

Chapter 3

Non-linear Oscillators

The study of non-linear oscillators has been important in the development of the theory of dynamical systems. Van der Pol and Van der Mark (1927) [1] studying a simple non-linear electronic circuit (a neon tube was the non-linear element) experimentally found, but were not much interested in, “noisy behavior” that we would now identify as chaos, and Carwright and Littlewood (1945) [2] studied chaos like behavior in a non-linear oscillator, predating Lorenz’s work by decades. In this chapter the equations for two famous non-linear oscillators will be studied: the “Van der Pol oscillator”, and (mainly through exercises) the “Duffing oscillator”. Both are simple equations, and correspond quite closely to experimental phenomena.

The motivation for this study are:

- general interest;
- to introduce more of the language of dynamical systems;
- to extend our intuition on the diagnostic schemes such as Poincaré sections;
- to introduce the important phenomenon of frequency locking in non-linear oscillators;
- to present some analytic approaches for dealing with non-linear systems.

3.1 Van der Pol Oscillator

The simple harmonic oscillator is generalized to include a non-linear damping. The “dissipation” is in fact negative for small amplitudes, modelling instability and feeding in energy, and becomes positive for large amplitudes. In the circuit studied by Van der Pol and Van der Mark the negative dissipation comes from a negative resistance region of the $I - V$ characteristics of the neon tube. Because energy is fed in to the oscillator, spontaneous sustained oscillations occur even without periodic driving.

The equation is

$$\ddot{x} - \gamma (1 - x^2) \dot{x} + x = g \cos(\omega_D t). \quad (3.1)$$

We will first study the oscillations for no driving $g = 0$. Roughly we expect the amplitude of the oscillations to grow until the average damping is zero.

As usual we can introduce phase space variables

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= \gamma (1 - x^2) v - x \end{aligned} \quad (3.2)$$

For large γ a different choice of variables (the Liénard variables) is convenient: define

$$\dot{y} = \ddot{x} - \gamma (1 - x^2) \dot{x} \quad (3.3)$$

so that

$$y = \dot{x} - \gamma \left(x - \frac{1}{3} x^3 \right) \quad (3.4)$$

to give

$$\begin{aligned} \dot{x} &= y + \gamma \left(x - \frac{1}{3} x^3 \right) \\ \dot{y} &= -x \end{aligned} \quad (3.5)$$

3.1.1 Small γ : Secular perturbation theory

Let us write $\gamma = \varepsilon$ to remind ourselves that γ is a small parameter, and the equation as

$$\ddot{x} + x = \varepsilon (1 - x^2) \dot{x} \quad (3.6)$$

so that on the left hand side we have the simple harmonic oscillator terms, and on the right the small perturbation.

For $\varepsilon = 0$ the solution is oscillations at frequency 1:

$$x = x_o(t) = ae^{i(t+\phi)} + c.c., \quad (3.7)$$

with the constants a and ϕ undetermined. For $\varepsilon \neq 0$, we expect the right hand side of (3.6) to fix the amplitude, and perhaps perturb the frequency to $\omega = 1 + \varepsilon\Delta + \dots$. This means for the phase

$$\phi \simeq \phi_0 + \varepsilon\Delta t \quad (3.8)$$

so that even for small ε , the change to the solution is not small for long times

$$e^{i(t+\phi_0+\varepsilon\Delta t)} \neq e^{i(t+\phi_0)} (1 + \varepsilon\Delta t + \dots). \quad (3.9)$$

Such a perturbation is described as *secular*. We cannot therefore write a perturbation expansion directly for x i.e. $x \simeq x_0 + \varepsilon x_1 + \dots$. Instead we write

$$x = [A(T) e^{it} + c.c.] + \varepsilon x_1 + \dots \quad (3.10)$$

where we use a complex amplitude A to combine both unknowns a , ϕ , and we expect the amplitude to have slow time dependence $A = A(T = \varepsilon t)$ with $A_T = O(1)$ (the derivative with respect to T). This is a general example of secular perturbation theory [3] where a second slow time scale is introduced $x = x(t, T = \varepsilon t)$. With this form we can demand that the correction εx_1 is indeed small.

