Chapter 18

Driven Pendulum and the 2D Circle Map

An important route to chaos is from a quasiperiodic state of n incommensurate frequencies. The phase space structure of such a state is topologically an n-torus, and the onset of chaos corresponds to the break down of this torus to a more complex structure. Landau envisioned the development of fluid turbulence as occurring through transitions to higher and higher-n tori, leading to more and more complicated dynamics in space and time. A very influential work was that of Ruelle and Takens (refined with Newhouse), who showed that "generically" this might not be expected to happen, but a chaotic state would instead develop from a small-n torus. The interpretation of the idea of genericity involves some subtlety. An alternative phenomena is that the oscillators in the quasiperiodic motion may lock to give a simpler dynamics of fewer incommensurate frequencies or even periodic motion. In this chapter we set up a simple two dimensional map—the 2D circle map—that displays these phenomena, and in the next chapter this is further reduced to a one dimensional map. In chapter 20 the *universal* aspects of the onset of chaos from quasiperiodic motion are breifly described, and in chapter 21 the Ruelle-Takens-Newhouse theorem and its understanding are discussed.

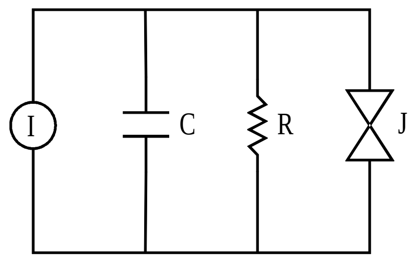

Figure 18.1: Resistively shunted Josephson junction. The capacitance C is the capacitance of the junction.

18.1 Driven Pendulum

A simple physical example showing these phenomena is the driven running pendulum, described by the equation

$$\ddot{\theta} + \gamma\dot{\theta} + \sin\theta = d + g\cos\omega_D t \tag{18.1}$$

where as well as the usual oscillating force a constant force d has been added which will tend to produce running solutions with nonzero $\langle\dot{\theta}\rangle$. Note that the motion can be considered as the oscillatory driven and damped motion of a particle in the "washboard potential" $V(\theta) = -\cos\theta - d\theta$. The frequency locking is between the drive frequency ω_D and the running frequency $< \dot{\theta} >$.

Equation (18.1) also describes the dynamics of the resistively shunted Josephson junction driven by a current source $I = A + B\cos\omega_D t$ (figure 18.1).

The only feature of the Josephson junction that we need to know is the current-voltage ($I - V$) relationship given by the two equations

$$\begin{aligned} I &= I_c \sin\theta \\ \dot{\theta} &= \tfrac{2e}{\hbar} V \end{aligned} \tag{18.2}$$

where θ is an internal variable (actually the difference in the phase of the superconducting order parameter across the junction), I_c is the maximum current the Josephson junction can support in the superconducting state, and e and $\hbar$ are the fundamental constants. Equating the current from the source to the sum of the currents through the three components gives

$$A + B\cos\omega_D t = \frac{\hbar C}{2e}\ddot{\theta} + \frac{\hbar}{2eR}\dot{\theta} + I_c\sin\theta \tag{18.3}$$

which can be rescaled to the form of Eq.(18.1). The running solution $\langle\dot{\theta}\rangle \neq 0$ corresponds to a mean voltage $\langle V \rangle \neq 0$ and so dissipation in the junction. The frequency locking yields a fixed voltage over some range of driving $V = (\hbar/2e)\,(p/q)\,\omega_D$ with the ratio of integers p/q giving the locking ratio: these are known as Shapiro steps. The alternative of chaotic dynamics in the circuit adds apparent noise to the circuit characteristics, increasing the effective "noise temperature" to much higher values than the true temperature, which can be a limitation in the application of such circuits.

Equation (18.1) can be reduced in the usual way to the flow in a three dimensional phase space

$$\begin{aligned} \dot{\theta} &= \omega \\ \dot{\omega} &= -\gamma\omega - \sin\theta + d\cos\theta_D \ . \\ \dot{\theta}_D &= \omega_D \end{aligned} \tag{18.4}$$

Alternatively the evolution can be reduced to a two dimensional map by strobing at the drive frequency ω_D (a Poincare section). Since the differential equation is second order, the dynamics in each period of the drive is determined by the values of θ and $\omega = \dot{\theta}$ at the beginning of the period, and by the force, which is the same in each period, i.e. the dynamics reduces to the map

$$\begin{pmatrix} \theta_{n+1} \\ \omega_{n+1} \end{pmatrix} = \begin{pmatrix} G_1(\theta_n, \omega_n) \\ G_2(\theta_n, \omega_n) \end{pmatrix} = G \begin{pmatrix} \theta_n \\ \omega_n \end{pmatrix} \tag{18.5}$$

where G is a two dimensional map that in principal is given by integrating from nT to $(n+1)\,T$ with T the period $2\pi/\omega_D$. The exact form of G is complicated, but since volumes contract uniformly at the rate γ in the flow and there is no contraction along the flow direction, it is clear that the Jacobean of the map is given by

$$J = \begin{vmatrix} \frac{\partial G_1}{\partial \theta_n} & \frac{\partial G_1}{\partial \omega_n} \\ \frac{\partial G_2}{\partial \theta_n} & \frac{\partial G_2}{\partial \omega_n} \end{vmatrix} = e^{-2\pi\gamma/\omega_D} = b \quad \text{say,} \tag{18.6}$$

i.e. there is uniform contraction of areas in the map (for $\gamma \neq 0$).

