Collective Effects

in

Equilibrium and Nonequilibrium Physics

Website: http://cncs.bnu.edu.cn/mccross/Course/

Caltech Mirror: http://haides.caltech.edu/BNU/

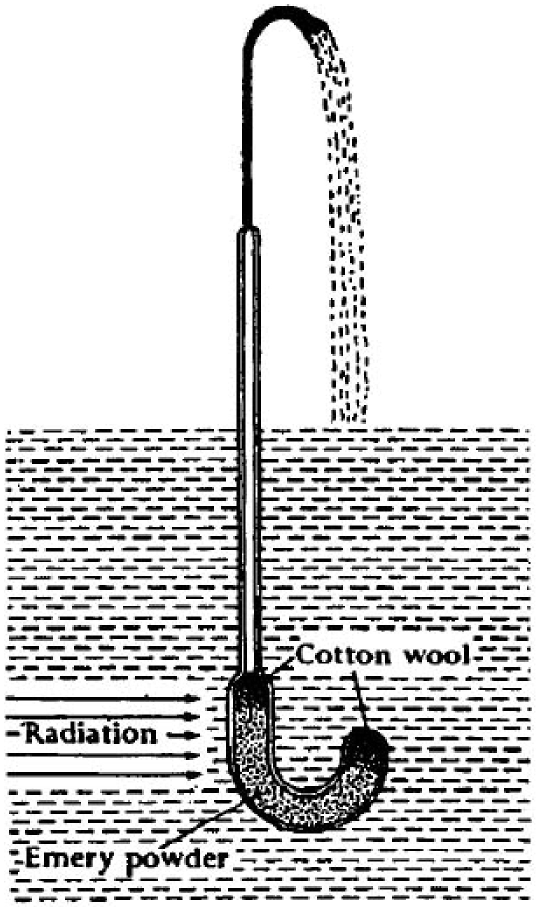
Today's Lecture

Superfluids and superconductors

- What are superfluidity and superconductivity?
- Review of phase dynamics
- Description in terms of a macroscopic phase
- Supercurrents that flow for ever
- Josephson effect
- Four sounds

The Amazing World of Superfluidity and Superconductivity

- Electric currents in loops that flow for ever (measured for $\sim$ decade)
- Beakers of fluid that empty themselves
- Fluids that flow without resistance through tiny holes
- Flow in surface films less than an atomic layer thick
- Flow driven by temperature differences (fountain effect)



History of Superfluidity and Superconductivity

1908 Liquefaction of ^{4}He by Kamerlingh Onnes

1911 Discovery of superconductivity by Onnes (resistance drops to zero)

1933 Meissner effect: superconductors expel magnetic field

1937 Discovery of superfluidity in ^{4}He by Allen and Misener

1938 Connection of superfluidity with Bose-Einstein condensation by London

1955 Feynman's theory of quantized vortices

1956 Onsager and Penrose identify the broken symmetry in superfluidity ODLRO

1957 BCS theory of superconductivity

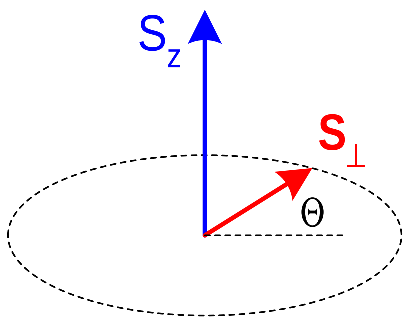
1962 Josephson effect

1973 Discovery of superfluidity in ^{3}He at 2mK by Osheroff, Lee, and Richardson

1986 Discovery of high-T_c superconductors by Bednorz and Müller

1995- Study of superfluidity in ultracold trapped dilute gases

Review of Phase Dynamics with a Conserved Quantity



Rotational symmetry in the XY plane (angle Θ)

The XY and Z components of the spin have different properties:

$$S_z = \Omega^{-1} \sum_{i \text{ in } \Omega} \langle s_{iz} \rangle \qquad \text{is a } \textit{conserved quantity}$$

$$\mathbf{S}_\perp = \Omega^{-1} \sum_{i \text{ in } \Omega} \langle \mathbf{s}_{i\perp} \rangle \qquad \text{is the XY } \textit{order parameter}$$