Now it is a simple matter of substituting

$$\dot{x} = (\varepsilon A_T + iA) e^{it} + c.c. + \varepsilon \dot{x}_1 + \dots \quad (3.11)$$

$$\ddot{x} = (\varepsilon^2 A_{TT} + 2i\varepsilon A_T - A) e^{it} + c.c. + \varepsilon \ddot{x}_1 + \dots \quad (3.12)$$

into the full equation and collecting terms at each order in ε .

At $O(\varepsilon^1)$ we get

$$\ddot{x}_1 + x_1 = (-2iA_T + iA - i|A|^2 A) e^{it} - iA^3 e^{3it} + c.c.. \quad (3.13)$$

If we can solve this equation for (a finite) x_1 we are done. However this is the equation for a driven, ideal oscillator, and the first term on the right is driving at the resonant frequency: hence to get a finite result we must put this driving strength

to zero. This “solvability condition” finally yields an equation for the unknown amplitude

$$A_T = \frac{1}{2} (1 - |A|^2) A \quad (3.14)$$

or going back to the unscaled time coordinate

$$\frac{dA}{dt} = \frac{\varepsilon}{2} (1 - |A|^2) A \quad . \quad (3.15)$$

This can be written in magnitude-phase form $A = ae^{i\phi}$:

$$\begin{aligned} \frac{da}{dt} &= \frac{\varepsilon}{2} (1 - a^2) a \\ \frac{d\phi}{dt} &= 0 \times \varepsilon + O(\varepsilon^2) \end{aligned} \quad (3.16)$$

and we see that the magnitude grows until saturation at $a = 1$, and in this particular case there is no frequency shift at $O(\varepsilon)$, so that the long time solution is $x = 2 \cos t$.

Formally, the solvability condition can be expressed by the statement that the right hand side of (3.13) must be “orthogonal to the zero mode of the adjoint of the linear operator” on the left hand side of the equation. (In this case the operator is $\partial^2/\partial t^2 + 1$, the adjoint operator is the same, and the zero eigenvalue mode is just $e^{\pm it}$). This formulation is illustrated in the appendix, working through the problem in the (x, v) phase-space description.

The small amplitude oscillations are illustrated in [demonstration 1](#).

3.1.2 Large γ : singular perturbations and relaxation oscillations

For large γ we rescale the variables $y = \gamma Y$, $t = \gamma T$, $x = X$, and introduce the small variable $\eta = \gamma^{-2}$ to give in the Liénard coordinates

$$\eta \frac{dX}{dT} = Y + \left(X - \frac{1}{3} X^3 \right), \quad (3.17)$$

$$\frac{dY}{dT} = -X. \quad (3.18)$$

If we first try setting $\eta = 0$, we run into contradictions: e.g. starting off at positive X the second equation says Y decreases in time (until X hits zero), whereas the

