Chapter 22

Mathematical Chaos

The sets generated as the long time attractors of “physical” dynamical systems described by ordinary differential equations or discrete evolution equations in the case of maps such as the Hénon map appear to be very complicated, often with unpleasant properties such as the lack of robustness (the parameters leading to chaotic and nonchaotic solution may be intertwined on an arbitrarily small scale). Few proofs are available for the conjectured properties of these sets. To make mathematical progress it is necessary to make restrictions on the type of dynamical systems considered, that go far beyond standard smoothness assumptions. Often the assumptions needed to construct proofs are so strict that it is *known* that all common “physical” attractors violate the assumptions. (However, numerical tests of the results proven for the restricted systems often show that they apply, at least as an excellent approximation, to the physical systems.) In this chapter the goal is to present some of the flavor of this mathematical approach, since many of our ideas on chaotic systems have arisen in this context.

A key assumption in much of the mathematical development is the notion of “hyperbolicity”. To define this concept it we first introduce the general idea of stable and unstable manifolds. Useful references are: Eckmann and Ruelle [1], sections *IIIE*, *F*; Ott [2], sections 4.3 and 9.5; Aligood et al. [3], section 2.6 and chapter 10; and Guckenheimer and Holmes [4] section 5.2.

22.1 Stable and Unstable Manifolds

22.1.1 Saddle Fixed Point

The idea of stable and unstable manifolds is most easily introduced in the context of a saddle fixed point of a two dimensional map. They are the natural extensions of the linear eigenvectors of the stability analysis of the fixed point into the nonlinear regime.

Consider a fixed point x_f of a differentiable two dimensional map F with a differentiable inverse map. If x_f has one unstable eigenvalue s with $|s| > 1$ and corresponding eigenvector E^s, and one stable eigenvalue u with $|u| < 1$ and corresponding eigenvector E^u, it is called a saddle fixed point. The stable manifold of x_f denoted $W^s(x_f)$ is the set of points y such that $\left|F^n(y) - F^n(x_f)\right| \to 0$ as $n \to \infty$; the unstable manifold of x_f denoted $W^u(x_f)$ is the set of points y such that $\left|F^{-n}(y) - F^{-n}(x_f)\right| \to 0$ as $n \to \infty$. Both are one dimensional manifolds containing x_f. A manifold is basically a nice set (e.g. without fractal properties): a one dimensional manifold is defined as a set that is locally a curve and can be produced locally by bending a line. The letters D (in a sans serif font!) and O are 1-manifolds, the letters A and X are not, since there are points where no small neighborhood looks like a line (Aligood et al. [3]). Also $W^s(x_f)$ is tangent to E^s and $W^p(x_f)$ is tangent to E^u at x_f.

The extension to periodic saddles is given by noting that these are fixed points of F^q for some q, and there is a straightforward generalization to higher dimensional maps. The idea of stable and unstable manifolds can be defined locally at an arbitrary point x in the phase space: the stable manifold W^s of x is the set of points y such that $|F^n(y) - F^n(x)| \to 0$ as $n \to \infty$, and the unstable manifold W^u of x is the set of points y such that $\left|F^{-n}(y) - F^{-n}(x)\right| \to 0$ as $n \to \infty$. Useful results can be proven for stable and unstable manifolds. For example if x is in an attracting set Σ then $W^u(x)$ is contained in Σ. Also the number of positive Lyapunov exponents of the set is a lower bound for the capacity dimension of Σ.

22.2 Hyperbolic Invariant Sets

A hyperbolic invariant set in a sense is the generalization of a saddle fixed point and can be defined in terms of the properties of the linearized map about the point x on the set, i.e. the tangent space T_x at the point x. Note that the definition applies to both attracting and nonattracting sets.

