Chapter 11

Reconstructing the attractor

Many of the diagnostics for chaos we have looked at depend on knowing the phase space dynamics. At first sight this seems to limit their application to numerical systems. However it turns out to be possible to *reconstruct* the phase space attractor, in a way that preserves topological diagnostics such as Lyapunov exponents and dimensions, from measurements of a *single* dynamical variable. This renders experimental systems, at least with attractors of moderately low dimension, accessible to the diagnostic tools. This idea was introduced by Packard et al. [1] and the mathematical proof of the validity of this idea is due to Takens [2].

The scheme is the following. From the measurement of a single variable $x(t)$ form the m-dimensional vector $\vec{\xi}(t)$ given by $x(t)$ at successive time delays $m\tau$, where τ is a parameter to be chosen:

$$\vec{\xi}(t) = (x(t), x(t+\tau), x(t+2\tau), \ldots x(t+(m-1)\tau)) .$$

The vector $\vec{\xi}(t)$ defines a trajectory in a m-dimensional space. Then if $m \geq 2D_C+1$ with D_C the capacity dimension of the attractor, the trajectory gives a faithful reconstruction of the flow in the physical phase space. Technically this means that distances $\delta\vec{\xi}$ in the reconstruction space and the corresponding distances in the physical phase space have a ratio that is uniformly bounded and uniformly bounded away from zero. Now we can do the same sort of tricks we do on numerical data on real experimental data, e.g. Poincaré sections, return maps, dimension estimates etc.

In practice D_C is not known *a priori*, and so usually increasing values of m are taken until dimension estimates saturate. The factor of 2 multiplying D_C in the necessary "embedding dimension" is to eliminate spurious crossings. For example, a circle corresponding to a limit cycle ($D_C = 1$) may be reconstructed as a figure-8 in a two dimensional embedding space, and this does not satisfy the requirements of distance bounds in the faithfulness condition. In three dimensions however, typically the line will not cross itself. Often a finite number of crossing points is not considered a problem, and $m \geq D_C + 1$ is expected to be sufficient.

The size of τ does not enter the mathematical proof—in principal any value may be used. In practice often there is some range of τ that gives the best reconstruction in the presence of experimental limitations on the data. For too small τ the reconstruction is swamped by lack of resolution in the data or by noise; for too large τ there are often many crossings of the trajectory in the range $D_C+1 \leq m \leq 2D_C+1$ usually used. Often a value of τ around the correlation time (the time δt over which the correlation function $< x(t)\, x(t+\delta t) >_t$ decays appreciably) is used, although the choice is typically an art rather than a science.

In the demonstrations the reconstruction method is applied to numerically generated data for the Lorenz and Rossler models.

January 30, 2000

Bibliography

[1] N. Packard, J. Crutchfield, D. Farmer, and R. Shaw, Phys. Rev. Lett. **45,** 712 (1980).

[2] F. Takens, Lecture Notes in Mathematics **898** (Springer, 1981).