Collective Effects

in

Equilibrium and Nonequilibrium Physics

Website: http://cncs.bnu.edu.cn/mccross/Course/

Caltech Mirror: http://haides.caltech.edu/BNU/

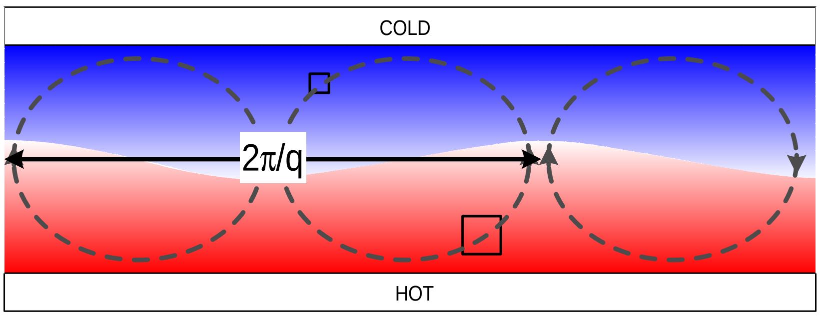
Today's Lecture: Nonlinear Theory of Patterns near Onset

Outline

- Review: linear instability towards patterns
- Qualitative picture of nonlinear, spatially periodic patterns
- General Patterns Near Onset
  - ◇ One dimensional amplitude equation
  - ◇ Generalizations to two dimensions

Analogies to and differences from equilibrium phase transition to broken symmetry state

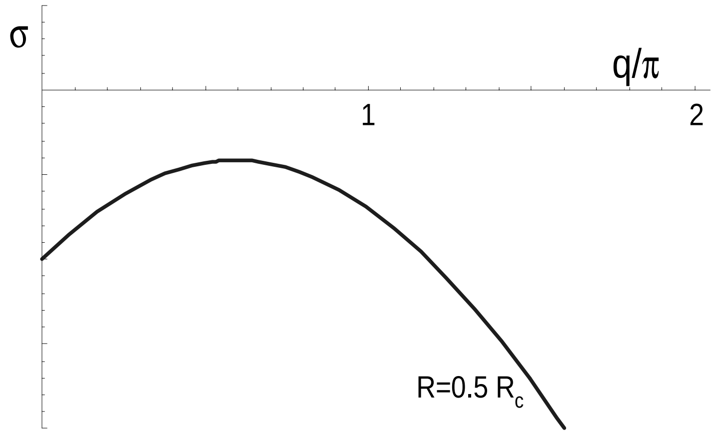
Review of Rayleigh-Bénard Instability



Linear Stability Analysis

- Driving strength: Rayleigh number $R \propto \Delta T$
- Look for linear mode $u, \theta \propto e^{\sigma(q)t} \cos(qx)$
- Calculate $\sigma(q)$ as a function of R
- $\sigma(q) > 0$ indicates exponential growth, i.e., instability towards a pattern with periodicity $2\pi/q$

Rayleigh's Growth Rate (for $\mathcal{P} = 1$)



$$R_c = \frac{27\pi^4}{4},\ q_c = \frac{\pi}{\sqrt{2}}$$

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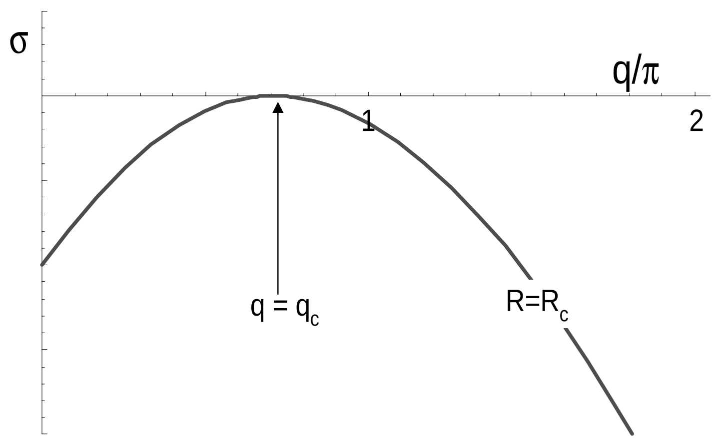

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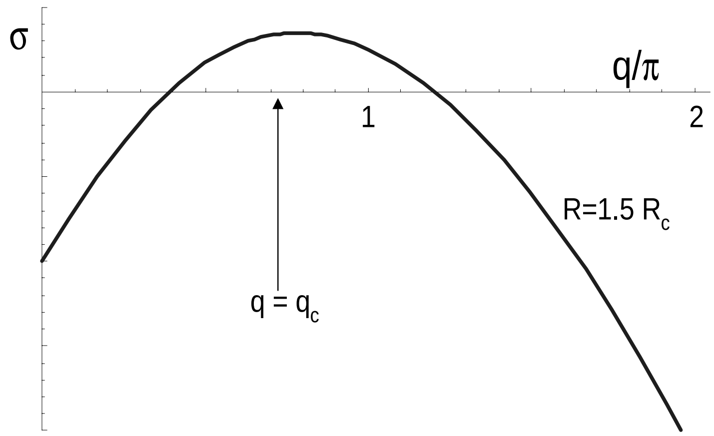