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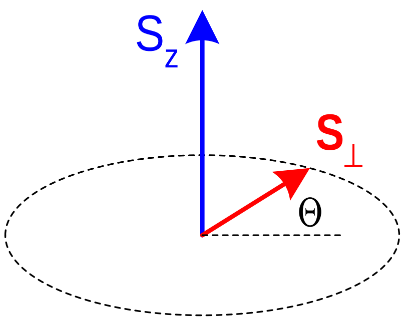

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S_z and Θ are canonically conjugate variables, so that with the free energy

$$F = \int d^d x \left[\frac{1}{2} K (\nabla \Theta)^2 + \frac{S_z^2}{2\chi} - S_z b_z \right]$$

we get

$$\dot{S}_z = -\frac{\delta F}{\delta \Theta} \qquad \text{giving} \qquad \dot{S}_z = -\nabla \cdot \mathbf{j}_{S_z} \quad \text{with} \quad \mathbf{j}_{S_z} = -K \nabla \Theta$$

$$\dot{\Theta} = \frac{\delta F}{\delta S_z} \qquad \text{giving} \qquad \dot{\Theta} = \chi^{-1} (S_z - \chi b_z)$$

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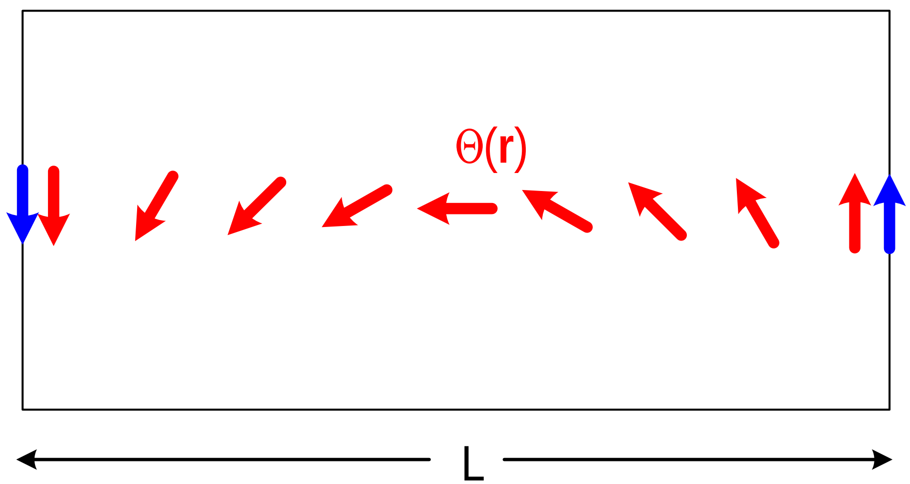

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- For formal proof see Halperin and Saslow, Phys. Rev. B 16, 2154 (1977), Appendix: *“the Larmor precession theorem”*

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$$d\varepsilon = T\,ds + \mu_z ds_z + \Phi \cdot d(\nabla\Theta) \text{ with}$$

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Derive

- dynamical equations for conserved quantities and broken symmetry variables for slowly varying disturbances

Rigidity and the Thermodynamic Identity

In terms of the energy density

$$d\varepsilon = T ds + \mu_z ds_z + \Phi \cdot d(\nabla\Theta)$$

- *conjugate fields* are

$$\mu_z = \left(\frac{\partial \varepsilon}{\partial s_z}\right)_{s,\nabla\Theta} \quad \text{and} \quad \Phi = \left(\frac{\partial \varepsilon}{\partial \nabla\Theta}\right)_{s,\mathbf{s}_z}$$

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Or with the free energy density $f = \varepsilon - Ts$

$$df = -sdT + \mu_z ds_z + \Phi \cdot d(\nabla\Theta)$$

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These give

$$\mu_z = \chi^{-1}(S_z - \chi b_z) \qquad \text{and} \qquad \Phi = K\nabla\Theta$$

Entropy Production

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- Identify the entropy current and production

$$\mathbf{j}^s = T^{-1}(\mathbf{j}^\varepsilon - \mu_z \mathbf{j}^{s_z})$$

$$RT = -T^{-1}(\mathbf{j}^\varepsilon - \mu_z \mathbf{j}^{s_z}) \cdot \nabla T - (\mathbf{j}^{s_z} + \Phi)\cdot\nabla\mu_z$$

Equilibrium Dynamics

Entropy Production

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We will consider adding dissipation later.

