Physics 161: Homework 2

(12 January 2000, due 19 January)

Further Reading:

The paper by R.M. May, *Simple mathematical models with very complicated dynamics*, Nature **261** 459 (1976), is a readable introduction to properties of one dimensional maps.

Problems

1. **The Shift Map**: I will define the Shift Map as

$$x_{n+1} = a\, x_n \mod 1 \tag{1}$$

For $a = 2$ it is known as the "Bernoulli Shift Map" and has particularly simple properties.

(a) Calculate the Lyapunov exponent for general a. (Note: the discontinuity at $y = 1/2$ does not affect the Lyapunov exponent.) For $a = 2$ if you start with two initial conditions separated by 10^{-6} estimate how many iterations it will take for the iterated points to differ by about 0.5 (e.g. to fall on opposite sides of $x = 0.5$).

(b) Carefully verify for $a = 2$ that the measure $\rho(x) = 1$ over the interval $0 \le x \le 1$ is indeed an invariant measure, i.e. is a fixed point of the "Frobenius-Peron" equation so that

$$\rho(y) = \int_0^1 \delta(y - F(x))\, \rho(x)\, dx \quad . \tag{2}$$

To understand the dynamics at $a = 2$ further it is useful to write the initial point in "binal" (c.f. decimal) representation $x_0 = 0.a_1 a_2 \ldots$ equivalent to

$$x_0 = \sum_{\nu=1}^{\infty} a_\nu 2^{-\nu} \tag{3}$$

with each a_ν either 0 or 1. The action of the map is then simply to shift the "binal" point to the right and discard the first digit

$$x_1 = F(x_0) = 0.a_2 a_3 \ldots \tag{4}$$

From this idea many properties of the dynamics generated by the Bernoulli shift map are easily predicted.

(c) Use the binal representation to argue that the answer to the question whether the mth iteration from an unknown initial condition is greater or less than $\frac{1}{2}$ is as random as a coin toss (e.g. no matter how long a sequence of 0's and 1's you measure, you cannot from the sequence alone predict whether the next answer will be 0 or 1).

(d) Show that an initial condition of $x = 0.2$ generates a period 4 orbit. (In fact any initial value x_0 that is rational will lead to a periodic orbit, since the sequence of a_ν in the binal representation of x_0 is then periodic. An irrational x_0 will lead to chaotic orbit. Since most x_0 in the unit interval are irrational, most initial conditions will lead to chaotic orbits—but ones you are likely to write down will be rational and lead to periodic orbits!)

(e) Verify these ideas numerically, e.g. with the *1DMap* applet used in the demonstrations. (Because of the difficulty that rational initial conditions will lead to periodic orbits for $a = 2$, it is better to use a value very near this e.g. $a = 1.99999$ to study the properties of the map numerically.)

2. **The quadratic map and the tent map**. The quadratic map

$$x_{n+1} = F(x_n) = a\, x_n(1 - x_n) \tag{5}$$

has played an important role in developing the understanding of chaos. The value $a = 4$ is the "most chaotic"—the iterations fill the whole of the unit interval, and for larger values of a the iterations diverge to infinity. It turns out that *for this particular value of* a the properties of the quadratic map can be understood analytically through a transformation that changes the map to the simpler tent map.

(a) Investigate the behavior of the map Eq.(5) at $a = 4$ numerically, e.g. using the *1DMap* applet of the demonstrations. Describe the properties of the dynamics at this value of a.

(b) Show that the transformation

$$x = \sin^2(\pi y/2) \tag{6}$$

can be used to convert this equation into the tent map

$$y_{n+1} = \begin{cases} 2y_n & \text{for} \quad y_n \le \frac{1}{2} \\ 2(1 - y_n) & \text{for} \quad y_n > \frac{1}{2} \end{cases} \tag{7}$$

(c) What is the Lyapunov exponent for this tent map?

(d) The dynamics of this tent map is equivalent to the Bernoulli shift map: if you iterate this tent map, from most initial conditions you will find a chaotic sequence with a uniform probability density $\rho(y) = 1$ of points in the interval $0 < y < 1$. Use the rule for transforming probability densities

$$\bar{\rho}(x) = \rho(y) \left| \frac{dy}{dx} \right| \tag{8}$$

to calculate the probability density $\bar{\rho}(x)$ for the original quadratic map. Is this consistent with what you saw numerically?

(e) Calculate the Lyapunov exponent for the given quadratic map by calculating the average:

$$\lambda = \int_0^1 dx\ \bar{\rho}(x)\ \log \left| \frac{dF}{dx} \right|. \tag{9}$$

You can transform to the y variable and use

$$\frac{2}{\pi} \int_0^{\pi/2} \ln(\cos x) dx = -\ln 2. \tag{10}$$

What do you notice about λ compared with part 2c? Diagnostic properties of chaos that are invariant under coordinate transformations are conceptually important tools (c.f. the power spectrum which does depend on such transformations) since they allow the possibility of identifying dynamical systems without having to use exactly the same coordinate systems.