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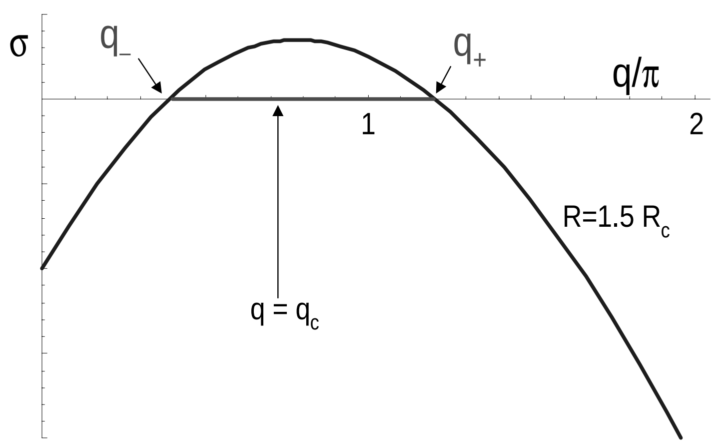
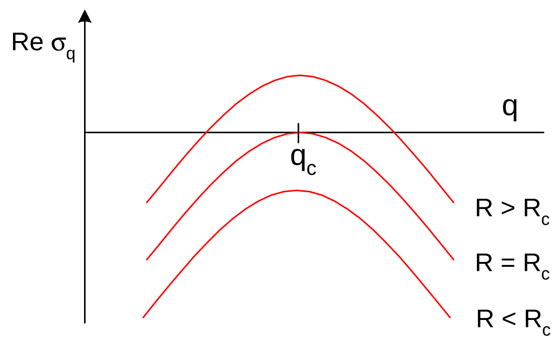


$$R_c = \frac{27\pi^4}{4}, \; q_c = \frac{\pi}{\sqrt{2}}$$

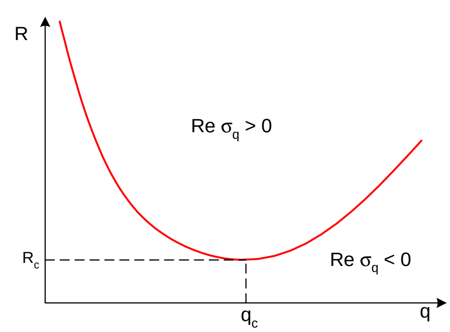
Parabolic approximation near maximum



For R near R_c and q near q_c

$$\text{Re }\sigma(q) = \tau_0{}^{-1}[\varepsilon - \xi_0{}^2\,(q - q_c)^2] \qquad \text{with} \qquad \varepsilon = \frac{R - R_c}{R_c}$$

Neutral stability curve



Re $\sigma(q) = 0$ defines the neutral stability curve $R = R_c(q)$ or $q = q_N(R)$

$$\text{Rayleigh}: \qquad R_c(q) = \frac{(q^2 + \pi^2)^3}{q^2}$$

Linear Stability Analysis

Linear stability theory is often a useful first step in understanding pattern formation:

- Often is quite easy to do either analytically or numerically
- Displays the important physical processes
- Gives the length scale of the pattern formation $1/q_c$

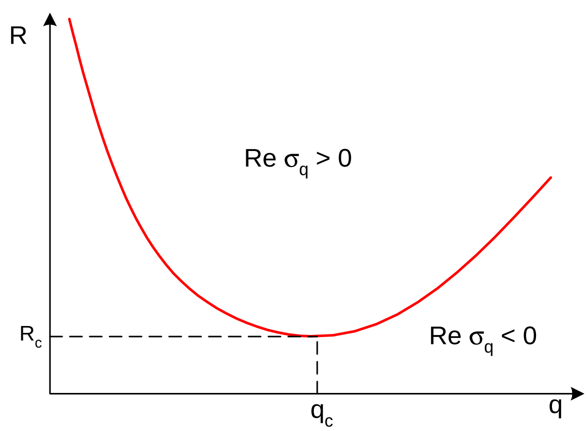
But:

- Leaves us with unphysical exponentially growing solutions

Nonlinear Theory

- Saturation of spatially periodic solution (bifurcation theory)
- General patterns (cf., broken symmetry at phase transitions)

Qualitative Picture of Nonlinear States



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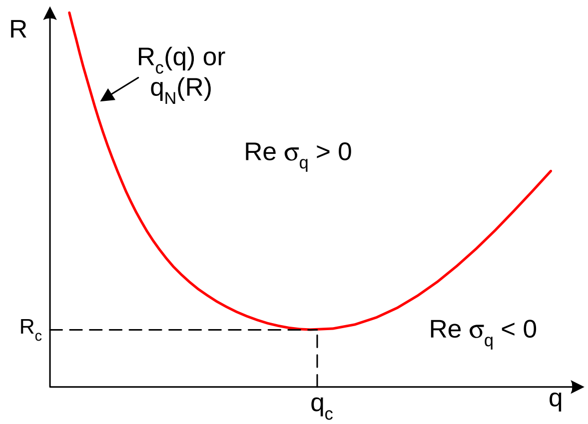