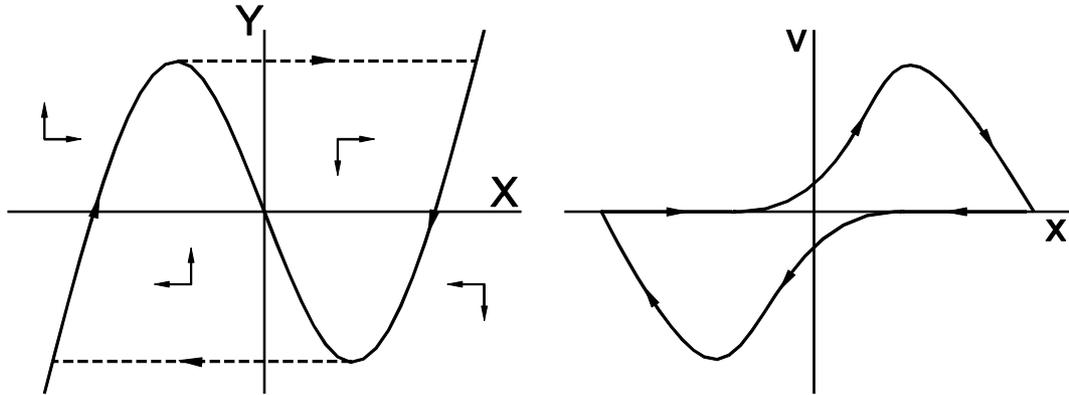


Figure 3.1: Relaxation Oscillations

first equation would say that Y has a minimum value of $-\frac{2}{3}$. Thus the term in η is not uniformly small, and completely changes the dynamics. A perturbation which increases the order of the differential equation is known as a singular perturbation. For small η we can divide the oscillation cycle into two pieces: either

- the right hand side of (3.17) is small so $Y \simeq -(X - \frac{1}{3}X^3)$ and the dynamics is on an $O(1)$ time scale (in T), or
- the right hand side of (3.17) is $O(1)$, then X changes very rapidly over a short time scale $O(\eta^{-1})$ and in this time, according to (3.18) we may take Y to be constant.

Thus the orbit follows the curve $Y \simeq -(X - \frac{1}{3}X^3)$ in phase space at a rate given by $dY/dT = -X$ until reaching the extrema, when the orbit “jumps branches” at constant Y (see figure 3.1).

Returning to the original variables, we see that $\dot{x} = v = \gamma [Y - (-X + \frac{1}{3}X^3)]$ is given by the height between the dashed and solid curves in the figure. It is then easy to sketch the orbit in the (x, v) phase space, and also x and v as a function of time.

The relaxation oscillations can be studied in [demonstration 2](#).

3.1.3 Driven oscillations: frequency locking

What happens if we now add driving at a frequency ω_D which is not the frequency of free oscillations? We might expect two possibilities. Firstly the two oscillations might continue independently, much like they did in the driven linear oscillator (chapter 2). A power spectrum should show signatures at both frequencies. Alternatively the internal oscillator might be captured by the drive, so that there are oscillations at a single frequency (and harmonics): this case is known as locked. A sharp way to discriminate between these two possibilities is to ask what happens as a parameter (e.g. the drive amplitude or the dissipation) is smoothly varied. In the locked case, the frequencies of the peaks in the spectrum should remain fixed over a finite range of parameter values. In the unlocked case, at least some frequencies should vary continuously. In this case the frequency ratios (of internal oscillator peaks to drive frequency) are necessarily passing through irrational values—such motion with two incommensurate frequencies is known as quasi-periodic motion.

In the perturbative case of small $\gamma = \varepsilon$, and with small drive amplitude $g = \varepsilon \times 2F$ near resonance $\omega_D = 1 + \varepsilon \Delta$ (with F and Δ of order unity) we can calculate the phenomenon of frequency locking analytically, using secular perturbation theory. The calculation is quite complicated, and is relegated to an appendix. The results are displayed in figure 3.2. The analysis is performed in terms of the behavior of an “amplitude equation” as in the secular perturbation theory above, and in particular stable fixed points of this equation corresponds to locked behavior of the oscillator. As might be expected, frequency locking occurs when the driving strength is large, and the frequency difference small. However the details are complicated, and we will see that this is generally true for frequency locking.

The quasiperiodic motion and phenomenon of frequency locking are illustrated in [demonstrations 3-4](#).

3.1.4 Chaos

As far as is known the periodically driven Van der Pol oscillator does not show chaos in the sense of a strange attractor—the attracting orbits are the limit cycles and quasiperiodic orbits discussed above. Cartwright and Littlewood [2] however studied chaotic like behavior for very special initial conditions on the boundaries of the basin of attraction of the limit cycles.