An invariant set Σ under the map F is said to be hyperbolic if there is a direct sum decomposition of T_x into stable and unstable spaces $T_x = E_x^s \oplus E_x^u$ for all x in Σ such that:

(i) the splitting into E_x^s, E_x^u varies continuously with x;

(ii) the splitting is invariant in the sense that $DF(x)E_x^{s,u} = E_{F(x)}^{s,u}$, i.e. the evolving the stable and unstable spaces at x with the tangent space map gives the same result as the stable and unstable spaces at the evolved point $F(x)$;

(iii) there are numbers $K > 0$ and $0 < \rho < 1$ such that for all $n > 0$

$$|DF^n(x)v| \le K\rho^n|v| \quad \text{for } v \text{ in } E_x^s \tag{22.1}$$

$$|DF^{-n}(x)v| \le K\rho^n|v| \quad \text{for } v \text{ in } E_x^u. \tag{22.2}$$

(In these expressions $DF(x)$ is the Jacobean matrix of F at x.) The latter condition says that the exponential decay rate of vectors in the stable subspace and the exponential growth rate in the unstable subspace are bounded away from zero.

Again the linear spaces E_x^s, E_x^u may be extended into the nonlinear regime far from x to give stable and unstable manifolds $W^s(x)$, $W^u(x)$ at each point x on the attractor that are tangent to E_x^s, E_x^u at x (e.g. Guckenheimer and Holmes [4], Theorem 5.2.8). Two points on the stable (unstable) manifold approach (separate from) each other exponentially.

Most of the mathematical understanding of chaotic attractors is restricted to hyperbolic attractors. Further restrictions are often needed on smoothness and other properties. For example an *Axiom A* attractor is an attractor of a differentiable map that is hyperbolic and mixing. The property of mixing is that for any two sets S_a and S_b in the phase space

$$\lim_{n\to\infty} \mu\left[S_a \cap F^n(S_b)\right] = \mu(S_a)\mu(S_b) \tag{22.3}$$

where μ is the natural measure of the attractor. The property of mixing is that initial conditions get spread over the attractor according to the measure. Axiom A attractors are particularly nice, and many results have been proven for these attractors, for example the existence of a natural invariant measure that is smooth along the expanding directions. Axiom A attractors are also structurally stable, which means that even the delicate chaotic structure survives a small perturbation of the map.

Most physical attractors are non-hyperbolic because there are points on the attractor where the stable and unstable manifolds are tangent to one another (see figure 22.1 for the Hénon map). Structural stability does not seem to be a property of many physical attractors. The bakers' map is hyperbolic, although the map is not differentiable so it is not Axiom A. In the next chapter the horseshoe map, that