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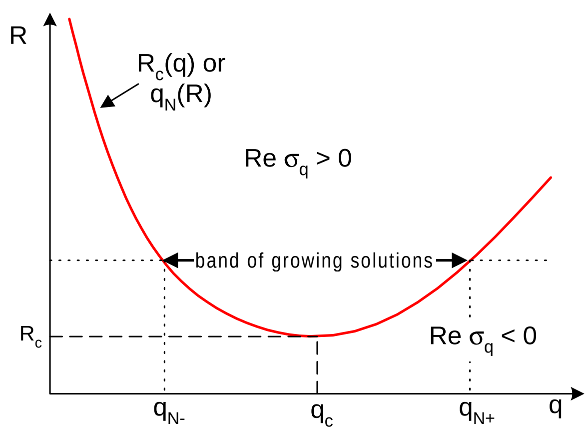
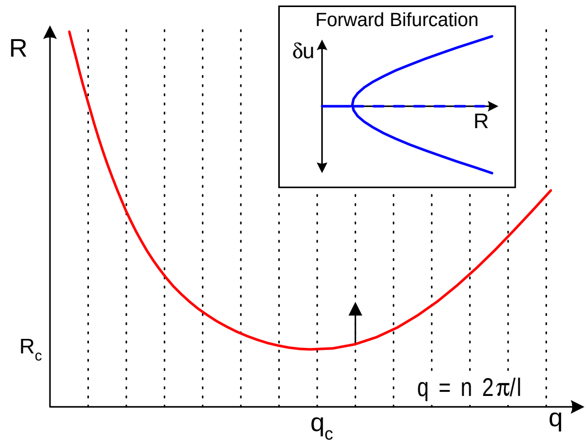


Qualitative Picture of Nonlinear States: Periodic BC



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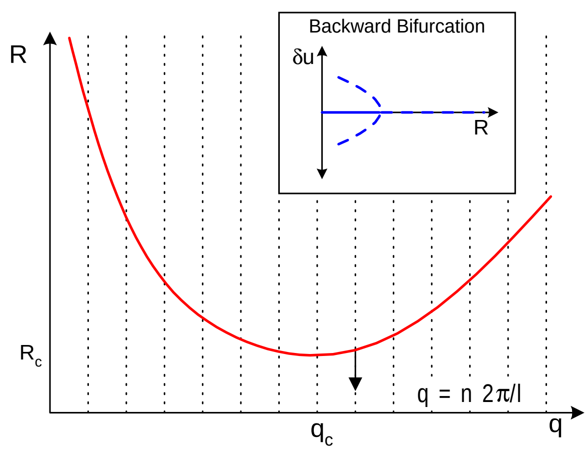
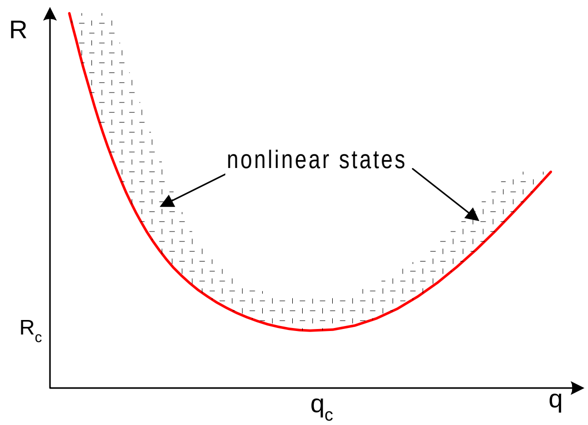


Qualitative Picture of Nonlinear States: Infinite System



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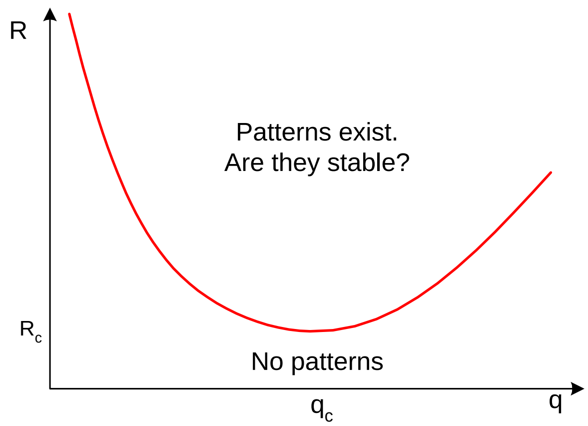
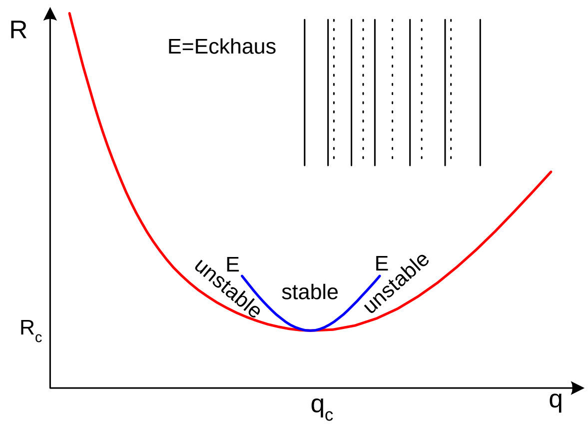


