Onset of Rayleigh-Bénard Convection: No-slip Boundaries

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The "free-slip" boundaries on the fluid at the upper and lower plate are unrealistic, but mathematically simplify the problem, rendering the equations separable. This handout describes the mathematical procedure for the realistic case of no-slip boundaries (all fluid velocities zero).

The basic linear differential equations are most conveniently formulated using the stream function ψ (with $u = -\partial_z\psi$, $w = \partial_x\psi$) and temperature deviation θ:

$$\begin{aligned} 0 &= \partial_x\theta + \nabla^4\psi \\ 0 &= R\partial_x\psi + \nabla^2\theta. \end{aligned} \tag{1}$$

To proceed we write

$$\begin{bmatrix} \psi(x,z,t) \\ \theta(x,z,t) \end{bmatrix} = e^{iqx} \begin{bmatrix} \psi_q(z) \\ \theta_q(z) \end{bmatrix} \tag{2}$$

and substitute into (1). The first equation leads to

$$\theta_q(z) = \frac{i}{q}\left(\partial_z^2 - q^2\right)^2 \psi_q(z) \tag{3}$$

and substituting into the second gives

$$\left[\left(\partial_z^2 - q^2\right)^3 + R_c(q)q^2\right]\psi_q(z) = 0 \tag{4}$$

which must be solved for the z dependence $\psi_q(z)$. This is an ODE with constant coefficients, and so the z-dependence is given by exponentials. The solutions will be either even or odd in z and the lowest onset turns out to be for an even solution, which therefore takes the form

$$\psi_q(z) = \sum_{i=1}^{3} a_i \cos(k_i z) \tag{5}$$

where a_i are as yet unknown coefficients and the k_i are the three roots (substitute into 4) of

$$\left(k^2 + q^2\right)^3 - R_c(q)q^2 = 0 \tag{6}$$

i.e.

$$k = \sqrt{-q^2 + \sqrt[3]{R_c(q)q^2}(1)^{1/3}} \tag{7}$$

with the 3 roots given by the three cube roots of unity $(1)^{1/3} = 1, -\frac{1}{2} \pm \frac{1}{2}i\sqrt{3}$. The a_i are determined by the three boundary conditions on $\psi_q(z)$ (equation 5) and $\theta_q(z)$ (equation 3), namely

$\psi_q(z) = \partial_z \psi_q(z) = \theta_q(z) = 0$ at $z = \pm\frac{1}{2}$. These give

$$\begin{aligned}
\sum_{i=1}^{3} a_i \cos(\frac{1}{2}k_i) &= 0 \\
\sum_{i=1}^{3} k_i a_i \sin(\frac{1}{2}k_i) &= 0 \\
\sum_{i=1}^{3} \left(k_i^2 + q^2\right)^2 a_i \cos(\frac{1}{2}k_i) &= 0.
\end{aligned} \tag{8}$$

The condition for a nonzero solution for the a_i is that the determinant is zero

$$\begin{vmatrix}
\cos\left(\frac{1}{2}k_1\right) & \cos\left(\frac{1}{2}k_2\right) & \cos\left(\frac{1}{2}k_3\right) \\
k_1 \sin\left(\frac{1}{2}k_1\right) & k_2 \sin\left(\frac{1}{2}k_2\right) & k_3 \sin\left(\frac{1}{2}k_3\right) \\
\left(k_1^2 + q^2\right)^2 \cos\left(\frac{1}{2}k_1\right) & \left(k_2^2 + q^2\right)^2 \cos\left(\frac{1}{2}k_2\right) & \left(k_3^2 + q^2\right)^2 \cos\left(\frac{1}{2}k_3\right)
\end{vmatrix} = 0. \tag{9}$$

Since all the k_i are known in terms of the single parameter $R_c(q)$ for each q (equation 7) this is a transcendental equation which must be solved numerically for $R_c(q)$. The minimum value of $R_c(q)$ turns out to be about 1707.76 and occurs at $q = q_0 \approx 3.11632$, very close to circular rolls (which would be $q_0 = \pi$).

Note the difference with the free-slip case is that we cannot use a single k_i to satisfy all the boundary conditions, and need to keep the fully general linear combination of modes with the three k_i consistent with the ODE.