Physics 161: Homework 6

(February 9, 2000; due February 16)

Problems

1. **One dimensional map with a power law "maximum"**: Although the quadratic map is the most natural, we can also consider maps of the unit interval with a different form of "maximum" at $x = \frac{1}{2}$, i.e. $x_{n+1} = \bar{f}(x_{n)}$ with

$$\bar{f}(x) = \bar{a}\left[\left(\frac{1}{2}\right)^p - \left|x - \frac{1}{2}\right|^p\right] \quad . \tag{1}$$

You can convince yourself that functional compositions of the map $\bar{f}^n(x)$ also have such a maximum. The "universality class" of the subharmonic cascade route to chaos depends on the power p. The limit $p = 1 + \epsilon$ with $\epsilon \to 0$ is convenient for analytic investigation, since for $\epsilon = 0$ the map reduces to the tent map where the behavior is much simpler, with chaos developing at $\bar{a} = 1$. In the following it is convenient to measure the x coordinate from $\frac{1}{2}$, and, because for small ϵ all the interesting dynamics (fixed points, chaos etc.) is limited to small $\left|x - \frac{1}{2}\right|$, it is convenient to rescale:

$$X = 2\left(x - \frac{1}{2}\right) / \left(\frac{\bar{a}}{2^\epsilon} - 1\right) \tag{2}$$

$$a = \bar{a}\left[\frac{1}{2}\left(\frac{\bar{a}}{2^\epsilon} - 1\right)\right]^\epsilon \quad .$$

This leads to the map $X_{n+1} = f(X_n)$ with

$$f(X) = 1 - a\,|X|^{1+\epsilon} \quad . \tag{3}$$

The maximum of f is now at $X = 0$. In addition chaos develops near $a = 1$, and so we will introduce the variable $\phi = a - 1$.

   (a) Investigate qualitatively the bifurcation diagram and Lyapunov exponent as a function of $\bar{a}$ for the maps with power law maxima $p = 1.2$ and $p = 1.5$ using *1dmap* or your own program. (Note that the parameters a, b of *1dmap* are related to the parameters of Eq.(1) by $\bar{a} \to a$, $p - 1 \to b$.) Now we look at the onset of chaos analytically.

   (b) Show that $f^2(X)$ defined as $f(f(X))$ is given by

$$f^2(X) \simeq (1 - a) + a^2(1 + \epsilon)\,|X|^{1+\epsilon} + \ldots \tag{4}$$

   with the coefficients of all higher powers of $|X|$ proportional to ϵ so that these terms are *small* for small ϵ. From now on ignore these higher order terms.

   (c) Calculate properties near the accumulation point of subharmonic bifurcations $a \to a^*$ by constructing $-sf(f(-X/s))$ with s a scale factor and requiring this to be in the form $1 - a'\,|X|^{1+\epsilon}$. Show that this leads to $s = 1/(a - 1)$ and

$$a' = a^2(1 + \epsilon)(a - 1)^\epsilon. \tag{5}$$

   If $f(X)$ with the parameter a gives a 2^n cycle then with these relationships $f(X)$ with the parameter a' gives a 2^{n-1} cycle i.e. Eq.(5) takes us successively through the parameters a_n for 2^n cycles for decreasing n.

(d) The accumulation point a^* is given by setting $a' = a = a^*$. For small ϵ show that $a^* = 1 + \phi^*$ with ϕ^* approximately satisfying

$$\epsilon + \phi^* + \epsilon \ln \phi^* = 0 \tag{6}$$

(ignoring terms of order ε^2 with logarithmic corrections). Solving this equation for $\phi^*(\varepsilon)$ then gives us the universal constant α which is the scale factor s evaluated at $a = a^*$, i.e. $\alpha = 1/\phi^*$.

(e) From the growth of small deviations of a from a^* given by linearizing Eq.(5) about $a = a^*$(i.e. linearize in small $\delta a = a - a^*$) we can calculate δ. Show that this gives

$$\delta = 2 + \frac{\epsilon a^*}{a^* - 1} \simeq 2 + \frac{1}{\ln(1/\phi^*) - 1} \tag{7}$$

where the last expression is good for small ϕ^*.

(f) Sketch the approach of a^* to 1 and δ to 2 as $\epsilon \to 0$. Note the nonanalytic behavior at small ε.

(g) By looking at the first few 2^n cycles numerically find crude approximations of α and δ for the maps with $p = 1.2$ and $p = 1.5$ and compare with the predictions of the expressions you have calculated. You can use *1dmap* on the website or your own program. (If you are using *1dmap* note that clicking on a running plot will allow you to read off values of points on the plots.)

2. **Correlation function for the period doubling route to chaos in the quadratic map:** The *correlation function* for the iterations of a map f describes the decay of correlations with map iteration number m. It is defined as

$$C(m; f) =< (x_{n+m} - \bar{x})(x_n - \bar{x}) >_n \tag{8}$$

where $<>_n$ denotes the average over iterations n and $\bar{x} =< x_n >_n$ is the mean of x. $C(m)$ is related to the power spectrum via a Fourier transform.

It can be shown that effect of the transformation T, the functional composition plus rescaling operation of the renormalization group approach, is

$$C(m; T[f]) = \alpha^2 C(2m; f) \tag{9}$$

where the factor of α^2 is simply the rescaling of the two powers of x in C and there is a factor of 2 multiplying the iteration number coming from the composition. Hence we can find $C(m; T^q[f])$ for any q. Use this with appropriate choices of q to show:

(a) For a fixed type of orbit (e.g. at band merging) the dependence on m for a fixed value of the control parameter R of the map is

$$C(m, R)/C(0, R) = \Phi(m/\tau(R)) \tag{10}$$

with Φ a universal function, and the "correlation time" τ varying as a power law

$$\tau \propto |R - R_\infty|^{p_1} \tag{11}$$

with R_∞ the value of R for the onset of chaos. Find the exponent p_1.

(b) For $R = R_\infty$ show that the correlations decay as a power law

$$C(m, R_\infty) \sim m^{-p_2} \tag{12}$$

for large m and find the exponent p_2.

Useful analogies can be drawn between the scaling of the correlation function (in time) at the onset of chaos via the period doubling cascade and the scaling of the correlation functions (in space) at second order thermodynamic phase transitions.