Chapter 13

Bifurcation Theory

The change in the qualitative character of a solution as a control parameter is varied is known as a *bifurcation*. This occurs where a *linear stability analysis* yields an instability (characterized by a growth rate σ of a perturbation of the base solution with $\mathrm{Re}\,\sigma = 0$). The connection is through the "implicit function theorem"—the solution can be continued smoothly except where the Jacobean is singular. Typically a new solution develops at this point. For parameter values near the bifurcation values the properties of the solutions are given by the method of *normal forms*. These are the ideas introduced in the present chapter.

13.1 Bifurcation from a steady solution

13.1.1 Linear analysis

Consider a set of ordinary differential equations (flow) for a vector U of variables

$$\dot{U} = f(U; r) \tag{13.1}$$

with r a control parameter that we will vary. Suppose $U = U_0$ is a steady state solution

$$f(U_0; r) = 0. \tag{13.2}$$

Look for a linear instability as r changes i.e. study the dynamics of a small perturbation δU linearizing about U_0, $U = U_0 + \delta U$ with

$$\delta\dot{U}^{(i)} = K_{ij}\delta U^{(j)} \quad \text{and} \quad K_{ij} = \left.\frac{\partial f^{(i)}}{\partial U^{(j)}}\right|_{U=U_0}. \tag{13.3}$$

Since U_0 is a time independent state, K_{ij} is a constant matrix, and its eigenvalues σ_α (ordered so that $\mathrm{Re}\,\sigma_1 \geq \mathrm{Re}\,\sigma_2 \ldots$) give the growth rates of perturbations:

$$\delta U \propto \sum_\alpha A_\alpha e^{\sigma_\alpha t} u^{(\alpha)}, \tag{13.4}$$

with A_α a set of initial amplitudes. The $u^{(\alpha)}$ are the eigenvectors, and tell us the character of the exponentially growing or decaying solutions.

Stability requires *all* $\mathrm{Re}\,\sigma < 0$. As r changes the onset of instability occurs when $\mathrm{Re}\,\sigma_1 = 0$ at $r = r_c$ say. There are two possible classes of behavior that will typically occur as the single control parameter r is changed, based on the fact that the differential equations are real:

1. A single real eigenvalue passes through zero—this is the case of a "stationary bifurcation"

2. A complex conjugate pair of eigenvalues passes through the imaginary axis in the complex σ plane—a Hopf bifurcation. In this case $\mathrm{Im}\,\sigma$ gives an oscillating component to the time dependence.

(It is assumed that various special cases do not occur, e.g. that the eigenvalues do not move up to the axis and then reverse without crossing—the assumption of "transversality". Such special cases might be engineered by carefully tuning the equations, but will not in general be robust to small changes of the equations. Similarly by varying two parameters, it may be possible to tune two unconnected eigenvalues to cross the imaginary axis together. Or physical symmetries in the problem may lead to degenerate eigenvalues. Such a degenerate bifurcation may have interesting properties (for example in some cases it is possible to predict nearby chaos), but will not be considered here. For a review see Crawford [1].)

Example: the Van der Pol oscillator. The equation for the Van der Pol oscillator may be written

$$\ddot{x} - \left(r - x^2\right)\dot{x} + x = 0, \tag{13.5}$$

or in phase space coordinates

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= \left(r - x^2\right) v - x \end{aligned}, \tag{13.6}$$

where a slightly different scaling of the x variable has been used than in chapter 3 ($x \to \gamma^{-1/2}x$) and γ is then rewritten as r. Clearly $x = 0,\ v = 0$ is a solution. Linearizing about this solution gives

$$\begin{aligned} \delta\dot{x} &= \delta v \\ \delta\dot{v} &= r\delta v - \delta x \end{aligned} \tag{13.7}$$

so that the Jacobean matrix is

$$K = \begin{bmatrix} 0 & 1 \\ -1 & r \end{bmatrix} \tag{13.8}$$

and the eigenvalues (growth rates) are $\frac{1}{2}\left(r \pm i\sqrt{4-r^2}\right)$. As r increases, a complex pair of eigenvalues passes through the imaginary axis—a Hopf bifurcation—when $r = r_c = 0$ with an imaginary part to σ (the oscillation frequency) equal to 1. The eigenvectors are $(1, \pm i)$, so that for r near 0 the solutions is

$$\begin{pmatrix} x \\ v \end{pmatrix} = \left[A_1 \begin{pmatrix} 1 \\ i \end{pmatrix} e^{it} + A_2 \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{-it}\right] e^{rt/2} \tag{13.9}$$

with $A_2 = A_1^*$ for a real solution and then A_1 is a constant set by initial conditions.

