Collective Effects

in

Equilibrium and Nonequilibrium Physics

Website: http://cncs.bnu.edu.cn/mccross/Course/

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Today's Lecture: Hydrodynamics

- Systematic equations for the time evolution of systems near equilibrium
- Collective dynamics at low frequencies and long wavelengths of conserved quantities and broken symmetry variables
- Captures essential physics of new phases (Goldstone modes, etc.)

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- Collective dynamics at low frequencies and long wavelengths of conserved quantities and broken symmetry variables
- Captures essential physics of new phases (Goldstone modes, etc.)
- Outline
  - ◇ Idea: two coupled systems
  - ◇ Continuum systems
  - ◇ Applications
    - ⋆ Spin wave hydrodynamics
    - ⋆ Equations of fluid dynamics and heat flow
  - ◇ Equilibrium, near equilibrium, and far from equilibrium

Equilibrium

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$$dS = \frac{1}{T}dE + \frac{P}{T}dV - \frac{\mu}{T}dN + \sum_i \frac{X_i}{T}dx_i$$

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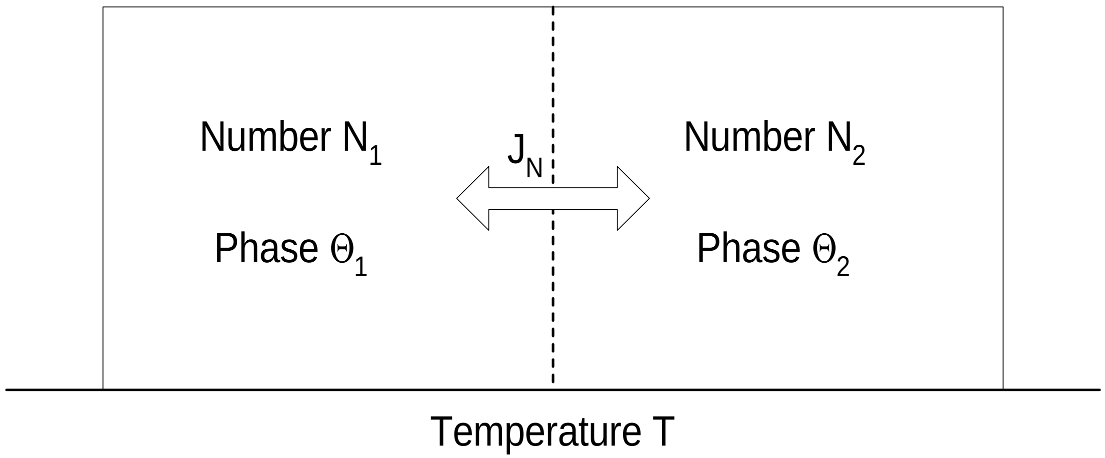
- In the maximum entropy (microcanonical) or minimum free energy (canonical) state the conjugate fields are zero

$$X_i = 0$$

and there is no dynamics.

Example: Josephson Junction



$$TdS = \cdots - \Delta\mu dN + \Phi\Delta\Theta$$

with

$$\Phi = -dE_J/d\Delta\Theta$$

In the minimum free energy state $\Delta\mu = \Phi = 0$ and there is no phase dynamics or superflow

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$\gamma_{ij} = \gamma_{ji}$	for x_i, x_j same time reversal signature	(e.g. N and T)
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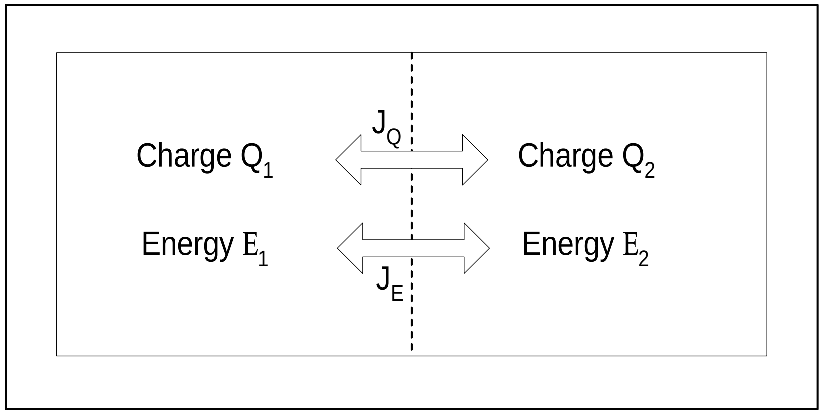
- Second law of thermodynamics places constraints on the matrix elements, e.g. $\gamma_{ii} \geq 0$.
- Kinetic matrix γ_{ij} can be related to correlation matrix $\langle \dot{x}_i(0)\dot{x}_j(t)\rangle$ via the fluctuation dissipation theorem

Example: Thermoelectric Effect



Two parts of an isolated system in contact via the exchange of energy E and charge Q.