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  - ◇ There are a macroscopic number of particles in a single wave function and so Θ is a macroscopic thermodynamic variable, and is the broken symmetry variable.

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- Currents are present *in equilibrium*, and so are *supercurrents*

Supercurrents by Analogy

- One-to-one correspondence at the quantum operator level

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(e.g., $\uparrow \equiv$ particle, $\downarrow \equiv$ no particle)

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- Or write in terms of flow of mass

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{g} \qquad \text{with} \qquad \mathbf{g} = \rho_s\mathbf{v}_s, \;\; \rho_s = mn_s$$

Hydrodynamic Derivation

- Free energy expression: generalized rigidity and energy in external potential V

$$f = \frac{\hbar^2 n_s}{2m}(\nabla\Theta)^2 + \frac{1}{2}Kn^2 + Vn$$

(K is bulk modulus)

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$$\Psi(V,t) = \Psi(0,t)e^{-i\hbar N\delta V t}$$

  gives

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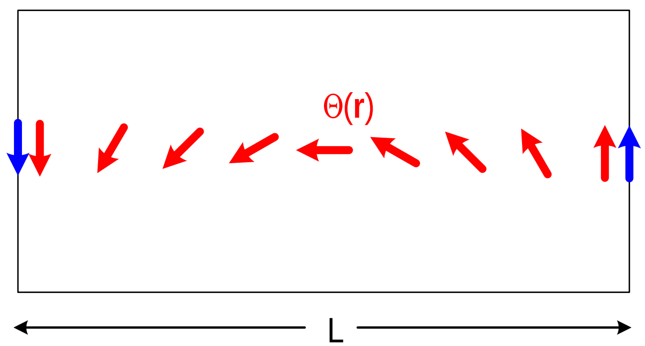
- Entropy production argument from the thermodynamic identity

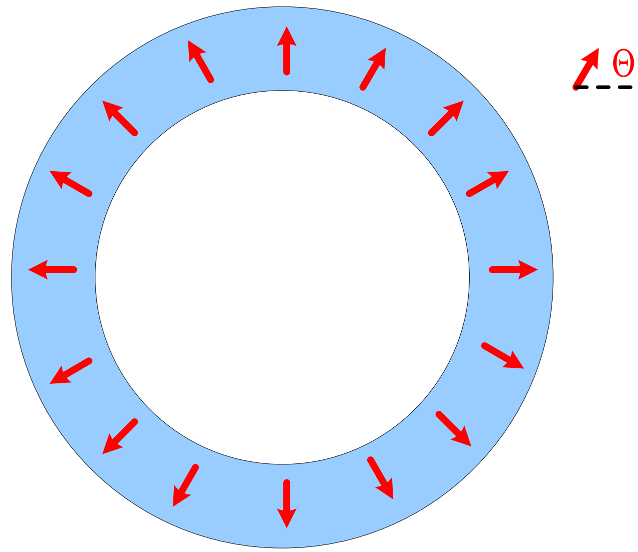
$$d\varepsilon = T\,ds + \mu\,dn + \Phi\cdot d(\nabla\Theta) \qquad \text{with} \qquad \Phi = (\hbar^2 n_s/m)\nabla\Theta$$

  gives the current of particles

$$\dot{n} = -\nabla\cdot\mathbf{j} \qquad \text{with} \qquad \mathbf{j} = n_s(\hbar/m)\nabla\Theta$$