13.1.2 Nonlinear analysis

For $r > r_c$ there is at least one exponentially growing solution to the linearized equation (13.3). The long time solution of the full equations will clearly be affected by nonlinearity. These may either saturate the growth, so that a new solution grows continuously away from the bifurcation point, or may further enhance the growth, taking the state far away from the initial one even for r close to r_c.

The behavior for $r \simeq r_c$ and $U \simeq U_0$ (i.e. "near the bifurcation point") falls into one of a few possibilities, determined by the symmetries of the equations and the signs of a few coefficients. These possibilities are displayed by the *normal forms*, essentially dynamical equations for the amplitudes of the unstable eigenvectors, after suitable transformations, perhaps nonlinear, to put the equations into standard forms. (Since the growth rates of the unstable eigenvectors are small near the bifurcation point the dynamics of the amplitudes are *slow* here: these modes control the evolution and other degrees of freedom follow adiabatically.) These normal forms allow an important connection to be made between the *existence* of solutions and their *stability*.

Stationary bifurcation

There is a single growing eigenvector, with real amplitude $X(t)$. In the linear regime $X(t) \propto e^{\sigma t}$: we want to extend the knowledge of the dynamics into the nonlinear regime. The normal forms are (defining $\varepsilon = r - r_c$ and then the equations are valid for small ε):

1. Transcritical: The nonlinearity appears at the first possible order, i.e. X^2:

$$\dot{X} = \varepsilon X - X^2. \tag{13.10}$$

   Note that this equation may be constructed on the principle of "what else could it be?". If we imagine being able to develop a formal expansion in small ε and X for the effect of the nonlinearity on the growth of X for a specific example, there is in general no reason to expect there *not* to be a term in X^2. For a particular unlucky choice of the equations, the coefficient might happen to be zero, but this is not likely to be robust against a tiny change of the parameters of the equation, and so will not typically occur. If the coefficient is nonzero, by appropriately rescaling X (which, as the amplitude of a linear mode, we are free to do) we can set the coefficient to unity. Certainly there will be higher order terms (e.g. X^3, εX^2), but these do not affect the behavior near enough to the bifurcation point.

   It is now easy to find the stationary solutions to this equation, and to evaluate the stability of these solutions. The solution $X = 0$ is stable for $\varepsilon < 0$, and becomes unstable for $\varepsilon > 0$—this is the linear instability that we started with. A second solution $X = \varepsilon$ intersects this solution at $\varepsilon = 0$ and "exchanges stability" i.e. is unstable for $\varepsilon < 0$ and stable for $\varepsilon > 0$ (figure 13.1, panel a). (Note that time has been rescaled to make the coefficient of εX unity, and the amplitude X has been rescaled to make the coefficient of X^2 unity.)

2. Pitchfork: If there is a symmetry in the equations $X \to -X$ the X^2 nonlinearity must be absent. Typically the next order term is will be present

$$\dot{X} = \varepsilon X \pm X^3. \tag{13.11}$$

   Again the coefficient of the X^3 term may be scaled to unity, but the physical behavior depends on the sign.