- Thermodynamic identity

$$T\,dS = -(\Delta T/T)\,dE - \Delta\Phi\,dQ$$

where $\Delta\Phi$ is the voltage difference $\Phi_2 - \Phi_1$ and $\Delta T = T_2 - T_1$.

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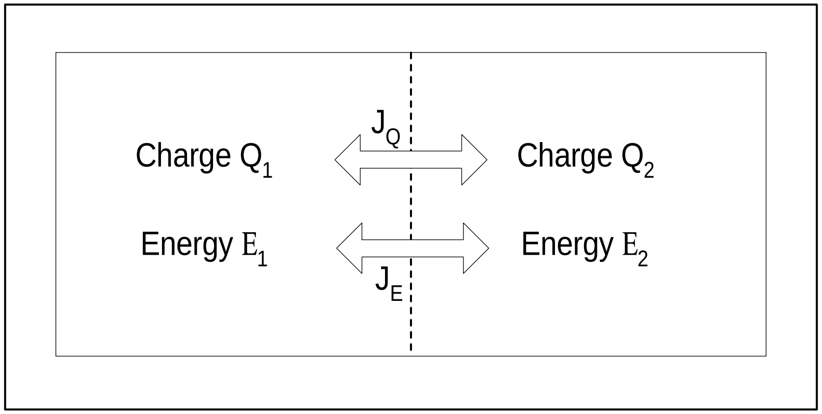

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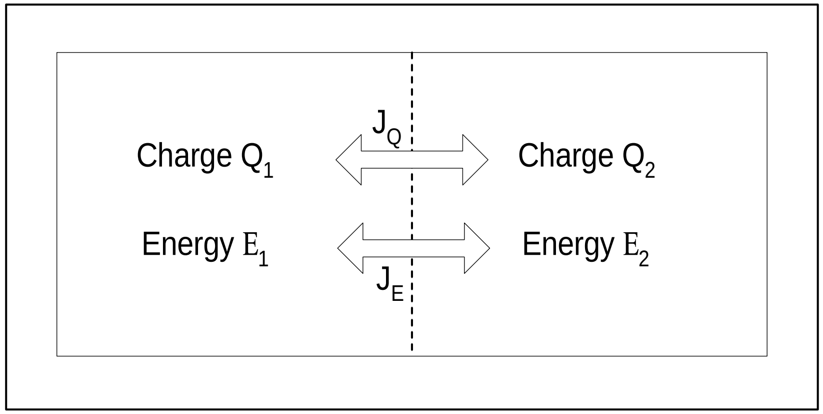

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- Equilibrium is given by the equality of temperature and electric potential, $\Delta T = \Delta\Phi = 0$.

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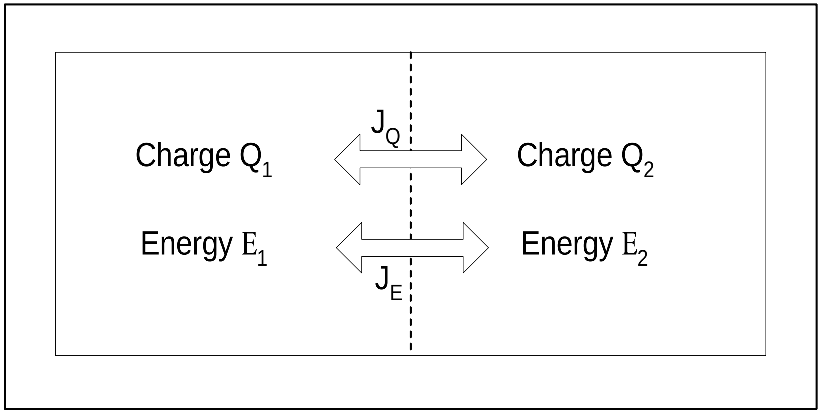

Relaxation of small perturbations from equilibrium is described by the equations for the electric current I and energy (heat) current H

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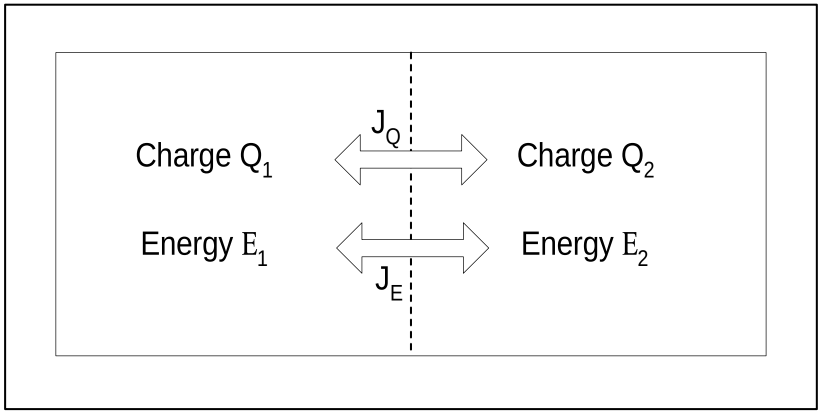

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We would like to learn something about the coefficients $\boldsymbol{\gamma}$

Onsager Symmetry Relations

- Derivation of the Onsager relationships depends on the relationship between fluctuations and dissipation.
- First review fluctuations from a thermodynamic point of view.