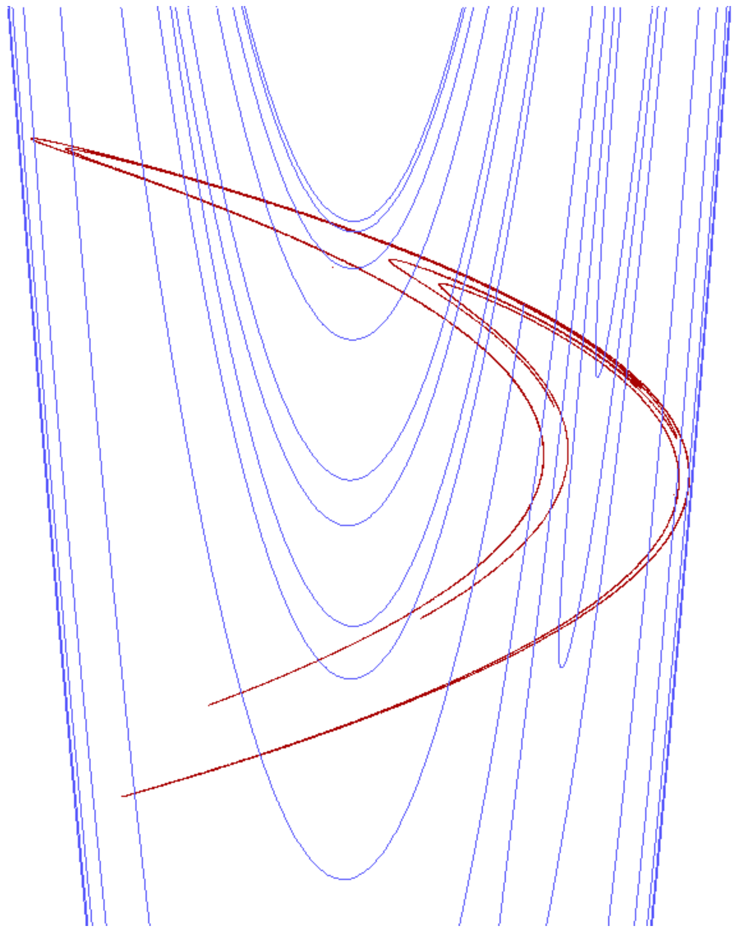

Figure 22.1: Plot of the Hénon attractor (red) and the stable manifold (blue) of the fixed point that lies within the attractor. The stable component E_x^s of the tangent space (i.e. the contracting direction) lies along this curve at the points x where it intersects the attractor. Since the expanding direction E_x^u lies along the attractor, we see that E_x^s and E_x^u are parallel at points where the blue curve is tangent to the attractor. At these points they do not span the tangent space T_x indicating a breakdown of hyperbolicity. The calculation of the stable manifold is discussed in reference [5], and the figure was constructed using the program supplied there.

is both differentiable and hyperbolic, is introduced. This map shows a chaotic set, but this set is not an attractor. The Ansov map or "cat map"

$$
\begin{aligned}
x_{n+1} &= x_n + y_n \quad \mathrm{mod} \quad 1 \\
y_{n+1} &= x_n + 2y_n \quad \mathrm{mod} \quad 1
\end{aligned}
\tag{22.4}
$$

is a differentiable, area preserving hyperbolic map (the eigenvalues of the Jacobean $(3 \pm \sqrt{5})/2$ and the eigenvectors are independent of position). The orbit from a typical initial condition fills the whole unit square with uniform measure. The map is therefore Axiom A. The Sinai map

$$
\begin{aligned}
x_{n+1} &= x_n + y_n + \delta \cos(2\pi y_n) \quad \mathrm{mod} \quad 1 \\
y_{n+1} &= x_n + 2y_n \quad \mathrm{mod} \quad 1
\end{aligned}
\tag{22.5}
$$

can be considered a perturbation of this map, and so by structural stability for small enough δ will also be hyperbolic and the attractor will be of capacity dimension 2. The measure of the Sinai map is however no longer uniform and in fact shows interesting structure, and diagnostics involving the measure (e.g. the information dimension) will vary with δ. The properties of the Sinai map are illustrated in the demonstration.

22.3 Illustration

To illustrate the flavor of the use of hyperbolicity to prove properties of chaotic attractors consider the following [6].

For a map F with an Axiom A (i.e. *hyperbolic* and *mixing*) attractor the natural measure of the attractor contained in some closed set S is

$$\mu(S) = \lim_{n\to\infty} \sum_i L_i^{-1} \tag{22.6}$$

where the sum is over the unstable fixed points of F^n which lie in S, and L_i is the product of the unstable eigenvalues of the linearization of F^n at the ith fixed point. In particular if the set S is the entire attractor so that $\mu(S) = 1$ we have

