Chapter 20

Universal Onset of Chaos from a 2-Torus

The approach to chaos at irrational winding numbers displays universal properties that however depend sensitively on the winding number (and in fact on the structure of the "continued fraction expansion", as we will see below). At criticality the $f(\alpha)$ curve is universal, and the power spectrum has universal structure. A renormalization group theory [1],[2] analogous to the theory for the period doubling cascade but somewhat more complicated in technical details can be developed to explain these results. Careful experiments have verified the predictions.

The 1d circle map provides the basis for the discussion, much as the quadratic map does for the period doubling cascade. Now however there are two parameters, Ω and K. Unfortunately the simple behavior is not given by say increasing K at fixed Ω, but rather increasing K at fixed winding number W of the solution (or alternatively varying the winding number at fixed K). The two parameters makes the theory more difficult, and only the outline will be sketched here.

20.1 Irrational numbers

The onset of chaos from an quasiperiodic orbit is the object of study, but it is useful to approach the irrational winding number through a sequence of rational approximants p/q with p and q integers.

Any irrational number can be uniquely written in a continued fraction expansion

$$W = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}} \tag{20.1}$$

where the integers $(a_1, a_2, \ldots)$ define the expansion. The nth rational approximant W_n to W is defined by truncating this expansion at the nth level, e.g.

$$W_2 = \frac{1}{a_1 + \frac{1}{a_2}}. \tag{20.2}$$

The quadratic irrationals (solutions to a quadratic equation) yield a sequence $(a_1, a_2, \ldots)$ which is *periodic*. The renormalization group theory makes use of this structure, and has therefore only been implemented for the quadratic irrationals. Most attention has been focused on the "Golden Mean" $G = \left(\sqrt{5} - 1\right)/2 \simeq 0.61803\ldots$ which has the simple continued fraction expansion $(1, 1, 1, \ldots)$. This number can be thought of as the "most irrational", since the rational approximants to this number converge most slowly. The rational approximants are

$$G_n = \frac{F_n}{F_{n+1}}, \tag{20.3}$$

where $F_0 = 1,\ F_1 = 1,\ F_{n+1} = F_n + F_{n-1}$ are the Fibonacci numbers. The successive rational approximants to the Golden Mean are therefore

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \frac{8}{13}, \ldots \tag{20.4}$$

The "silver mean" $\sqrt{2} - 1$ with continued fraction expansion $(2, 2, 2, \ldots)$ has also been studied both theoretically and experimentally.

20.2 Scaling for the Circle Map

The following results are obtained numerically (originally by Shenker [3]) for the circle map:

1. Define $\Omega_n(K)$ as the value of Ω giving a periodic orbit of winding number G_n and containing the origin $x = 0$. (For reference look at the figure for winding number $\frac{5}{8}$.) This is defined by

$$f^{F_{n+1}}_{K,\Omega_n}(0) = F_n, \tag{20.5}$$

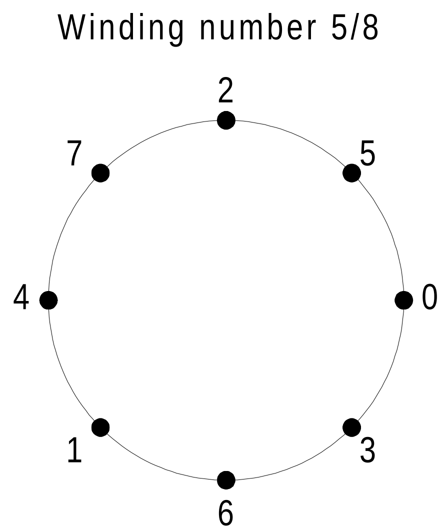

Figure 20.1: Orbit on circle for winding number 5/8.

since after F_{n+1} (8 for the figure) iterations the point returns to the starting value $x = 0$ but has orbited F_n times (5 for the figure). Then it is found for large n

$$\Omega_n(K) = \Omega_\infty(K) - c\tilde{\delta}^{-n} \tag{20.6}$$

with

$$\tilde{\delta} = \begin{cases} G^{-2} = 2.618\ldots & \text{for } K < 1 \quad \text{“trivial”} \\ 2.8336\ldots & \text{for } K = 1 \quad \text{“nontrivial”} \end{cases}, \tag{20.7}$$

where the “trivial result” follows for a smooth map, but the nontrivial result is obtained numerically [3]. The trivial result can be found from the result for a uniform rotation.