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- For macroscopic variables the fluctuations will be small, and we can expand the entropy about its maximum. For ease of notation redefine x_i so that its mean is zero.

$$S \approx S_0 - \frac{1}{2} \sum_{i,j=1}^{n} \beta_{ij} x_i x_j$$

and then

$$p(\{x_i\}) = A \exp\left[-\frac{1}{2k_B} \sum_{ij} \beta_{ij} x_i x_j \right]$$

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- The conjugate force is given by

$$\frac{X_i}{T} = (\partial S/\partial x_i) = -\sum_j \beta_{ij} x_j$$

Equal Time Correlations

- The equal time correlations are

$$\langle x_i x_j \rangle = k_B (\beta^{-1})_{ij}$$

$$\langle X_i X_j \rangle = k_B T^2 \beta_{ij}$$

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- The last result will be important below
- The same results would be obtained by considering the free energy in the canonical ensemble, etc.

Correlation Functions: General Properties

The *correlation function* tells us how the fluctuations decay in time (putting $\langle x_i \rangle = 0$ again)

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- Typically we expect the deviations from the mean to become uncorrelated at long times

$$C_{ij}(\tau \to \pm\infty) \to 0.$$

- By *shifting* the time coordinate we can relate the correlation function for negative times to the values for positive times

$$\begin{aligned} C_{ij}(-\tau) &= \langle x_i(t) x_j(t-\tau)\rangle \quad \text{definition} \\ &= \langle x_i(t+\tau) x_j(t)\rangle \quad \text{add } \tau \text{ to } t \end{aligned}$$

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From the fluctuation-dissipation theorem we can expect that this gives symmetry results for the kinetic matrix (dissipation)

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- Using $\langle x_i X_k \rangle \propto \delta_{ik}$ this gives

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- Another way to get this result is from the fluctuation dissipation theorem proved in lecture 5. From those results it can be shown

$$\gamma_{ij} = \frac{1}{k_B T} \int_{-\infty}^{0} \langle \dot{x}_i(t) \dot{x}_j(t + \tau) \rangle d\tau$$

  and the Onsager relation follows.

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- Positive entropy production for any X_i places *constraints* on the $\gamma^{(e)}, \gamma^{(o)}$ matrices

$$\gamma_{ii}^{(e)} \geq 0$$

$$\gamma_{ij}^{(e)} \leq \sqrt{\gamma_{ii}^{(e)} \gamma_{jj}^{(e)}}$$

  and similar results for $\boldsymbol{\gamma}^{(o)}$

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- *Seebeck effect:* voltage from a temperature difference with no current

$$\Delta\Phi = -\left(\frac{1}{T}\frac{\gamma_{EQ}}{\gamma_{QQ}}\right)\Delta T$$

Thermoelectric Effect: Results

$$I = \dot{Q} = -\gamma_{QQ}\Delta\Phi - \gamma_{EQ}\Delta T/T$$

$$H = \dot{E} = -\gamma_{EQ}\Delta\Phi - \gamma_{EE}\Delta T/T$$

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- Second law requires $\gamma_{EQ} \leq \sqrt{\gamma_{QQ}\gamma_{EE}}$ so that the electrical and thermal conductances limit the magnitude of the thermoelectric effects.

Continuum Systems

Thermodynamic identity

$$Tds = d\varepsilon - \mu dn + \sum_i X_i d\xi_i$$

or in terms of the free energy

$$df = -sdT + \mu dn - \sum_i X_i d\xi_i$$

with s, ε, ξ_i the corresponding densities of conserved quantities or gradients of angle variables, e.g. for the superfluid

$$Tds = d\varepsilon - \mu dn + \mathbf{j}_s \cdot d\mathbf{v}_s$$

with $\mathbf{v}_s = (\hbar/m)\nabla\Theta$ and $\mathbf{j}_s = n_s(\hbar/m)\nabla\Theta$ the supercurrent.