Qualitative Picture of Nonlinear States: Instability of Stripes



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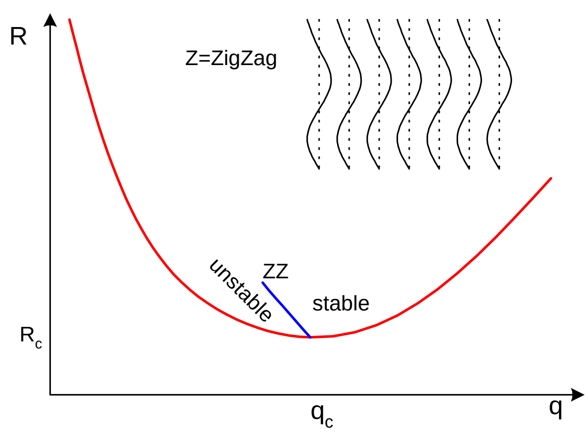

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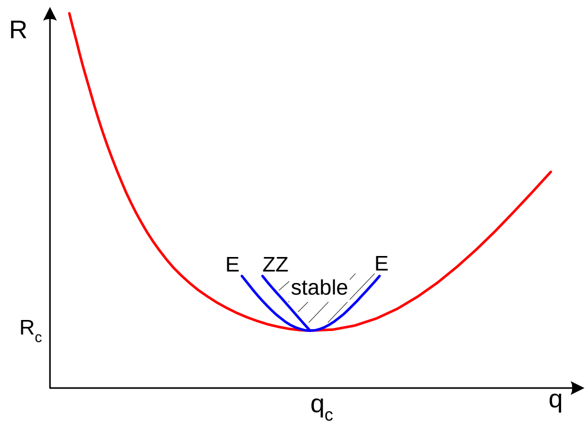
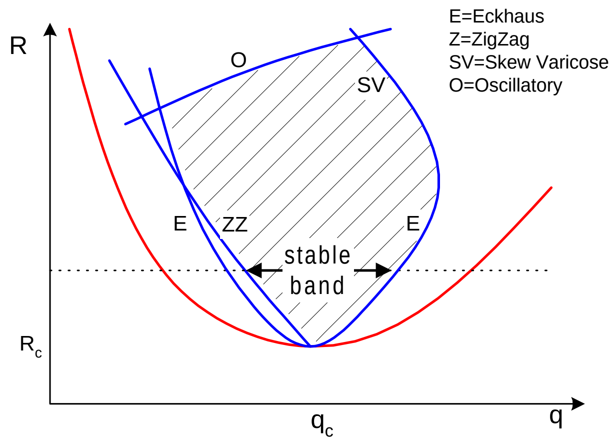


Qualitative Picture of Nonlinear States: Stability Balloon



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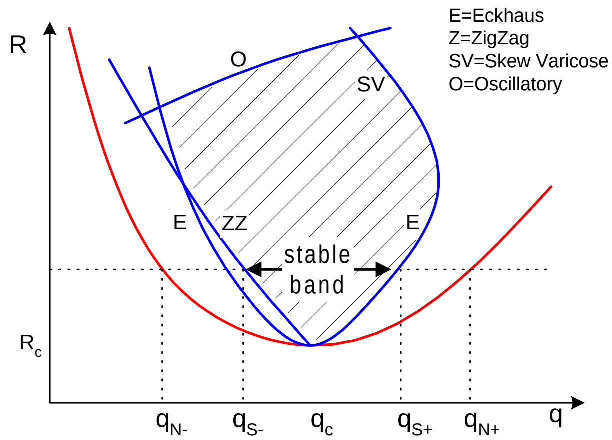
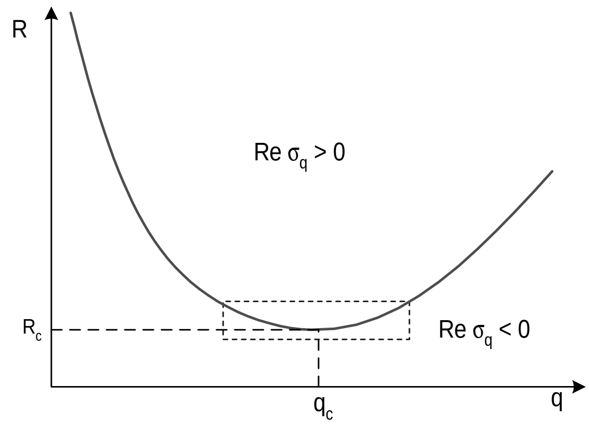