   (a) Supercritical (negative sign): the nonlinearity is saturating. For $\varepsilon < 0$ the solution $X = 0$ is stable, and no other (real) solutions exist. For

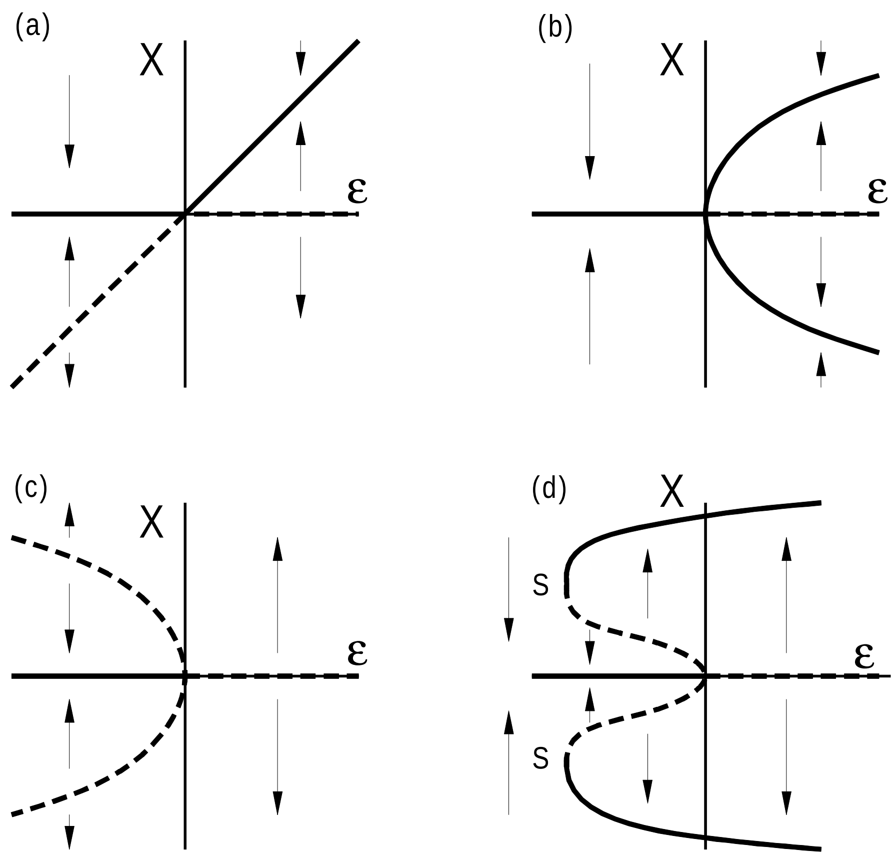

Figure 13.1: Normal forms for stationary bifurcations. Full lines are stable solutions, dashed lines unstable solutions. The arrows show the evolution of $X(t)$ at fixed ε. (a) transcritical; (b) supercritical pitch fork; (c) subcritical pitch fork; (d) subcritical pitch fork with phenomenological quintic stabilizing terms. In (d) the saddle-node bifurcations are denoted by s.

$\varepsilon > 0$ the $X = 0$ solution becomes unstable. Two new solutions develop $X = \pm\varepsilon^{1/2}$. A linear stability analysis about *these* solutions ($X = \varepsilon^{1/2} + \delta X$ etc.) shows them to be stable, (panel b).

(b) Subcritical (positive sign): the nonlinear term is destabilizing. For $\varepsilon < 0$ the $X = 0$ solution is stable, but there are also two unstable solutions $X = (-\varepsilon)^{1/2}$. For $\varepsilon > 0$ the $X = 0$ solution is unstable, and there are no other solutions to the equation (13.11). A small perturbation to $X = 0$ will grow to large values, where presumably further nonlinear terms come in to saturate the growth in a way that is not controlled in the perturbation expansion about $\varepsilon = 0$, $X = 0$, (panel c). Crudely we might anticipate the behavior to be qualitatively given by

$$\dot{X} = \varepsilon X + X^3 - gX^5 \tag{13.12}$$

as shown in panel d. However since the stable solution occurs at $X = O(1)$, there is no reason to expect a truncated power series expansion to be adequate, unless other parameters (e.g. g^{-1}) are small for some reason.

3. Saddle node: The behavior in figure 13.1d displays a further type of stationary bifurcation where two new solutions form from no solution as r increases. This is expressed by the normal form

$$\dot{Y} = \bar{\varepsilon} - Y^2$$

(where $Y = X - X_s$ and $\bar{\varepsilon} = r - r_s$, with r_s, X_s the position of the “nose” in figure 13.1d). This bifurcation does not correspond to the instability of a pre-existing (i.e. at $\bar{\varepsilon} < 0$) solution, but shows how new solutions can develop “far away”.