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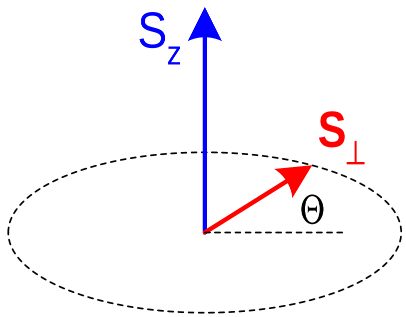
- Positive entropy production places *constraints* on the coefficients $\boldsymbol{\gamma}$

Examples of Applications

1. Hydrodynamic theory of spin waves
2. Heat and mass flow in a fluid

Hydrodynamics Theory of Spin Waves



Thermodynamic identity

$$T ds = d\varepsilon - \mu_z ds_z - \mathbf{\Phi} \cdot d(\nabla\Theta) \text{ with } \mathbf{\Phi} = K\nabla\Theta$$

Phase dynamics

$$\dot{\Theta} = \mu_z + h_z^d \text{ with } h_z^d = 0 \text{ in uniform state}$$

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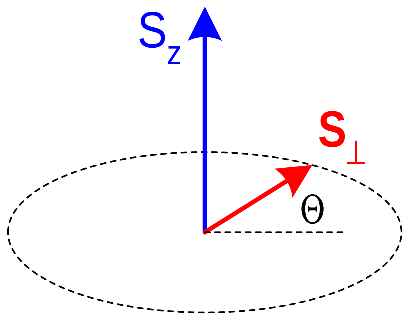

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- Form time derivative of entropy density, using conservation laws and phase dynamics

$$\frac{ds}{dt} = -\frac{1}{T}\boldsymbol{\nabla}\cdot\mathbf{j}^{\varepsilon} + \frac{\mu_z}{T}\boldsymbol{\nabla}\cdot\mathbf{j}^{s_z} - \frac{\boldsymbol{\Phi}}{T}\cdot\boldsymbol{\nabla}(\mu_z + h_z^d)$$

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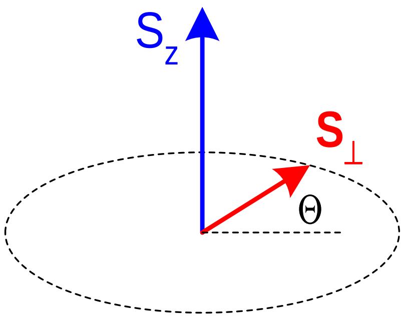

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- Entropy production equation

$$\frac{ds}{dt} = -\nabla \cdot \mathbf{j}^s + R \qquad \text{with} \qquad R \geq 0$$

with the entropy current and production

$$\mathbf{j}^s = T^{-1}(\mathbf{j}^{\varepsilon} - \mu_z \mathbf{j}^{s_z} + \mathbf{\Phi} h_z^d)$$

$$RT = -T^{-1}\mathbf{j}^s \cdot \nabla T - (\mathbf{j}^{s_z} + \Phi) \cdot \nabla \mu_z + h_z^d \, \nabla \cdot \mathbf{\Phi}$$

Dynamics

Entropy Production

$$RT = -T^{-1}\mathbf{j}^{s}\cdot\boldsymbol{\nabla}T - (\mathbf{j}^{s_z}+\Phi)\cdot\boldsymbol{\nabla}\mu_z + h_z^d\,\boldsymbol{\nabla}\cdot\boldsymbol{\Phi}$$

Dynamics

Entropy Production

$$RT = -T^{-1}\mathbf{j}^{s} \cdot \nabla T - (\mathbf{j}^{s_z} + \Phi)\cdot\nabla\mu_z + h_z^d \, \nabla \cdot \mathbf{\Phi}$$

Expand dissipative parts of $\mathbf{j}^{s}$, $\mathbf{j}^{s_z}$, h_z^d in gradients of conjugate fields $\nabla T, \nabla\mu_z, \nabla \cdot \mathbf{\Phi}$

$$\mathbf{j}^{s_z} = -\Phi - D\nabla\mu_z$$

$$h_z^d = \zeta\nabla \cdot \mathbf{\Phi}$$

$$\mathbf{j}^{s} = -T^{-1}K\nabla T$$

(Off-diagonal terms are absent by mirror and time symmetries)

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For positive entropy production D, ζ, K must be positive

Spin Waves

Coupled S_z, Θ equations

$$\dot{S}_z = K\nabla^2\Theta + \chi^{-1}D\nabla^2 S_z$$

$$\dot{\Theta} = \chi^{-1}S_z + K\zeta\nabla^2\Theta$$

Spin Waves

Coupled S_z, Θ equations

$$\dot{S}_z = K\nabla^2\Theta + \chi^{-1}D\nabla^2 S_z$$
$$\dot{\Theta} = \chi^{-1}S_z + K\zeta\nabla^2\Theta$$

The dispersion relation now gives a *damped* wave

$$\omega = \pm ck - \tfrac{1}{2}i\gamma k^2 + O(k^4)$$

with $c = \sqrt{K/\chi}$ and $\gamma = \chi^{-1}D + K\zeta$

Equations of Fluid Motion and Heat Transfer

Thermodynamic identity (ε, s are per mass)

$$d\varepsilon = T\,ds + \frac{p}{\rho^2}d\rho + \mathbf{v}\cdot d\mathbf{g}$$

Mass conservation (LL1.2)

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{g} = 0 \qquad \text{with} \qquad \mathbf{g} = \rho\mathbf{v}$$