3.2 Duffing Oscillator

The Duffing oscillator is the motion of a particle in a quartic potential well

$$V(x) = \pm \frac{1}{2}x^2 + \frac{1}{4}x^4 + \frac{1}{4}$$

giving the equation of motion (including dissipation and periodic driving)

$$\ddot{x} + \gamma \dot{x} \pm x + x^3 = g \cos(\omega_D t). \quad (3.19)$$

The positive sign corresponds to an anharmonic well, and can be used to study the phenomenon of frequency pulling. With the negative sign (the “inverted” Duffing equation) the fixed point $x = 0$ is unstable, and there are potential minima at $x = \pm 1$. Small amplitude oscillations occur near these minima, and larger amplitude motion extends between the two minima. An experimental implementation of the Duffing equation is the “Moonbeam”[3] consisting of a flexible beam that oscillates in the double well potential formed by the elastic restoring forces and two magnets. Again introducing the phase space coordinates $(x, v = \dot{x}, \theta_D = \omega_D t)$ gives the autonomous equations

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= -\gamma v \mp x - x^3 + g \cos \theta_D . \\ \dot{\theta}_D &= \omega_D \end{aligned} \quad (3.20)$$

The Duffing oscillator, and the Moonbeam, do appear to show chaos.

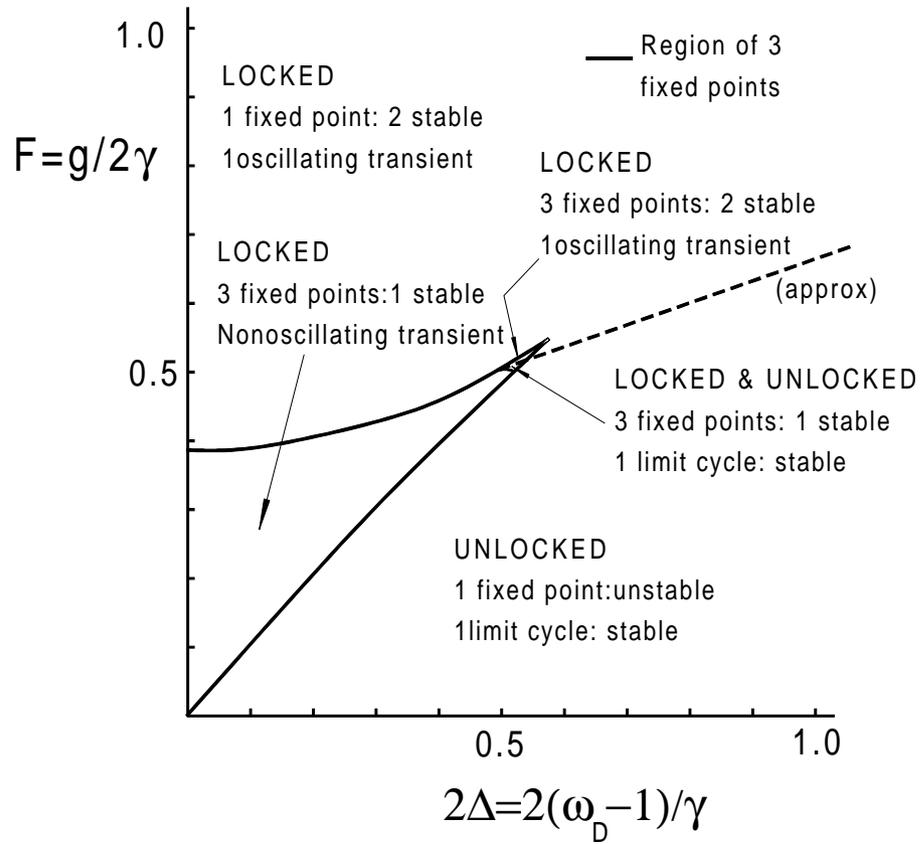


Figure 3.2: Locking in the van der Pol Oscillator. (The “fixed points” in the labels refer to results in the perturbation analysis, not to the orbits themselves.)