Currents that Flow Forever





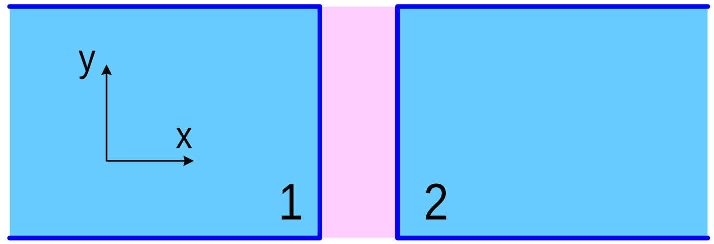
$$g = \rho_s(\hbar/m)(\pi/L)$$

$$g = \rho_s(\hbar/m)(2\pi/L)$$

$$\oint \mathbf{v}_s \cdot \mathbf{dl} = \frac{h}{m}$$

quantum of circulation

Josephson Effect



- Energy depends on phase difference. For weak coupling

$$E = -J_c \cos(\Theta_2 - \Theta_1)$$

- Change in number of particles: current $I = dN_2/dt$

$$I = \frac{dE}{d\Theta_2}$$

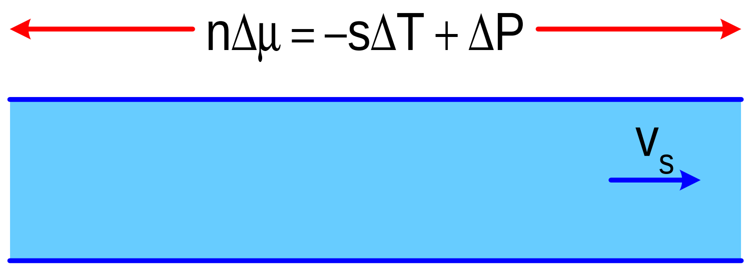
d.c. Josephson effect $\quad I = J_c \sin(\Theta_2 - \Theta_1)$

- Time dependence of phase is given by the potential

$$\hbar \dot{\Theta}_i = -\mu_i$$

a.c. Josephson effect $\quad \hbar(\dot{\Theta}_2 - \dot{\Theta}_1) = -\Delta\mu$

Breakdown of Superfluidity



$$\hbar \Delta \dot{\Theta} = -\Delta \mu, \qquad v_s = (\hbar/m) n_s \Delta \Theta / L$$

- pressure or temperature difference accelerates superflow
- constant superflow does not require pressure drop

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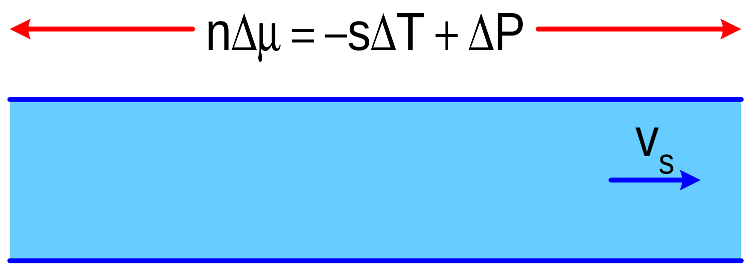

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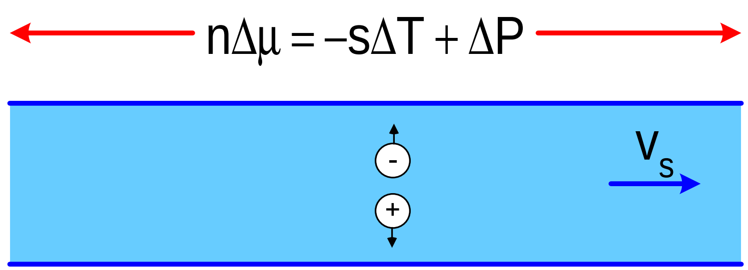

- pressure drop (dissipation) requires passage of vortex topological defects (“quantized vortex lines”) across flow channel
- presence of dissipation depends on whether vortices can be produced by thermal activation or other mechanism

Josephson Effect for a Superconductor

- Θ is phase of *pair* wave function
- expressions must be *gauge invariant* in presence of vector potential

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For bulk material

$$\text{Supercurrent:} \quad \mathbf{j} = n_s \frac{\hbar}{2m} \left(\nabla \Theta(\mathbf{x}) + \frac{2e}{\hbar c} \mathbf{A} \right)$$

$$\text{Josephson equation:} \quad \hbar \dot{\Theta} = 2eV$$

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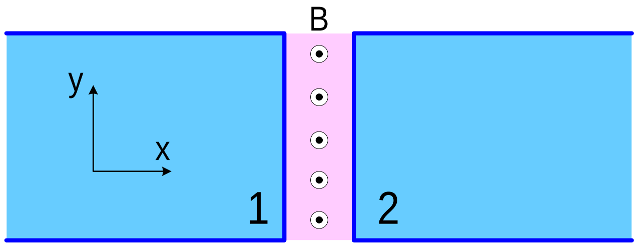
For Josephson junction, current is $I = \int j(y,z)\,dy\,dz$ with