Hopf bifurcation

The eigenvalue $\sigma = \varepsilon + i\omega$ (with $\varepsilon = r - r_c$ and ω approximately constant for r near r_c) is complex, and the conjugate σ^* is also an eigenvalue. Similarly the eigenvector will be complex. Write the amplitude of the eigenvector with eigenvalue σ as $Z = X + iY = |Z|e^{i\phi}$ (perhaps with nonlinear corrections). There are two normal forms:

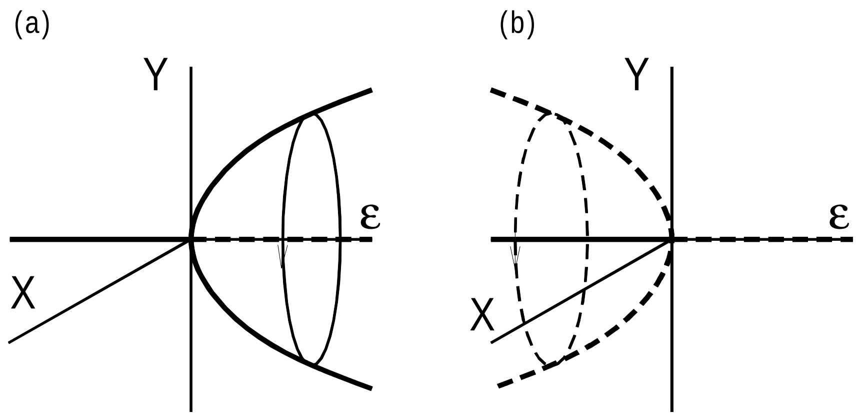

Figure 13.2: Normal forms for Hopf bifurcation: (a) supercritical; (b) subcritical.

1. Supercritical: the nonlinearity is saturating

$$\dot{Z} = (\varepsilon + i\omega) Z - (1 + ib) |Z|^2 Z. \tag{13.13}$$

For $\varepsilon > 0$ the $Z = 0$ solution is unstable, and a new, stable solution develops

$$\begin{array}{rcl} |Z| & = & \varepsilon^{1/2} \\ \dot{\phi} & = & \omega - b\varepsilon \end{array}, \tag{13.14}$$

i.e. with amplitude that grows continuously as $\varepsilon^{1/2}$ and with a frequency that is the Hopf frequency ω with corrections linear in ε. The motion in these scaled coordinates is a circle, and the orbit is described as a limit cycle, (figure 13.2a). In fact it may be shown to all orders in perturbation theory that the motion is conjugate to uniform rotation on a circle, i.e. after suitable smooth transformations

$$\begin{array}{rcl} \left|\dot{Z}\right| & = & |Z| \left[\sigma_R(\varepsilon) - \sum_j a_j(\varepsilon) |Z|^{2j}\right] \\ \dot{\phi} & = & \omega(\varepsilon) + \sum_j b_j(\varepsilon) |Z|^{2j} \end{array}. \tag{13.15}$$

so that the qualitative nature of the new solution is not an artifact of the truncation of the expansion.

2. Subcritical: the nonlinearity is destabilizing

$$\dot{Z} = (\varepsilon + i\omega)\, Z + (1 + ib)\, |Z|^2\, Z. \qquad (13.16)$$

In this case an unstable limit cycle exists for $\varepsilon < 0$. For $\varepsilon > 0$ the solution $Z = 0$ is unstable to a growing limit cycle, but there is no saturated nonlinear solution nearby, (panel b).

Example: continuing the example of the Van der Pol oscillator, we have already derived the equation for the complex amplitude of the unstable eigenvector (see equations 3.10 or 3.22 and equation 3.14). For the scaling introduced in (13.5) this reads (writing $B = Ae^{it}$ for the full amplitude)

$$\dot{B} = \left(\frac{r}{2} + i\right) B - \frac{1}{2}\, |B|^2\, B \qquad (13.17)$$

and is easily put into the standard form (with $\varepsilon = r$) by rescaling the time variable. Note that in this case there is no correction to the frequency at order ε ($b = 0$). Also in the general case it may be necessary to perform a nonlinear transformation on the phase space variables to define the "amplitude" Z so as to arrive at the canonical form of the equation (e.g. motion on a circle rather than an ellipse)..