Tools for the Nonlinear Problem

- The instability to a pattern is another example of a broken symmetry transition, now in the context of nonequilibrium systems
- The same basic ideas we discussed in the context of equilibrium phase transitions apply:
  - ◇ near the transition ($R \simeq R_c$, $q \simeq q_c$ or slow modulations of a pattern at q_c) describe the behavior using an order parameter
  - ◇ away from the transition use a phase variable description to describe the behavior resulting from the broken symmetry
- There will be similar general behavior:
  - ◇ new rigidity
  - ◇ Goldstone modes
  - ◇ importance of topological defects
- There will be important differences in formulation and behavior because we cannot start from a free energy, but must consider directly the dynamics

Amplitude Equations

Systematic approach for describing weakly nonlinear solutions near onset for solutions near a stripe state



Amplitude Equations

Linear onset solution for stripes

$$\delta \mathbf{u}_{\mathbf{q}}(\mathbf{x}_\perp, z, t) = \underbrace{\left[a_0 e^{i(\mathbf{q}-\mathbf{q}_c)\cdot\mathbf{x}_\perp} e^{\mathrm{Re}\,\sigma_{\mathbf{q}} t}\right]}_{\text{Small terms near onset}} \times \underbrace{\left[\mathbf{u}_{\mathbf{q}}(z)\, e^{i\mathbf{q}_c\cdot\mathbf{x}_\perp}\right]}_{\text{Onset solution}} + \text{c.c.}$$

Weakly nonlinear, slowly modulated, solution

$$\delta \mathbf{u}(\mathbf{x}_\perp, z, t) \approx \underbrace{A(\mathbf{x}_\perp, t)}_{\text{Complex amplitude}} \times \underbrace{\left[\mathbf{u}_{\mathbf{q}_c}(z)\, e^{i\mathbf{q}_c\cdot\mathbf{x}_\perp}\right]}_{\text{Onset solution}} + \text{c.c.}$$

$A(\mathbf{x}_\perp, t)$ is the order parameter for the stripe state

$A(\mathbf{x}_\perp, t)$ satisfies the amplitude equation. In 1d $[\mathbf{q}_c = q_c\hat{\mathbf{x}},\ A = A(x,t)]$:

$$\tau_0 \partial_t A = \varepsilon A + \xi_0^2 \partial_x^2 A - g_0 |A|^2 A, \qquad \varepsilon = (R - R_c)/R_c$$

Complex Amplitude

Magnitude and phase of A play very different roles

$$A(x,y,t) = a(x,y,t)e^{i\theta(x,y,t)}$$
$$\delta\mathbf{u}(\mathbf{x}_\perp,z,t) = ae^{i\theta} \times e^{iq_c x}\mathbf{u}_{\mathbf{q}_c}(z) + c.c.$$

- magnitude $a = |A|$ gives strength of disturbance
- phase change $\delta\theta$ gives shift of pattern (by $\delta x = \delta\theta/q_c$)— symmetry!
- x-gradient $\partial_x\theta$ gives change of wave number $q = q_c + \partial_x\theta$
  $A = ae^{ikx}$ corresponds to $q = q_c + k$
- y-gradient $\partial_y\theta$ gives rotation of wave vector through angle $\partial_y\theta/q_c$
  (plus $O[(\partial_y\theta)^2]$ change in wave number)

The amplitude equation describes

$$\tau_0\partial_t A = \varepsilon A - g_0|A|^2 A + \xi_0^2\partial_x^2 A$$

growth saturation dispersion/diffusion

Parameters

$$\tau_0 \partial_t A = \varepsilon A + \xi_0^2 \partial_x^2 A - g_0 |A|^2 A,$$

- control parameter $\varepsilon = (R - R_c)/R_c$
- system specific constants τ_0, ξ_0, g_0
  - ◇ τ_0, ξ_0 fixed by matching to linear growth rate $A = a\, e^{i\mathbf{k}\cdot\mathbf{x}_\perp} e^{\sigma_{\mathbf{q}} t}$ gives pattern at $\mathbf{q} = \mathbf{q}_c \hat{x} + \mathbf{k})$

$$\sigma_{\mathbf{q}} = {\tau_0}^{-1}[\varepsilon - {\xi_0}^2 (q - q_c)^2]$$

  - ◇ g_0 by calculating nonlinear state at small ε and $q = q_c$.

Scaling

$$\tau_0 \partial_t A = \varepsilon A + \xi_0^2 \partial_x^2 A - g_0 |A|^2 A, \qquad \varepsilon = \frac{R - R_c}{R_c}$$

Introduce scaled variables

$$x = \varepsilon^{-1/2} \xi_0\ X$$
$$t = \varepsilon^{-1} \tau_0\ T$$
$$A = (\varepsilon/g_0)^{1/2}\ \bar{A}$$

This reduces the amplitude equation to a *universal* form

$$\partial_T \bar{A} = \bar{A} + \partial_X^2 \bar{A} - \left|\bar{A}\right|^2 \bar{A}$$

Since solutions to this equation will develop on scales $X, Y, T, \bar{A} = O(1)$ this gives us scaling results for the physical length scales.