$$j(y,z) = j_c \sin\left(\Theta_2 - \Theta_1 + \frac{2e}{\hbar c}\int_1^2 A_x dx\right)$$

and

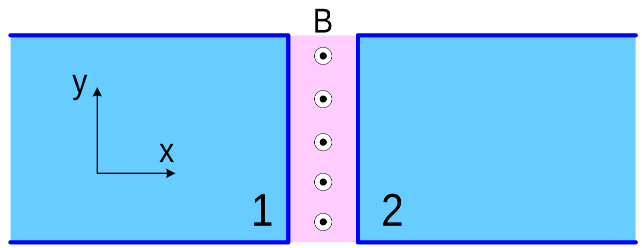
$$V = (\hbar/2e)(\dot{\Theta}_2 - \dot{\Theta}_1)$$

Josephson Junction in a Magnetic Field



Josephson current density: $$j(y,z) = j_c \sin\left(\Theta_2 - \Theta_1 + \frac{2e}{\hbar c}\int_1^2 A_x\, dx\right)$$

Josephson Junction in a Magnetic Field



Josephson current density: $$j(y,z) = j_c \sin\left(\Theta_2 - \Theta_1 + \frac{2e}{\hbar c}\int_1^2 A_x dx\right)$$

For field $B\hat{\mathbf{z}}$ in junction the vector potential is $\mathbf{A} = -By\hat{\mathbf{x}}$, so that

$$I \propto \int dy\ j_c \sin[\Theta_2 - \Theta_1 - (2e/\hbar c)Byd]$$

giving

$$I = I_c(B)\sin(\Theta_2 - \Theta_1) \qquad \text{with} \qquad I_c(B) = \frac{\sin(\pi\phi/\phi_0)}{\pi\phi/\phi_0}$$

where $\phi = Bld$ is the flux through the junction and $\phi_0 = hc/2e$ is the *flux quantum* $(2.1 \times 10^{-7} \text{gauss cm}^2)$

Experimental Discovery of the dc Josephson Effect

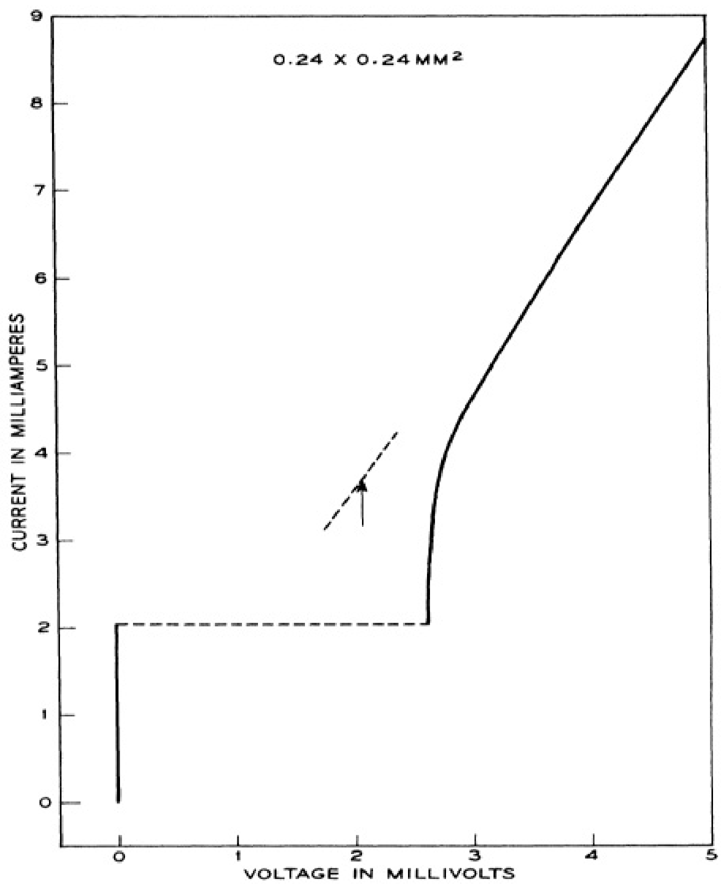


FIG. 1. Current-voltage characteristic for a Pb-I-Pb junction at 1.3°K. The arrow marks the predicted maximum magnitude of the Josephson current.