Momentum conservation (LL15.1)

$$\frac{\partial(\rho\mathbf{v})}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{\Pi} = 0 \qquad \text{or} \qquad \frac{\partial(\rho v_i)}{\partial t} + \nabla_j \Pi_{ij} = 0$$

with (LL15.3)

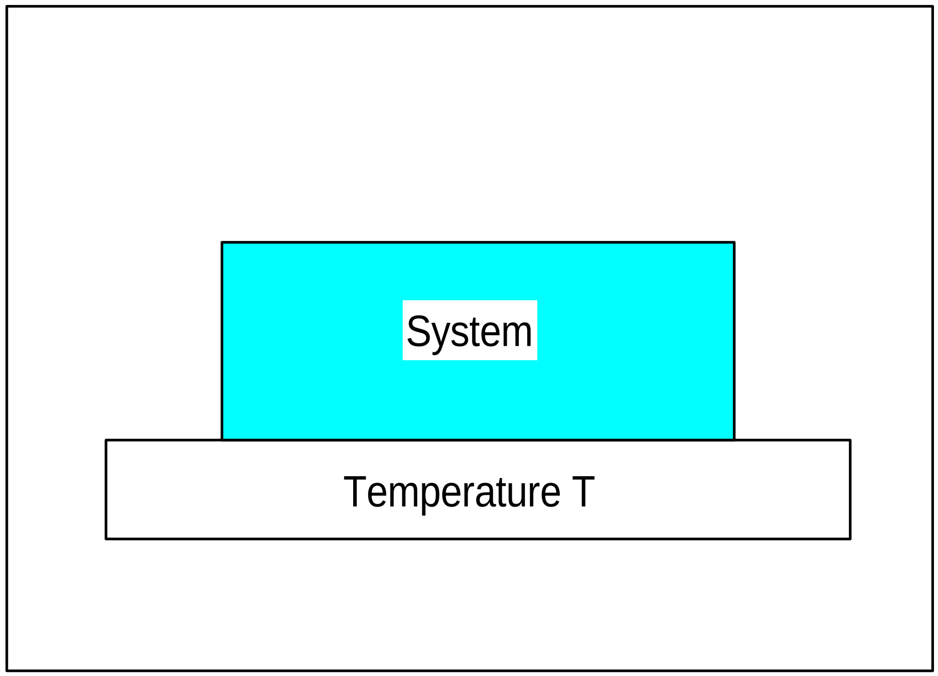
$$\Pi_{ij} = p\delta_{ij} + \rho v_i v_j - \eta\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial v_i}{\partial x_i}\right) - \zeta\delta_{ij}\frac{\partial v_i}{\partial x_i}$$

Entropy production (LL49.5-6)

$$\frac{\partial(\rho s)}{\partial t} + \nabla\cdot\left(\rho s\mathbf{v} - \frac{K}{T}\boldsymbol{\nabla}T\right) = \frac{K(\nabla T)^2}{T^2} + \frac{\eta}{2T}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial v_i}{\partial x_i}\right)^2 + \frac{\zeta}{T}\left(\frac{\partial v_i}{\partial x_i}\right)^2$$

Equilibrium, Near Equilibrium, and Far from Equilibrium

Equilibrium v. Nonequilibrium



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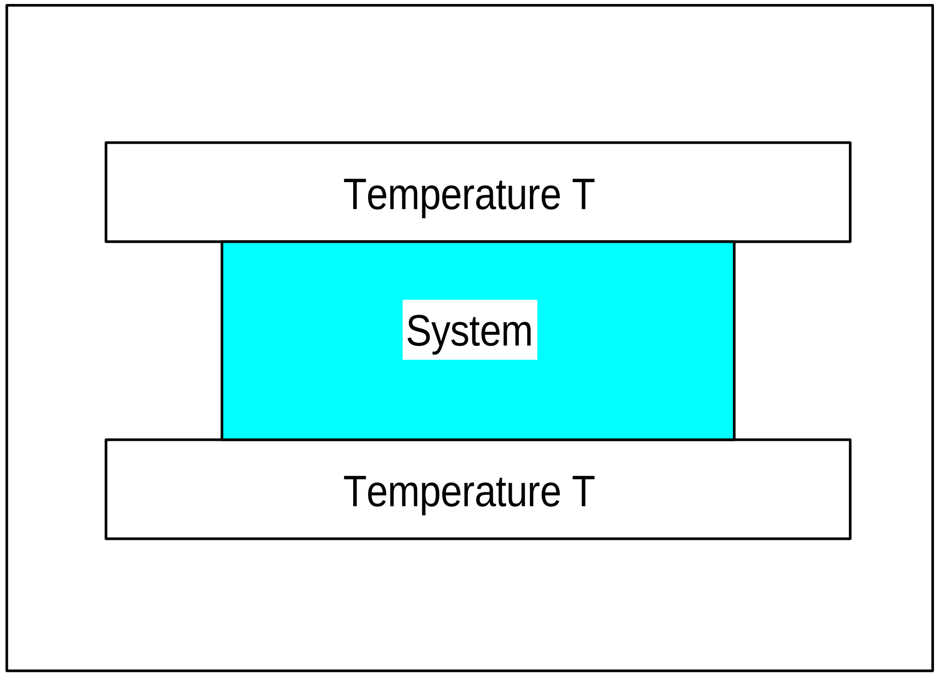

