

# Physics 161: Homework 4

(January 26, 2000; due February 2)

## Reading

Please read the [instructions](#) for using the dimension calculation part of the *2dmap* applet. The second question is a straightforward exercise to encourage you to understand the class notes on the “two scale factor Cantor set”. You can also read the section of the paper by Halsey et al., Phys. Rev. **A33**, 1141 (1986) (also in the reserve folder) on this topic.

## Problems

1. Construct a fractal similar to the  $\frac{1}{3}$  Cantor set, but instead remove the middle  $\frac{1}{2}$  instead of the middle  $\frac{1}{3}$  from each interval. Show that the capacity dimension is  $\frac{1}{2}$ .
2. Calculate the  $f(\alpha)$  curve and the generalized dimensions  $D_q$  for the two scale factor Cantor set formed using length rescaling parameters  $l_1 = \frac{1}{4}$ ,  $l_2 = \frac{1}{3}$ , and probabilities  $p_1 = \frac{3}{5}$  and  $p_2 = \frac{2}{5}$  (see class notes or the section from the paper by Halsey et al., Phys. Rev. **A33**, 1141 (1986) for notation). Calculate the intercepts on the  $f = 0$  axis and the maximum value of  $f$ .
3. The **bakers' map** is defined as the transformation of the unit square  $0 \leq x \leq 1$ ,  $0 \leq y \leq 1$  :

$$\begin{aligned}
 x_{n+1} &= \begin{cases} \lambda_a x_n & \text{if } y_n < \alpha \\ (1 - \lambda_b) + \lambda_b x_n & \text{if } y_n > \alpha \end{cases} \\
 y_{n+1} &= \begin{cases} y_n / \alpha & \text{if } y_n < \alpha \\ (y_n - \alpha) / \beta & \text{if } y_n > \alpha \end{cases}
 \end{aligned}$$

where  $\beta = 1 - \alpha$  and  $\lambda_a + \lambda_b \leq 1$

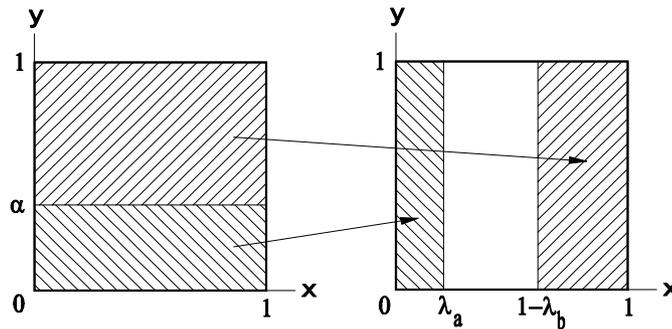


Figure 1: Bakers Map

Since the attractor is uniform in the  $y$  direction, the dimensions are given by  $D_q = 1 + \hat{D}_q$  with  $\hat{D}_q$  the dimensions of the intersection of the set with a horizontal line, which is given by the two scale factor Cantor set. Using the results derived from the “partition function” formulation for the two scale factor Cantor set or otherwise (see class notes) shows that the generalized dimensions are given by the transcendental equation

$$\alpha^q \lambda_a^{(1-q)\hat{D}_q} + \beta^q \lambda_b^{(1-q)\hat{D}_q} = 1$$

- (a) By expanding in small  $q - 1$  show that the information density is

$$D_1 = 1 + \frac{\alpha \ln(1/\alpha) + \beta \ln(1/\beta)}{\alpha \ln(1/\lambda_a) + \beta \ln(1/\lambda_b)}$$

- (b) Show that the Lyapunov dimension is equal to  $D_1$ , illustrating the “Kaplan-Yorke conjecture”.
- (c) See how well you can verify these analytic results using the *2dmap* applet on the website for two different choices of parameters.
- (d) Choose a set of parameters for the bakers’ map so that the capacity  $D_C$  is 2 but  $D_q \leq 2$  for  $q \geq 1$ . Look at the bakers’ map for this choice of parameters with *2dmap*: describe the attractor that you find, and find  $D_C$  and  $D_1$  numerically. Are they what you expected?

4. **The Kaplan-Yorke map** (see problem 2 of homework set 3)

- (a) What is the Lyapunov dimension for  $a = 3$  and (i)  $b = \frac{1}{2}$  (ii)  $b = \frac{1}{4}$ ? Does this explain the different appearance of the attractor for these two values?
- (b) Use the *2dmap* applet to find  $D_q$  for  $q = 0, 1$  and  $2$  and compare with the Lyapunov dimension. Use various choices of the number of iterations and the number of subdivisions in the box counting algorithm “*Div*” to make sure you are getting (reasonably) converged answers.
- (c) Take a subregion of the  $x - y$  plot and recalculate the dimensions for the attractor in this restricted region.