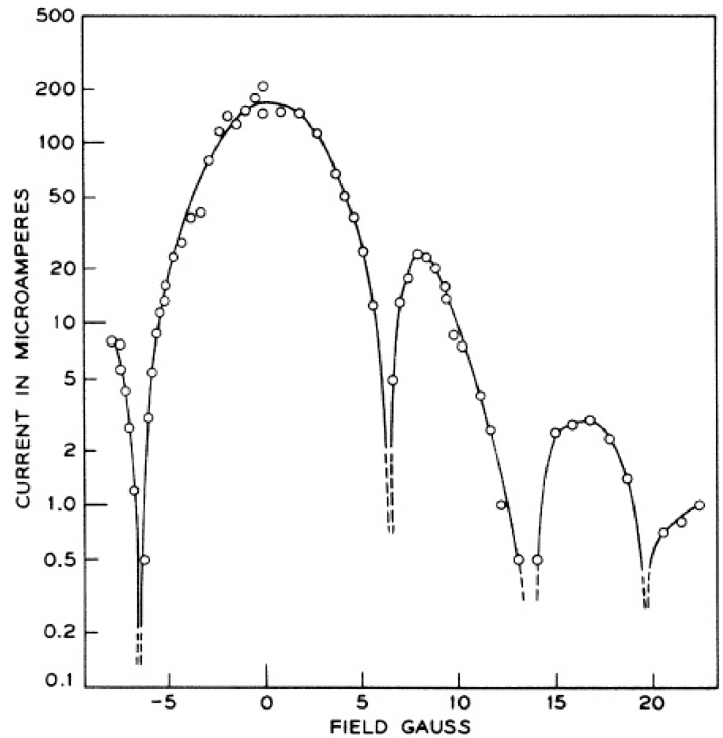
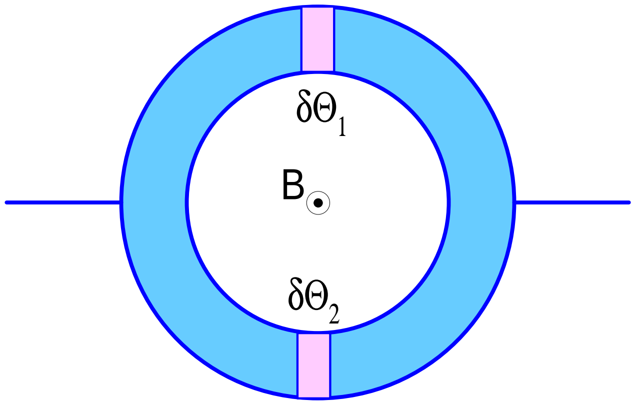


FIG. 3. The field dependence of the Josephson current in a Pb-I-Pb junction at 1.3°K.

Rowell, Phys. Rev. Lett. **11**, 200 (1963)

SQUID

Superconducting Quantum Interference Device



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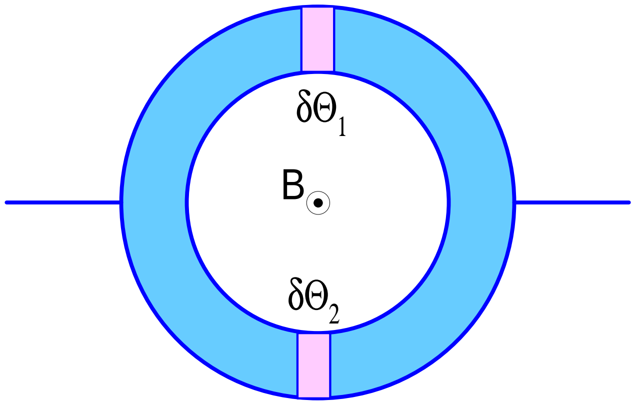

Integrate $\mathbf{j} = n_s \frac{\hbar}{2m}\left(\nabla\Theta(\mathbf{x}) + \frac{2e}{\hbar c}\mathbf{A}\right)$ around whole loop using fact that current $\mathbf{j}$ is small

$$\delta\Theta_1 - \delta\Theta_2 = \frac{2e}{\hbar c}\oint \mathbf{A}\cdot\mathbf{dl} = 2\pi\phi/\phi_0$$

with $\phi = B \times$ area, the flux through the loop.