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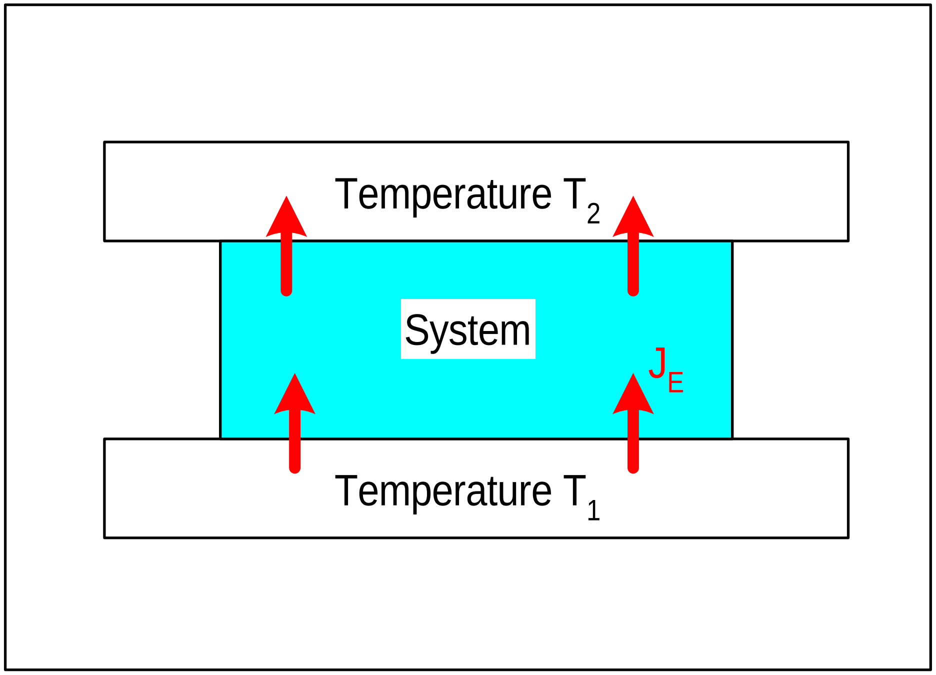

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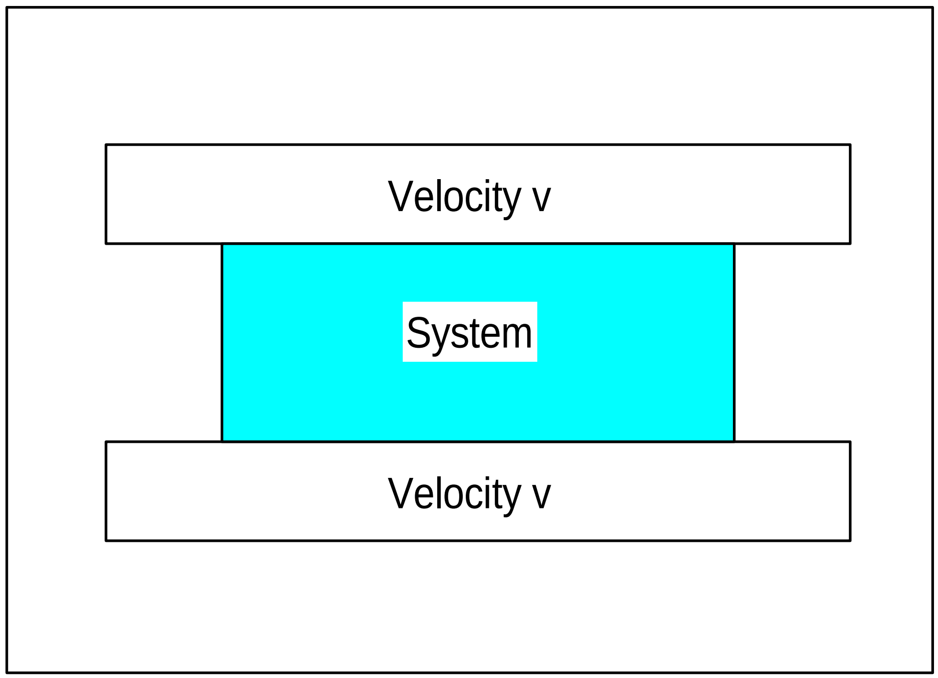