Derivation

$$\tau_0 \partial_t A = \varepsilon A + \xi_0^2 \partial_x^2 A - g_0 |A|^2 A, \qquad \varepsilon = \frac{R - R_c}{R_c}$$

- Expand dynamical equation in powers of A and use symmetry arguments (cf., equilibrium phase transitions where we expand free energy). Equation must be invariant under:
  - ◇ $A(\mathbf{x}_\perp) \to A(\mathbf{x}_\perp)e^{i\Delta}$ with Δ a constant, corresponding to a physical translation
  - ◇ $A(\mathbf{x}_\perp) \to A^*(-\mathbf{x}_\perp)$, corresponding to inversion of the horizontal coordinates (parity symmetry)
- Multiple scales perturbation theory (Newell and Whitehead, Segel 1969)
- Mode projection (MCC 1980)

Amplitude Equation = Ginzburg Landau equation

$$\tau_0 \partial_t A = \varepsilon A + \xi_0^2 \partial_x^2 A - g_0 |A|^2 A,$$

Familiar from other branches of physics:

- Good: take intuition from there
- Bad: no *really* new effects

e.g. equation is relaxational (potential, Lyapunov)

$$\tau_0 \partial_t A = -\frac{\delta V}{\delta A^*}, \qquad V = \int dx \left[-\varepsilon |A|^2 + \tfrac{1}{2} g_0 |A|^4 + \xi_0^2 |\partial_x A|^2 \right]$$

This leads to

$$\frac{dV}{dt} = -\tau_0^{-1} \int dx\, |\partial_t A|^2 \le 0$$

and dynamics runs “down hill” to a minimum of V— no chaos!

- We have arrived at the same Landau type formulation with an effective “potential” or “free energy” V!
- This is not fundamental, and is “luck” resulting from our expansion in ε to lowest order
  - ◇ no effective potential at higher order
  - ◇ no effective potential for some side-wall boundary conditions
  - ◇ no effective potential for rotating convection (and there is chaos at onset!)

Applications

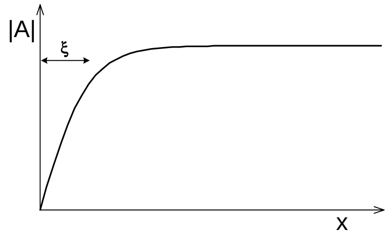
What we can calculate:

- Effect of distant sidewalls
- Eckhaus instability
- Propagation of pattern into no pattern region (e.g., from localized initial condition
- Evolution from random initial condition
- …

Example: Effect of Distant Sidewalls

One dimensional geometry with sidewalls that suppress the pattern (e.g. rigid walls in a convection system)

$$\partial_T \bar{A} = \bar{A} + \partial_X^2 \bar{A} - \left|\bar{A}\right|^2 \bar{A} \qquad \bar{A}(0) = 0$$



$$\bar{A} = e^{i\theta} \tanh(X/\sqrt{2})$$

Unscaled variables:

$$A = e^{i\theta}(\varepsilon/g_0)^{1/2} \tanh(x/\xi) \qquad \text{with} \qquad \xi = \sqrt{2}\varepsilon^{-1/2}\xi_0$$

Solution

$$A = e^{i\theta}(\varepsilon/g_0)^{1/2} \tanh(x/\xi)$$

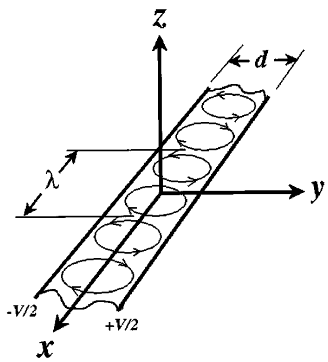
- suppression of pattern over length $\varepsilon^{-1/2}\xi_0$
- arbitrary position of rolls
- asymptotic wave number is $k = 0$, giving $q = q_c$: no band of existence

Extended amplitude equation to next order in ε (MCC, Daniels, Hohenberg, and Siggia 1980) shows

- discrete set of roll positions
- solutions restricted to a narrow $O(\varepsilon^1)$ wave number band with wave number far from the wall

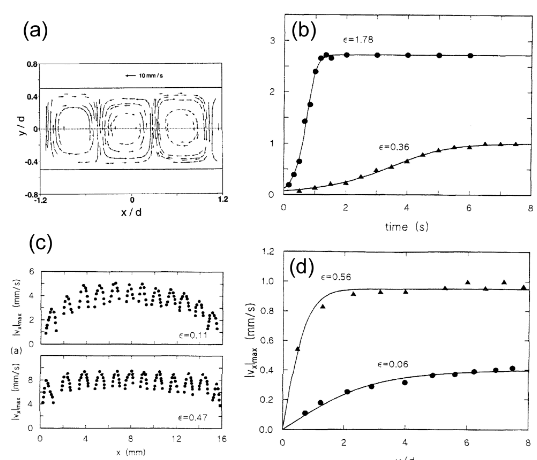
$$\alpha_- \varepsilon < q - q_c < \alpha_+ \varepsilon$$