13.2 Bifurcation from a periodic solution

13.2.1 Linear analysis

This can be considered from two points of view: the bifurcation of a limit cycle flow, or the bifurcation of a fixed point on the Poincaré section map.

Considered as a flow, we have a base solution that is time-periodic

$$U = U_0(\omega_0 t); \quad U_0(\phi + 2\pi) = U_0(\phi) \qquad (13.18)$$

with period $T = 2\pi/\omega_0$. The stability is given by a *Floquet analysis*, which is analogous to (and predates) Bloch theory for the wave function in a periodic potential in quantum mechanics: a solution is sought in the form

$$U = U_0(\omega_0 t) + e^{\sigma t}\, \delta U(\omega_0 t) \qquad (13.19)$$

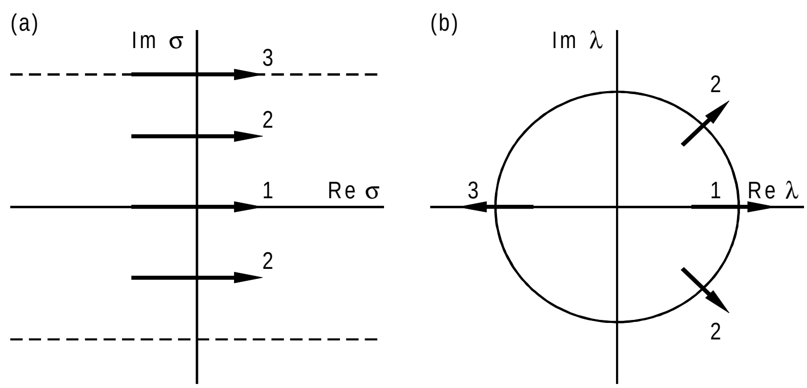

Figure 13.3: Instability of a periodic solution: (a) analysis of the flow showing the behavior of the complex growth rate σ as the control parameter is varied; (b) analysis of the map on the Poincaré section showing the dependence of complex eigenvalue λ as the control parameter is varied. Three possible types of instability $1-3$ are shown.

where δU is periodic with the *same* period as the base solution $\delta U(\phi + 2\pi) = \delta U(\phi)$. The parameter σ is the stability parameter, with $\mathrm{Re}\,\sigma$ giving the growth rate. Again $\omega = \mathrm{Im}\,\sigma$ gives the oscillation frequency of the linear perturbation. Now however, because the exponential multiplies a periodic function we can restrict $\mathrm{Im}\,\sigma$ to lie in the range

$$-\frac{\omega_0}{2} < \mathrm{Im}\,\sigma \le \frac{\omega_0}{2} \tag{13.20}$$

since a frequency ω outside of this range can be folded into this range with a redefinition of δU

$$e^{i\omega t}\delta U = e^{i(\omega - n\omega_0)t}\left[e^{in\omega_0 t}\delta U\right] = e^{i(\omega - n\omega_0)t}\overline{\delta U} \tag{13.21}$$

with $\overline{\delta U}$ another periodic function with period $2\pi/\omega_0$.

There are *three* possible types of behavior (figure 13.3a):

1. a single real eigenvalue crosses the imaginary axis in the complex σ plane;
2. a complex pair of eigenvalues crosses the imaginary axis;
3. a single complex eigenvalue with $\mathrm{Im}\,\sigma = \frac{1}{2}\omega_0$ crosses the imaginary axis, since now σ and σ^* correspond to the same solution by the folding procedure.