2. The distance from $x = 0$ to the nearest element of the G_n cycle is

$$d_n = f^{F_n}_{K,\Omega_n}(0) - F_{n-1} \tag{20.8}$$

and scales for large n as

$$\frac{d_n}{d_{n+1}} \to \tilde{\alpha} \tag{20.9}$$

with

$$\tilde{\alpha} = \begin{cases} -G^{-1} = -1.618\ldots & \text{for } K < 1 \quad \text{“trivial”} \\ -1.28857\ldots & \text{for } K = 1 \quad \text{“nontrivial”} \end{cases} . \tag{20.10}$$

3. The scaling parameters $\tilde{\delta}$ and $\tilde{\alpha}$ are *universal* for the given winding number i.e. are independent of the details of the map. In addition the power spectrum for $n \to \infty$ is universal: the principal peaks occur at powers of G (i.e. equally spaced peaks on a log plot) and the structure of secondary peaks is the same between all adjacent main peaks. In addition the $f(\alpha)$ curve is universal at the critical point..

20.3 Renormalization Group Theory

Motivated by the scaling of the point separation, define

$$\bar{f}^n(x) = f^{F_{n+1}}(x) - F_n \tag{20.11}$$

and the scaled version

$$\bar{f}_n(x) = \tilde{\alpha}^n f^{F_{n+1}}(\tilde{\alpha}^{-n} x) - F_n. \tag{20.12}$$

Then

$$\begin{aligned} \bar{f}^{n+1}(x) &= f^{F_{n+2}}(x) - F_{n+1} \\ &= f^{F_{n+1}}\left(f^{F_n}(x)\right) - (F_n + F_{n-1}) \\ &= f^{F_{n+1}}\left(f^{F_n}(x) - F_{n-1}\right) - F_n \end{aligned} \tag{20.13}$$

using the properties of Fibonacci numbers and functional composition in the second line and $f(x+1) = f(x) + 1$ in the third. This gives the composition rule

$$\bar{f}^{n+1}(x) = \bar{f}^n\left(\bar{f}^{n-1}(x)\right). \tag{20.14}$$

We now look for a fixed point structure using this functional composition and rescaling i.e. a fixed point function $\bar{f}^*$ defined by

$$\bar{f}^*(x) = \tilde{\alpha}\bar{f}^*\left(\tilde{\alpha}\bar{f}^*\left(\tilde{\alpha}^{-2}x\right)\right), \tag{20.15}$$

and linearize as in the analysis of the period doubling fixed point. One solution is $\bar{f}^*(x) = const + x$ which gives $\tilde{\alpha} = -G^{-1}$: this is the result for $K < 1$. For $K = 1$ the linear term in $\bar{f}^*(x)$ disappears, and a second solution is valid with a nontrivial value of $\tilde{\alpha}$ quoted above. Similarly the values of $\tilde{\delta}$ are given by linearizing about the fixed point.

20.4 Experiments

Considering the delicacy of the phenomena, it is remarkable that there are a number of experiments [4][5][6][7], on systems from fluids to electronic devices, that quantitatively verify the predictions of the theory. It is imperative that the systems have two “knobs” to tune the system—one to increase the nonlinearity, and the other to maintain the frequency ratio of the oscillators (either two internal, or one internal and one drive oscillator) at the chosen irrational ratio, usually the Golden Mean. Many features of the theory have been illustrated:

1. the structure of frequency locking and the Arnold tongues;
2. reconstructing the torus attractor and its “crinkling” at the onset of chaos;
3. the universal power spectrum at the onset of chaos;
4. the universal $f(\alpha)$ at the onset of chaos;
5. the period doubling cascade in locked tongues.

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