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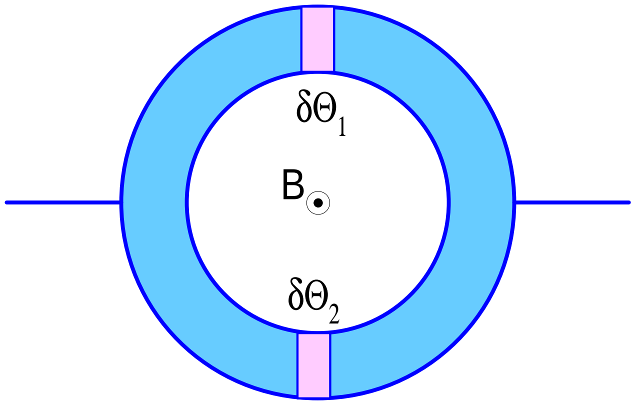

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Total current

$$\begin{aligned} I &= J_c\,[\sin\delta\Theta_1 + \sin\delta\Theta_2] \\ &= 2J_c \sin(\pi\phi/\phi_0)\sin[\tfrac{1}{2}(\delta\Theta_1 + \delta\Theta_2)] \end{aligned}$$

Maximum current varies periodically with applied field — very sensitive magnetometer.

Four Sounds in a Superfluid

Equations of motion for conserved quantities

$$\dot{\rho} = -\nabla \cdot \mathbf{g}$$
$$\dot{\mathbf{g}} = -\nabla P$$
$$\dot{s} = 0$$

and the dynamics of the broken symmetry variable

$$\hbar\dot{\Theta} = -\mu$$

which can be written as

$$\rho\dot{\mathbf{v}}_s = s\nabla T - \nabla P$$

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Need to connect the momentum density to the superfluid velocity.

Galilean Invariance

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Define the "normal fluid density" $\rho_n = \rho - \rho_s$ and write the transformed superfluid velocity $\mathbf{v}_s = \mathbf{v}_s^{(0)} + \mathbf{v}_n$

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- Momentum equation can be transformed to

$$\rho_s \dot{\mathbf{v}}_s + \rho_n \dot{\mathbf{v}}_n = -\nabla P$$

and using the equation for $\dot{\mathbf{v}}_s$ in the form

$$(\rho_s + \rho_n) \dot{\mathbf{v}}_s = s \nabla T - \nabla P$$

gives

$$\rho_n (\dot{\mathbf{v}}_s - \dot{\mathbf{v}}_n) = s \nabla T$$

First Sound

Usual coupled density and momentum equations

$$\dot{\rho} = -\nabla \cdot \mathbf{g}$$
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and the pressure-density relationship (K is the bulk modulus)

$$\delta P = K\, \delta\rho/\rho$$

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These give first sound waves $\propto e^{i(\mathbf{q}\cdot\mathbf{r}-\omega t)}$ propagating with the usual sound speed $\omega = c_1 q$ with

$$c_1 = \sqrt{\frac{K}{\rho}}$$

Second Sound

Coupled counterflow and entropy wave. Use $c_2 \ll c_1 \Rightarrow$ density constant, $\mathbf{g} = 0$

$$\rho_s \mathbf{v}_s + \rho_n \mathbf{v}_n = 0 \qquad \Rightarrow \qquad \mathbf{v}_s - \mathbf{v}_n = -(\rho/\rho_s)\mathbf{v}_n$$

(remember $\rho_s + \rho_n = \rho$).

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Entropy-temperature relationship (C is the specific heat): $\delta s = C\delta T/T$

$$C\dot{T} = sT(\rho_s/\rho)\nabla \cdot (\mathbf{v}_s - \mathbf{v}_n)$$

Counterflow equation

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Counterflow equation

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These give propagating second sound waves with the speed

$$c_2 = \sqrt{\frac{\rho_s}{\rho_n}\frac{s^2 T}{\rho C}}$$

Fourth Sound

Fluid confined in porous media: no conserved momentum, no Galilean invariance (no $\mathbf{v}_n$), temperature constant