Appendix

Secular perturbation theory in phase space coordinates

Redoing the secular perturbation theory for the Van der Pol oscillator illustrates the more general formulation of the theory. The ideas are sketched very briefly here. For more details see e.g. Bender and Orszag [4]. In phase space coordinates the Van der Pol equation becomes

$$\left[\frac{d}{dt} - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right] \begin{pmatrix} x \\ v \end{pmatrix} = \varepsilon \begin{pmatrix} 0 \\ (1-x^2)v \end{pmatrix} \quad . \quad (21)$$

The solution is expanded

$$\begin{pmatrix} x \\ v \end{pmatrix} = A(T) e^{it} \begin{pmatrix} 1 \\ i \end{pmatrix} + c.c. + \varepsilon \begin{pmatrix} x_1 \\ v_1 \end{pmatrix} + \dots \quad (22)$$

where the vector $(1, i)$ is the eigenvector of the problem with $\varepsilon = 0$. Substituting into the equation, and collecting terms at each order in ε gives at $O(\varepsilon^1)$

$$\left[\frac{d}{dt} - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right] \begin{pmatrix} x_1 \\ v_1 \end{pmatrix} = e^{it} \left[-\frac{\partial A}{\partial T} \begin{pmatrix} 1 \\ i \end{pmatrix} + \begin{pmatrix} 0 \\ i(1-|A|^2 A) \end{pmatrix} \right] + c.c. + \text{terms in } e^{\pm 3it} \quad (23)$$

The adjoint of the operator on the left is

$$L^+ = \left[-\frac{d}{dt} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right]$$

and the zero eigenvalue eigenvectors satisfying $L^+ u^{(+)} = 0$ are

$$u^{(+)} = \begin{pmatrix} 1 \\ \pm i \end{pmatrix} e^{\pm it} \quad . \quad (24)$$

The solvability condition is then $\langle u^{(+)} | RHS \rangle = 0$, where the scalar product involves an integration over one time period and the vector scalar product, i.e.

$$\int_0^{2\pi} dt (1, \pm i) e^{\pm it} \left\{ e^{it} \left[-\frac{\partial A}{\partial T} \begin{pmatrix} 1 \\ i \end{pmatrix} + \begin{pmatrix} 0 \\ i(1-|A|^2 A) \end{pmatrix} \right] + \dots \right\} = 0 \quad (25)$$

The time integration over one period extracts the $e^{\mp it}$ terms from $\{ \}$ and the matrix product then yields the same amplitude equation as before.

Frequency Locking in the Driven Van der Pol Oscillator

This appendix goes beyond the level of the course, and is here for your interest only.

The equation of motion is

$$\ddot{x} + x = \epsilon(1 - x^2)\dot{x} + g \cos(\omega_D t) \quad (26)$$

Now assume $\omega_D = 1 + \epsilon\Delta$ and $g = 2\epsilon F$ with $\Delta, F = O(1)$ i.e. *weak driving near resonance*, and write as before

$$x = A(T)e^{it} + c.c. + \epsilon x_1 + \dots$$

with $T = \epsilon t$ to give the slow time scale of the dynamics around the unperturbed simple harmonic motion, and the driving term as $\epsilon F e^{i\Delta T} e^{it} + c.c.$. Collect all terms at $O(\epsilon)$ to give

$$\ddot{x}_1 + x_1 = (-2iA_T + iA - i|A|^2A + F e^{i\Delta T})e^{it} - iA^3 e^{3it} + c.c. \quad (27)$$

The condition that x_1 be finite is again that the coefficient of the “resonant” term e^{it} vanish (or demand orthogonality to the zero eigenvalue eigenvector of the adjoint operator i.e. multiply by e^{-it} and integrate over the 2π period and require the result to be zero):

$$A_T = \frac{1}{2}A(1 - |A|^2) - \frac{i}{2}F e^{i\Delta T} \quad (28)$$

Note that the absence of any explicit ϵ dependence in this equation shows that the original choice of scaling of g and Δ with ϵ was correct. We want to look for the possibility of locking to the external drive i.e. $A_T \propto e^{i\Delta T}$. It is then convenient to introduce \bar{A} defined by $A = \bar{A}e^{i\Delta T}$ which then satisfies the equation

$$\bar{A}_T + i\Delta\bar{A} = \frac{1}{2}\bar{A}(1 - |\bar{A}|^2) - \frac{i}{2}F \quad (29)$$