Electroconvection in a Smectic Film



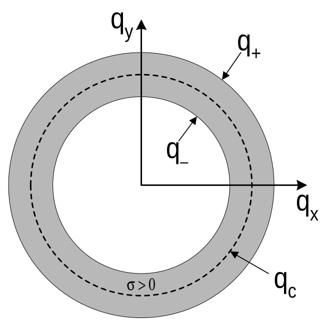
V. B. Deyirmenjian, Z. A. Daya, and S. W. Morris (1997)

Electroconvection in a Smectic Film



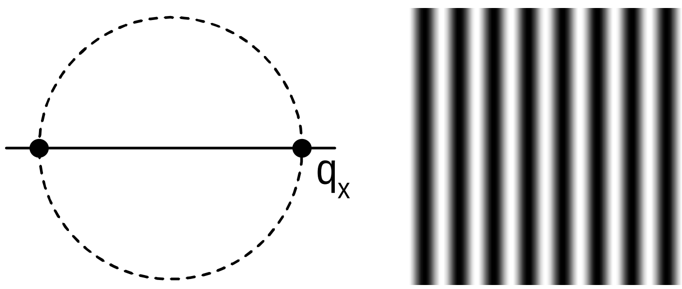
From Morris et al. (1991) and Mao et al. (1996)

Onset in Systems with Rotational Symmetry



- Two dimensional amplitude equation for stripes
- Amplitude equations for lattice states
- Rotationally invariant “model equation”

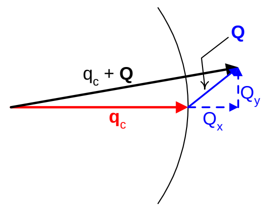
Stripe state



Rotational symmetry: amplitude equation for stripes

For a 2d, rotationally invariant system the gradient term is more complicated

$$\tau_0 \partial_t A = \varepsilon A + \xi_0^2 \left(\partial_x - \frac{i}{2q_c} \partial_y^2 \right)^2 A - g_0 |A|^2 A$$

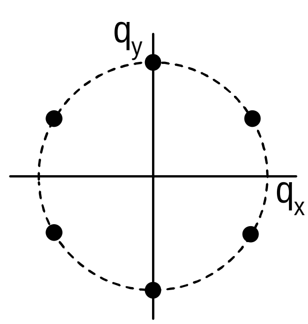


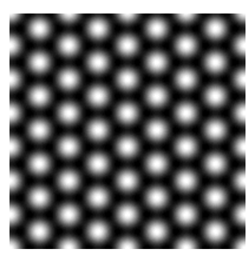
$$q - q_c = \sqrt{(q_c + Q_x)^2 + Q_y^2} - q_c \approx Q_x + \frac{Q_y^2}{2q_c}$$

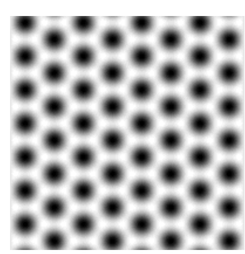
Note: the complex amplitude can only describe *small* reorientations of the stripes.

Isotropic system gives anisotropic scaling: $x = \varepsilon^{-1/2} \xi_0 \, X$; $y = \varepsilon^{-1/4} (\xi_0 / q_c)^{1/2} \, Y$

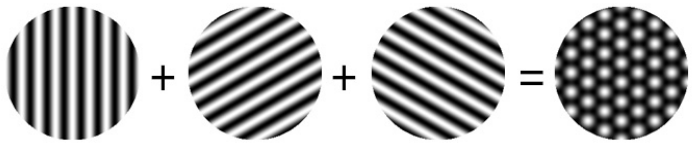
Hexagonal state







Amplitude theory of hexagons



Amplitudes of rolls at 3 orientations $A_i(\mathbf{r}, t),\ i = 1 \ldots 3$

$$dA_1/dt = \varepsilon A_1 - A_1(A_1^2 + gA_2^2 + gA_3^2) + \gamma A_2 A_3$$
$$dA_2/dt = \varepsilon A_2 - A_2(A_2^2 + gA_3^2 + gA_1^2) + \gamma A_3 A_1$$
$$dA_3/dt = \varepsilon A_3 - A_3(A_3^2 + gA_1^2 + gA_2^2) + \gamma A_1 A_2$$

- $A_1 \neq 0$, $A_2 = A_3 = 0$ gives stripes
- $A_1 = A_2 = A_3 \neq 0$ gives hexagons

For $A_i \to -A_i$ symmetry, $\gamma = 0$ and stripes v. hexagons depends on g

For no $A_i \to -A_i$ symmetry, $\gamma \neq 0$ and always get hexagons at onset

Swift-Hohenberg Equation

Rotationally invariant formulation in terms of a scalar field $\psi(x, y, t)$ that captures the same physics as the amplitude equation

$$\partial_t \psi = \left[\varepsilon - (\nabla_\perp^2 + 1)^2\right]\psi - \psi^3 \qquad [\boldsymbol{\nabla}_\perp = (\partial/\partial x, \partial/\partial y)]$$