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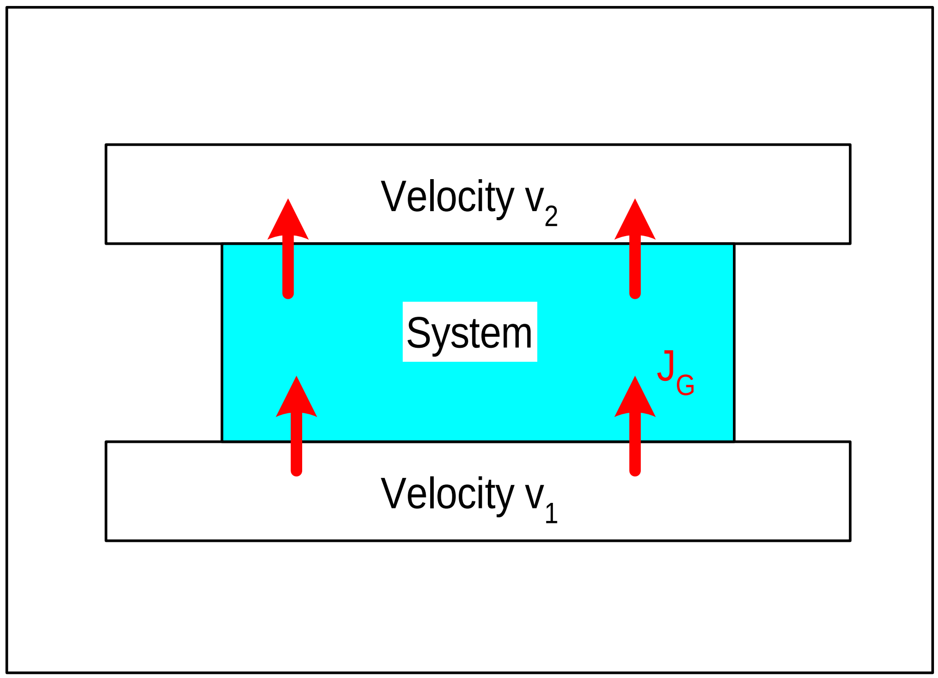
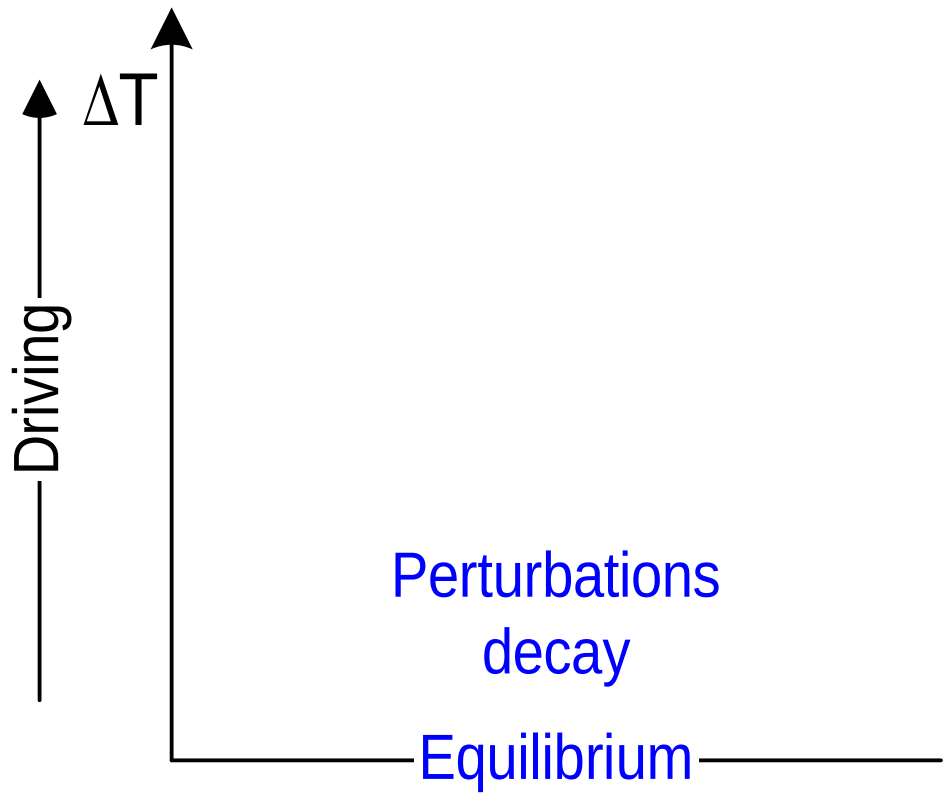