$$1 = \lim_{n\to\infty} \sum_i L_i^{-1}. \tag{22.7}$$

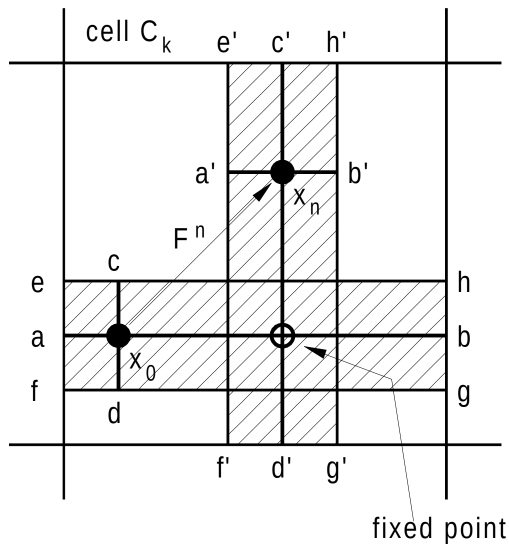

Figure 22.2: Point x_0 is mapped into x_n under F^n. Then if $ab \to a'b'$ and $cd \to c'd'$ the parallelogram $efgh \to e'f'g'h'$, and by continuity there must be a single saddle fixed point of F^n in the intersection region. In all cases the horizontal (vertical) lines in the figure are segments of the stable (unstable) manifolds. The figure is drawn with orthogonal lines for simplicity, but this is not essential to the argument.

22.3.1 Proof

We will illustrate the proof for two dimensional maps. Partition the phase space into small cells C_i where each cell has as its boundaries the stable and unstable manifolds. Small enough cells may be considered parallelograms. Consider the iteration of a large number of initial conditions distributed over a particular cell C_k according to the natural measure of the attractor. Iterate a large number of times n then a fraction given by $\mu(C_k)$ will return to the cell (the mixing property). Suppose x_0 is an initial condition that returns to x_n in C_k. If ab is the stable manifold segment passing through x_0 and $a'b'$ its image after n iterations, and $c'd'$ is the unstable manifold segment passing through x_n and cd its preimage under n iterations, then the parallelogram $efgh$ is mapped to the parallelogram $e'f'g'h'$ after n iterates, where the boundaries of $efgh$ and $e'f'g'h'$ are segments of stable and unstable manifolds (see figure 22.2). This means there must be a single saddle fixed point of F^n in the intersection region, and conversely any saddle fixed point of F^n can be surrounded by similar parallelograms. If the unstable eigenvector of the fixed point is $\lambda_u > 1$, then the height of $efgh$ is a fraction λ_u^{-1} of the height of C_k. The measure of an Axiom A attractor is smooth along the unstable directions, and since C_k is small so that the distribution of the measure across the height can be considered uniform, the fraction of the measure of C_k in the strip $efgh$ is then just λ_u^{-1}. Thus the fraction of initial conditions that return to C_k under F_n, which by the mixing property is $\mu(C_k)$ for large n, is given by summing the λ_u^{-1} over the saddle fixed points of F_n within C_k. Summing over the C_k in S then proves the result 22.6 since for a two dimensional map L is just equal to the unstable eigenvalue λ.

Further results may be derived relating the properties of chaotic attractors to the unstable periodic orbits. A few of these results, proven for hyperbolic attractors, are quoted here for a two dimensional map.

1. The Lyapunov exponents are
$$\lambda_{1,2} = \lim_{n\to\infty} \frac{1}{n} \sum_i \frac{1}{\lambda_u^{(i)}} \ln \lambda_{u,s}^{(i)} \tag{22.8}$$
with $\lambda_u^{(i)}, \lambda_s^{(i)}$ the unstable and stable eigenvalues at the ith period n point.

2. The capacity dimension D_C is given by
$$\lim_{n\to\infty} \sum_i \left(\lambda_s^{(i)}\right)^{D_C - 1} = 1. \tag{22.9}$$

3. The eigenvalues of the Perron-Frobenius operator $\delta(y - f(x))$ for the evolution of the measure can be expanded in the unstable fixed points [7].

Auerbach et al. [8] have tested some of these results on the (nonhyperbolic) Hénon map, which has 1, 3, 1, 7, 1, 15, 29, 63, 55, 103 order 1 to 10 cycles respectively. For example the calculate values for D_C based on (22.9) and get values of 1.26, 1.29, 1.30, 1.26, 1.27 using cycles of order 6 to 10. There is a large literature in both math and physics on this area.

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