Physics 161: Homework 7: Frequency Locking

(February 16, 2000; due February 23)

Reading:

The pedagogical articles by Bak et al. (Physica Scripta **T9**, 50 (1985)), and by Glazier and Libchaber (IEEE Transactions on Circuits and Systems, **35** 790 (1988)) which reviews experimental tests, are well worth looking at. (Both are in Vol.2 of the reprint folder in Millikan).

Problems:

1. **The 1D Circle Map:** The one-dimensional circle map is $x_{n+1} = f(x_n)$ with

$$f(x) = x + \Omega - \frac{K}{2\pi}\sin(2\pi x) \quad \mathrm{mod} \quad 1.$$

   (a) By considering fixed points of the map f^m for $m = 1$ and $m = 2$ show that the range in Ω for locked tongues near $\Omega = 0, \frac{1}{2}, 1$ grows with small K as K^1, K^2, K^1 respectively. You might want to use the *1dmap* program to see what the issues are.

   (b) What is the nature of the bifurcation at the edges of the ranges?

   (c) For Ω slightly outside the locking range the phase becomes almost stationary $x_{n+1} \simeq x_n$ for a large number of iterations. How does this number vary with $\Omega - \Omega_L$ with Ω_L the edge of the locking band.

   (d) Use either *1dmap* or *2dmap* (or your own program) to sketch the width of the locked tongues near $\Omega = 0, \frac{1}{2}, 1$ as K varies from 0 to 1. (Note: one way is to look at the two dimensional version of the circle map using *2dmap* setting $b = 0$ to get the strong damping limit which is the $1d$ circle map. The $c \to \Omega$ and $a \to K$. It is easy to set the value of a and then step through values of c with the change buttons, clearing the plot if necessary to see if the orbit is locked or not.)

2. **Fireflies:** As you might know (e.g. if you have lived on the East Coast) fireflies tend to synchronize their flashes. This question is related to a simple model of Ermentrout and Rinzel that discusses this phenomenon. I adapted this question from ideas in the nice book *Nonlinear Dynamics and Chaos* by Strogatz.

   Suppose each firefly emits a flash at phase 0 (or $2n\pi$) of an internal oscillator. Let the observed flashes occur at integer times n (i.e. define the time scale to be the period of the observed flashes). Consider a particular firefly that would, undisturbed, emit flashes at a frequency slightly higher the other fireflies, so that on its own its flashes would occur at a period slightly less than 1. If we sample its phase variable at each "external" flash, it would evolve independently as

$$\theta_{n+1} = \theta_n + \varepsilon. \tag{1}$$

   However the firefly tends to adjust its phase depending on the observed flashes. We could model this with a periodic function f which tends to move the phase towards zero (or 2π etc.) at the external flashes

$$\theta_{n+1} = \theta_n + \varepsilon - f(\theta_n). \tag{2}$$

For example we could consider the case

$$f(\theta) = \begin{cases} A\theta & \text{for} \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \\ A(\pi - \theta) & \text{for} \quad \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2} \end{cases} \tag{3}$$

(with the periodic extensions) where A is some constant.

(a) For fixed A, for what is the range of the frequency mismatch ε for which the flashes are frequency locked to the external flashes.

(b) Although the flashes are frequency locked in this band of ε, they do not occur at exactly the same time as the external flashes. The phase is locked to the external flashes but is not zero there. Calculate the time delay or advance as a function of ε in the locked regime.

(c) For ε just outside the locking band, there will be long times over which the flashes are almost locked. Calculate how this time (e.g. defined as the time the phase takes to drift by an amount of order π) depends on small ε near the edge of the locking band.