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$$\rho \dot{\mathbf{v}}_s = -\nabla P$$

$$\delta P = -K \delta\rho / \rho$$

These gives a fourth sound wave propagating with the speed

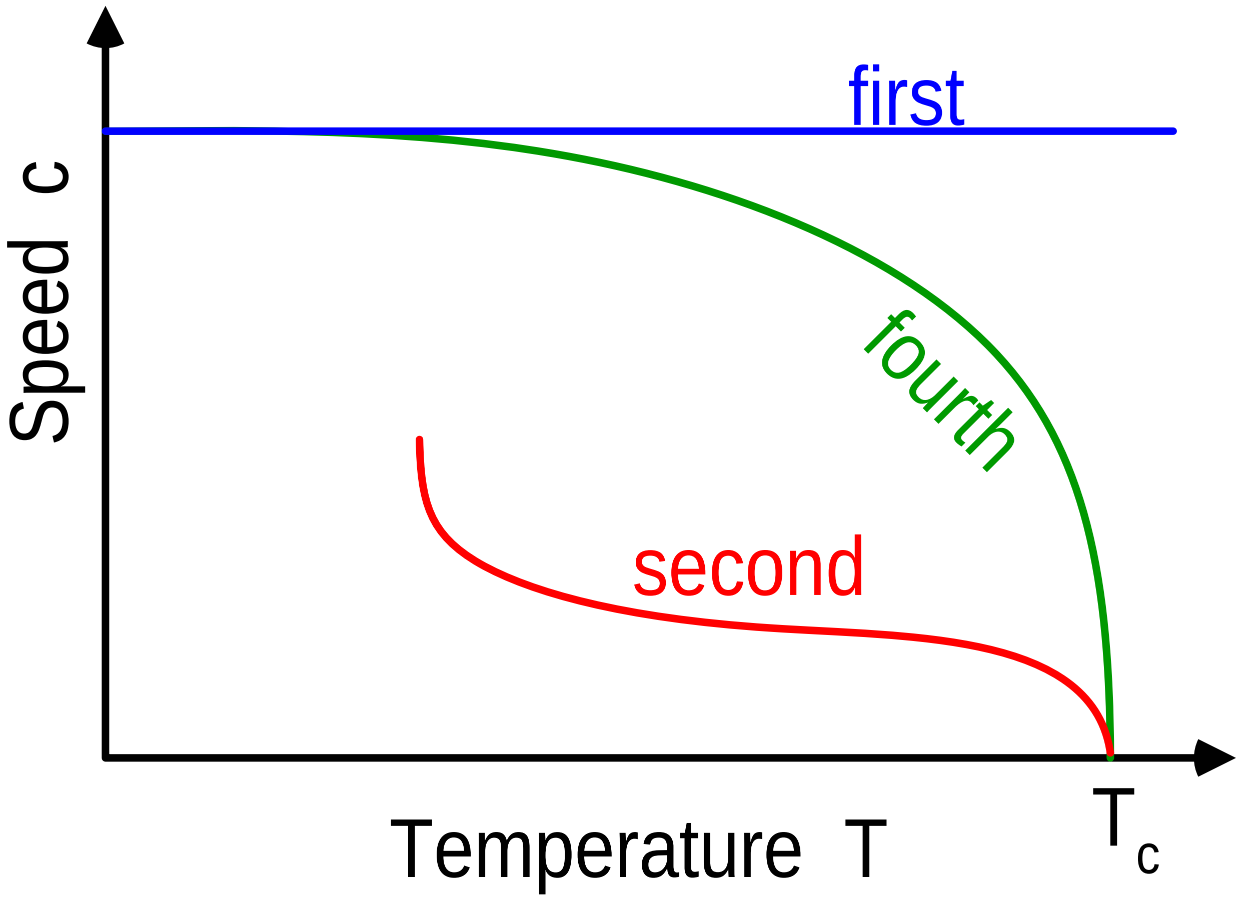
$$c_4 = \sqrt{\frac{\rho_s}{\rho} \frac{K}{\rho}}$$

Third Sound

Wave propagating in thin films down to atomic layer thickness.

Like fourth sound, but involve changes of thickness rather than density, and effective compressibility depends on strength of interaction with surface.

Sounds in Helium-4



$$c_1 = \sqrt{\frac{K}{\rho}}, \qquad c_2 = \sqrt{\frac{\rho_s}{\rho_n}\frac{s^2 T}{\rho C}}, \qquad c_4 = \sqrt{\frac{\rho_s}{\rho}\frac{K}{\rho}}$$

Next Lecture

Onsager theory and the fluctuation-dissipation theorem

- Derivation and discussion
- Application to nanomechanics and biodetectors