The question of locking or entrainment now reduces to the nature of the solutions of (29). Fortunately since this is the equation for dynamics in a two dimensional phase space (real and imaginary \bar{A}) the possibilities for the asymptotic long time dynamics are very limited. In fact a theorem called the Poincaré-Bendixson theorem tells us the only possibilities are fixed points, heteroclinic orbits connecting unstable fixed points, or limit cycles. A stable *fixed point* for \bar{A} corresponds to a locked

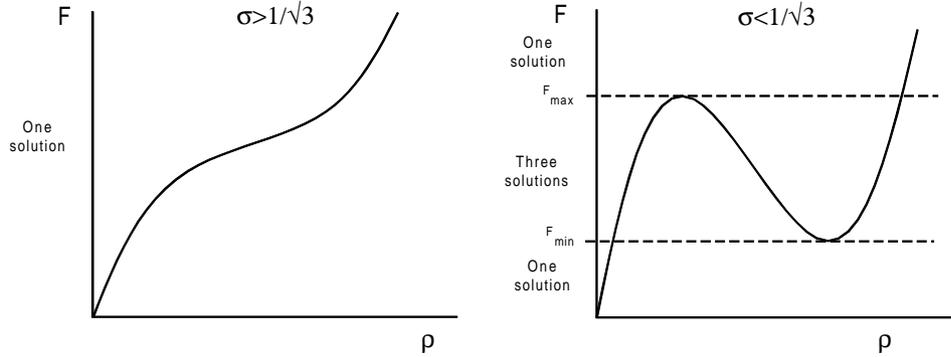


Figure 3: Multiplicity of solutions

solution or entrainment of x ; a stable *limit cycle* corresponds to unlocked solution or no entrainment (two frequencies). A full analysis of these possibilities in (28) is still quite complicated because the possibility of more than one fixed point or limit cycle has to be addressed.

We can at least easily address the existence of fixed points ($\bar{A}_T = 0$). In this case taking the modulus squared of the equation gives the cubic equation for $\rho = |\bar{A}|^2$:

$$\sigma^2 \rho + \rho(1 - \rho)^2 = F^2 \quad (30)$$

where $\sigma = 2\Delta$. There will be three fixed points if this cubic equation has three real roots, and one fixed point if there is only one root. The number of roots can be determined by considering the extrema of $f(\rho) = \sigma^2 \rho + \rho(1 - \rho)^2$ determined by

$$\frac{df}{d\rho} = 3\rho^2 - 4\rho + 1 + \sigma^2 = 0$$

i.e. $\rho = \rho_{\pm} = \frac{2}{3} \pm \frac{1}{3}\sqrt{1 - 3\sigma^2}$ which has real solutions providing $\sigma < \frac{1}{\sqrt{3}}$. In this case we can substitute ρ_{\pm} back into (30) to find the values of F at the extrema $F_{\max}(\sigma)$ and $F_{\min}(\sigma)$. So we have the following cases (see figure 3):

- (i) $\sigma > \frac{1}{\sqrt{3}}$: one fixed point
- (ii) $\sigma < \frac{1}{\sqrt{3}}$ and then

- (a) $F_{\min} < F < F_{\max}$: three fixed points
- (b) $F > F_{\max}$ or $F < F_{\min}$: one fixed point.

This gives the regions of one or three fixed points separated by the solid lines in figure 2. We can then in principle test the stability of the fixed points: any *stable* fixed point tells us that entrainment is a solution; a *single* fixed point that is *unstable* tells us that there must be a stable limit cycle (cf. the Poincaré-Bendixson theorem) so that the oscillators are definitely unlocked. Other possibilities, and the coexistence of stable limit cycles (unlocked orbits) with the stable fixed points (locked orbits) need more detailed analysis. The full analysis is described in section 2.1 of Guckenheimer and Holmes [5], and leads to the results plotted in figure 2. (Note that the stability lines are qualitative sketches.)

In this particular case we have effectively calculated the Poincaré section of the driven oscillator: since ϵ is small we can calculate the Poincaré section from the change in A due to the small increment $\delta T = \epsilon \times 2\pi$.

December 24, 1999

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