For large damping (small b) we would expect the velocity ω to relax rapidly to the value determined by the instantaneous θ, and then

$$\theta_{n+1} = f(\theta_n)\,, \tag{18.7}$$

a one dimensional map.

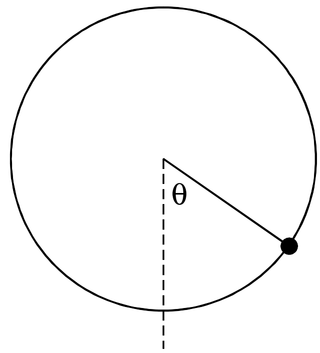

Figure 18.2: Periodically kicked rotor

18.2 The Periodically Kicked Rotor

Since a quantitative reduction of the pendulum to a 2d map is complicated, it is convenient to study a simpler model, namely the damped motion of a ball on a ring kicked *periodically* in time with a strength that depends on the *angle*

$$\ddot{\theta} + \gamma\dot{\theta} = (A - B\sin\theta)\sum_n \delta(t-n)\,. \tag{18.8}$$

The motion can be easily integrated between kicks: if $\dot{\theta}_n$ is the angular velocity just *before* the nth kick, the velocity just after the kick is $\dot{\theta}_n + A - B\sin\theta_n$, which then decays exponentially, because of the damping, up to the time of the next kick, so that for $n < t < n+1$ we have

$$\dot{\theta} = e^{-\gamma(t-n)}\left(\dot{\theta}_n + A - B\sin\theta_n\right). \tag{18.9}$$

In particular, just before the next kick

$$\dot{\theta}_{n+1} = e^{-\gamma}\left(\dot{\theta}_n + A - B\sin\theta_n\right) \tag{18.10}$$

and integrating Eq.(18.9) gives

$$\theta_{n+1} = \theta_n + \frac{1-e^{-\gamma}}{\gamma}\left(\dot{\theta}_n + A - B\sin\theta_n\right). \tag{18.11}$$

Now defining

$$\Omega = \frac{A}{2\pi\gamma},\, r_n = \frac{e^{\gamma}-1}{\gamma}\dot{\theta}_n - 2\pi\Omega,\; \frac{K}{2\pi} = \frac{1-e^{-\gamma}}{\gamma}B \quad \text{and} \quad b = e^{-\gamma} \tag{18.12}$$

reduces the equations to a standard form that is known as the “dissipative 2D circle map”

$$\begin{aligned} \theta_{n+1} &= \theta_n + 2\pi\Omega - K\sin\theta_n + br_n \\ r_{n+1} &= br_n - K\sin\theta_n \end{aligned} \quad . \tag{18.13}$$

The determinant of the Jacobean of the map is b so that areas contract at a uniform rate for $b < 1$, and the map is *area preserving* for $b = 1$.

An alternative motivation of the $2d$ circle map is through a naïve discretization of the pendulum equation (18.1) at times $t = n2\pi/\omega_D$ (when the angle is θ_n):

$$(\theta_{n+1} - 2\theta_n + \theta_{n-1}) + (1-b)\,(\theta_n - \theta_{n-1}) + K\sin\theta_n = (1-b)\,\Omega \tag{18.14}$$

where

$$(1-b) = \frac{2\pi\gamma}{\omega_D}, \quad K = \frac{2\pi}{\omega_D}, \quad \Omega = \frac{d+g}{\gamma}, \quad r_n = \theta_n - \theta_{n-1} - 2\pi\Omega. \tag{18.15}$$

The dissipative circle map is often written in terms of $x_n = \theta_n/2\pi$ and $y_n = r_n/2\pi$:

$$\begin{aligned} x_{n+1} &= x_n + \Omega - \tfrac{K}{2\pi}\sin 2\pi x_n + by_n \\ y_{n+1} &= by_n - \tfrac{K}{2\pi}\sin 2\pi x_n \end{aligned} \quad . \tag{18.16}$$

For $b \to 0$ (large damping of the kicked rotor) we get the one dimensional circle map

$$x_{n+1} = x_n + \Omega - \tfrac{K}{2\pi}\sin 2\pi x_n \; . \tag{18.17}$$

The behavior of the dissipative circle map is studied in the demonstrations.

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