$

  Typically $D_p(x)$ is independent of x for almost all x (with respect to the measure of the attractor) and is then equal to the information dimension D_1. However characterizing the set of x giving other pointwise dimensions is another way of characterizing the strange attractor, as we will see in the next chapter.

Correlation Dimension $q = 2$

For D_2 we have

$$D_2 = \lim_{\varepsilon \to 0} \frac{\log \sum_i p_i^2}{\log \varepsilon}. \tag{9.16}$$

But $\sum_i p_i^2$ is the probability that two points lie within cells of length ε, and scales in the same way as the probability that two points in the data set are separated by a distance less than ε, which is determined by the pair correlation function. Thus D_2 is called the correlation dimension, and can be estimated from a pairwise manipulation of the data, a much easier task than box counting: define

$$C(\varepsilon) = \lim_{N \to \infty} \frac{1}{N^2} \sum_{i,j}^{N} \Theta(\varepsilon - |\vec{x}_i - \vec{x}_j|) \tag{9.17}$$

with Θ the unit step function. Then

$$D_2 = \lim_{\varepsilon \to 0} \frac{\log C(\varepsilon)}{\log \varepsilon}. \tag{9.18}$$

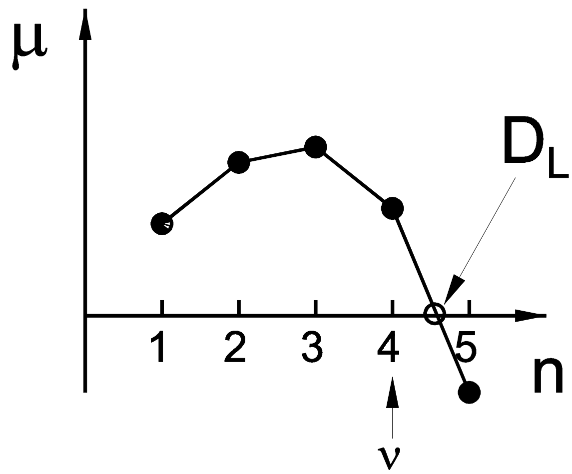

Figure 9.2: Lyapunov Dimension. In the example shown the first three Lyapunov exponents are positive, and so $\mu(n) = \sum_{i=1}^{n} \lambda_i$ is an increasing function of n up to $n = 3$. The sum μ is positive up to $n = 4$, giving $\nu = 4$. D_L is the intersection with the $\mu = 0$ axis given by linear interpolation.

9.3 Lyapunov Dimension

Dimensions are a static characterization of the attractor. Nevertheless the attractor is formed by the dynamical system, and so it is interesting to look for a connection between dimensions and dynamical diagnostics. Kaplan and Yorke [2] proposed a dimension based on the Lyapunov exponents:

$$D_L = \nu + \frac{1}{|\lambda_{\nu+1}|} \sum_{i=1}^{\nu} \lambda_{\nu}, \tag{9.19}$$

where ν is the largest integer for which the sum of the first ν exponents $\mu(\nu) = \sum_{i=1}^{\nu} \lambda_{\nu}$ is positive. (If ν equals the dimension of the phase space, then $D_L = \nu$.) This result can be motivated by noting that $\lambda_1 + \lambda_2 + \cdots \lambda_n$ gives the rate of expansion or contraction of an n dimensional volume in phase space. D_L is the estimate of the dimension of the volume that neither grows nor decays (figure 9.2). Since the exponents are defined by averaging over the attractor, with more weight given to regions visited more often, Kaplan and Yorke conjectured that D_L might be equal to the information dimension D_1:

$$\text{Conjecture: } D_L = D_1. \tag{9.20}$$

This result is true for the Bakers' map, and appears to be true numerically for the Hénon map, but is probably not a general result.

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Bibliography

[1] A. Arneodo, G. Grasseau, and E.J. Kostelich, Phys. Lett. **124A**, 426 (1987)

[2] J.L. Kaplan and J.A. Yorke, Lecture Notes in Math **730**, 204 (1979)