Considered as a map on the Poincaré section of dimension $n - 1$ for an n dimensional phase space we have

$$R_{n+1} = F(R_n; r). \tag{13.22}$$

Stability of the fixed point R_f with $R_f = F(R_f; r)$ is given by writing $R = R_f + \delta R$ and then

$$\delta R_{n+1} = K\delta R_n \quad \text{with} \quad K_{ij} = \left.\frac{\partial F^{(i)}}{\partial R^{(j)}}\right|_{R=R_f}. \tag{13.23}$$

The solution is

$$\delta R_n = \sum_\alpha A_\alpha\, u^{(\alpha)}\, (\lambda_\alpha)^n \tag{13.24}$$

with $u^{(\alpha)}$ the eigenvectors and λ_α the eigenvalues of K and A_α amplitudes set by initial conditions. Now the onset of instability corresponds to the largest $|\lambda_\alpha|$ passing through the unit circle. Again there are three possibilities for the way this occurs (figure 13.3b):

1. a real λ passes through the unit circle at $+1$;
2. a complex conjugate pair of λ passes through the unit circle;
3. a real λ passes through the unit circle at -1.

The relationship between the two descriptions is $\lambda = e^{\sigma T}$ with T the base period, and the correspondence of the three possible types of behavior should be apparent.

13.2.2 Nonlinear analysis

In case (1) a single real eigenvalue is involved. A new periodic solution of the same frequency develops at the bifurcation and there are the same possibilities for the bifurcation behavior as for the stationary bifurcation of a fixed point.

In case (2), at the linear level oscillations at a new frequency $\omega_1 = \operatorname{Im}\sigma$ develop. The nonlinear behavior however is complicated. It can be shown that an *invariant circle* analogous to the growth of the limit cycle at a Hopf bifurcation, develops continuously on the Poincaré section near the bifurcation point. An invariant circle is one for which any point on the circle is mapped by the dynamics to another point on the circle This corresponds to the formation of an invariant 2-torus in the flow. However the behavior of the iterations on the circle, or flow on the torus, is complicated because of the possibility of frequency locking between the two frequencies ω_0 and ω_1. This depends sensitively on whether ω_1/ω_0 is a rational or irrational, and, if irrational, how close the irrational ratio is to a rational one. Also the order in the perturbation theory expansion at which locking is detected depends on the order of the rationality, i.e. whether the rational is simple, such as $1/2$ (when the locking is easily captured in low order perturbation theory) or complicated, such as $21/34$. This phenomena was briefly studied in chapter 3, and will be investigated more deeply in chapter 18.

In case (3) the growing solution can be understood by strobing at the base frequency (or equivalently looking at the Poincaré section)

$$\delta U(nT) = e^{(\operatorname{Re}\sigma)nT}\,(-1)^n \tag{13.25}$$

which is a growing perturbation that oscillates in sign. This means that we still have a periodic orbit, but that the period has doubled—a *period doubling* bifurcation. The normal form on the Poincaré section is (assuming a subcritical bifurcation)

$$X_{n+1} = -X_n - \varepsilon X_n + X_n^3 \tag{13.26}$$

so that

$$X_{n+2} \simeq X_n + 2\varepsilon X_n - 2X_n^3. \tag{13.27}$$

For $\varepsilon < 0$ the only fixed point solution of the second iteration $X_{n+2} = X_n$ is $X = 0$; for $\varepsilon > 0$ a new (stable) fixed point of the second iteration develops with $X = \varepsilon^{1/2}$ corresponding to the period doubled solution $X_n = (-1)^n \varepsilon^{1/2}$.

13.3 Further Bifurcations

We could now imagine asking questions about the bifurcations of, say, a quasiperiodic solution formed from an irrational ω_0 and ω_1. Being physicists, we might imagine that the arguments can be repeated, so that successive bifurcations to states with more and more irrational frequencies occur. (Bifurcations such as case 1 in section 13.2, may occur as well, but do not increase the complexity of the dynamics.) In early versions of Landau and Lifshitz's *Fluid Dynamics* text, this was proposed as a schematic of the way in which the spatially and temporally disordered flow of strongly driven fluids known as turbulence might develop. In fact bifurcations of the invariant torus are much more subtle, and cannot be as simple categorized. Many bifurcations of the quasiperiodic state are possible—for example to chaos, to a frequency locked state, or to other quasiperiodic states. Various aspects of this topic are taken up again in chapters 18, 19, 19, and 20.

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Bibliography

[1] J.D. Crawford, Rev. Mod. Phys. **63**, 991 (1991)