Nonequilibrium

Equilibrium - Far From Equilibrium



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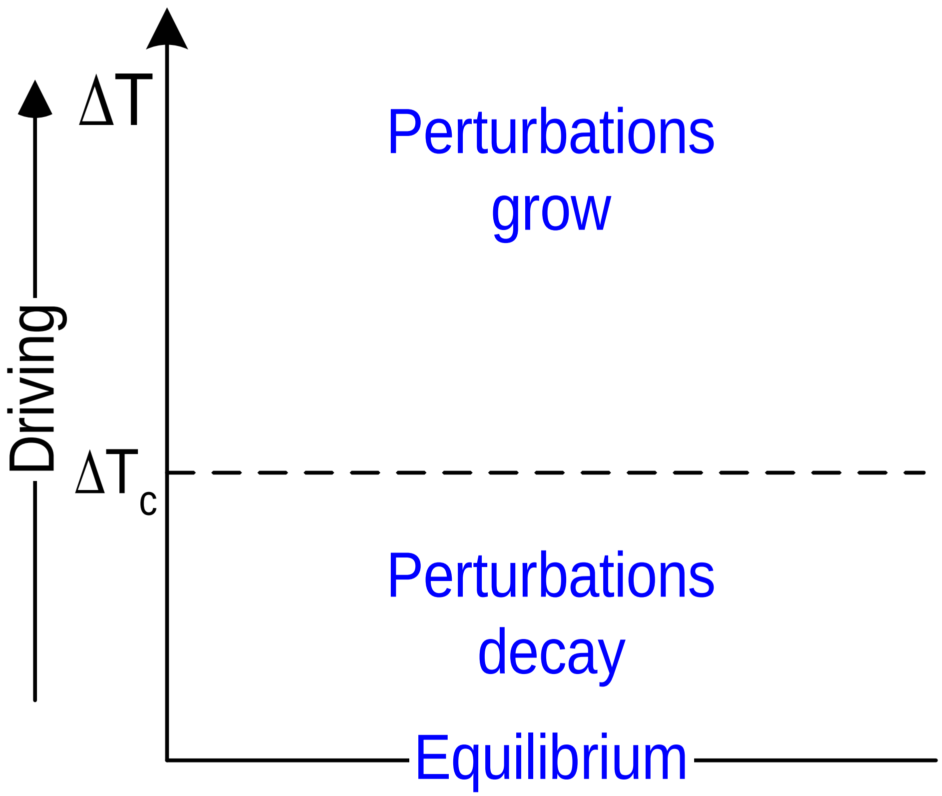

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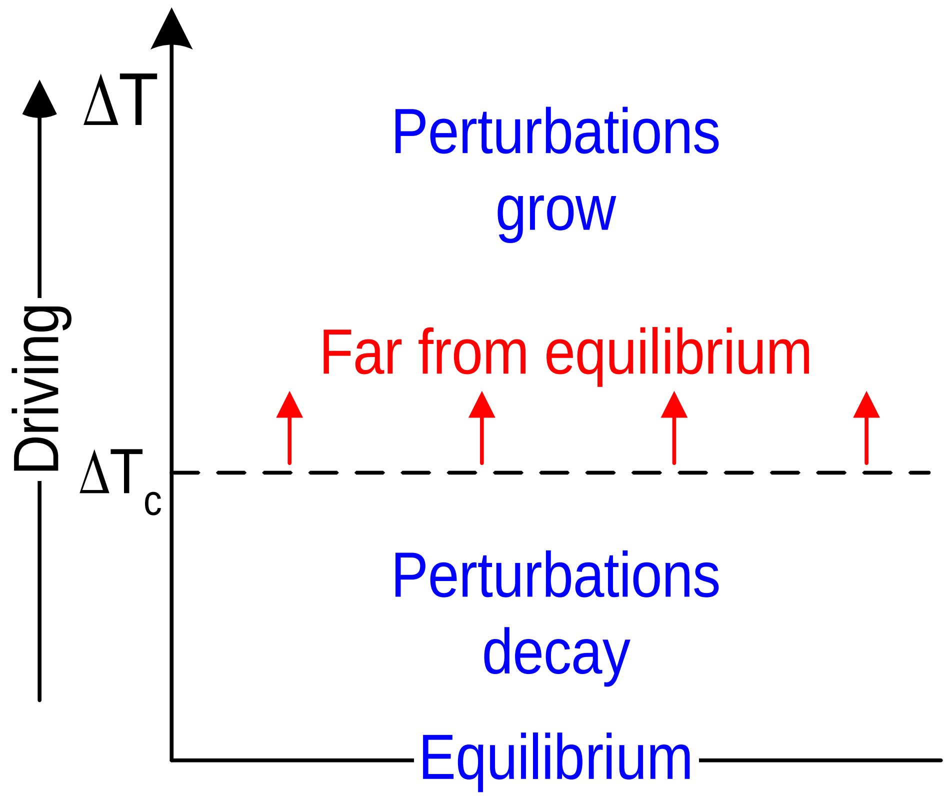

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- These systems allow a quantitative understanding of phenomenon far from equilibrium
- Other systems far from equilibrium may not be near local equilibrium, e.g. biology, chemistry. For these quantitative descriptions are harder.