- originally introduced to investigate *universal* aspects of the transition to stripes
- later used to study qualitative aspects of stripe pattern formation
- no systematic derivation: model rather than controlled approximation
- equation is again relaxational

$$\partial_t \psi = -\frac{\delta V}{\delta \psi}, \qquad V = \iint dx dy \left\{-\tfrac{1}{2}\varepsilon\psi^2 + \tfrac{1}{2}\left[(\nabla_\perp^2 + 1)\psi\right]^2 + \tfrac{1}{4}\psi^4\right\}$$

Motivation

- Mode amplitude $\psi_{\mathbf{q}}(t)$ at wave vector $\mathbf{q}$ satisfies linear equation (for $q \simeq q_c$)

$$\dot{\psi}_{\mathbf{q}} = \tau_0^{-1}[\varepsilon - \xi_0^2(q - q_c)^2]\psi_{\mathbf{q}}$$

- To be able to write this as a local equation for the Fourier transform $\psi(x, y, t)$ approximate this by

$$\dot{\psi}_{\mathbf{q}} = \tau_0^{-1}[\varepsilon - (\xi_0^2/4q_c^2)(q^2 - q_c^2)^2]\psi_{\mathbf{q}}$$

- In real space this gives

$$\tau_0\dot{\psi}(x, y, t) = \varepsilon\psi - (\xi_0^2/4q_c^2)(\nabla_\perp^2 + q_c^2)^2\psi$$

  Simplest linear pde that gives the ring of unstable modes (for $\varepsilon > 0$)

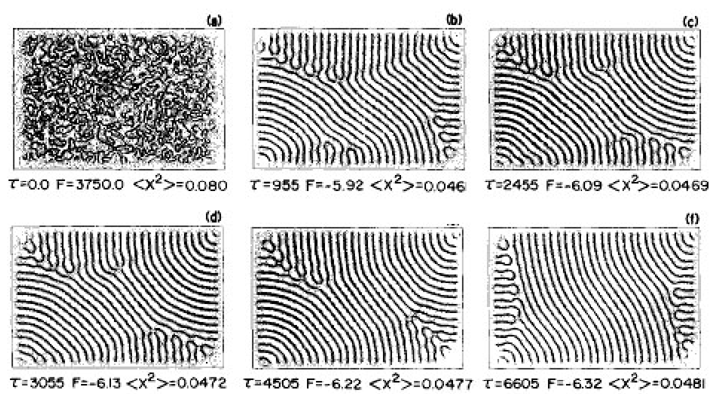
- Add simplest possible nonlinear saturating term

$$\tau_0\dot{\psi}(x, y, t) = \varepsilon\psi - (\xi_0^2/4q_c^2)(\nabla_\perp^2 + q_c^2)^2\psi - g_0\psi^3$$

- Alternative motivation:

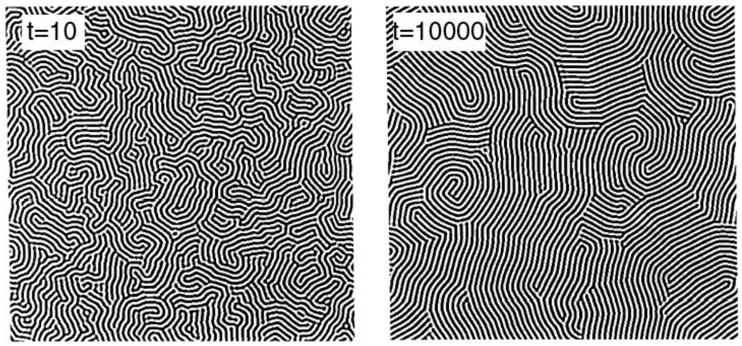
$$A(x, y)e^{iq_c x} \Rightarrow \psi(x, y)$$

Relaxation to steady state



(from Greenside and Coughran, 1984)

Coarsening in a periodic geometry



(From Elder, Vinals, and Grant 1992)

Generalized Swift-Hohenberg models

Qualitatively include other physics:

- break $\psi \to -\psi$ symmetry

$$\partial_t \psi = \left[r - (\nabla_\perp^2 + 1)^2\right] \psi + \gamma \psi^2 - \psi^3$$

- change nonlinearity to make equation non-potential, e.g.

$$\partial_t \psi = \left[r - (\nabla_\perp^2 + 1)^2\right] \psi + (\nabla_\perp \psi)^2 \nabla_\perp^2 \psi$$

- model effects of rotation

$$\partial_t \psi = \left[r - (\nabla_\perp^2 + 1)^2\right] \psi - \psi^3 + g_2 \hat{\mathbf{z}} \cdot \nabla_\perp \times [(\nabla_\perp \psi)^2 \nabla_\perp \psi] + g_3 \nabla_\perp \cdot [(\nabla_\perp \psi)^2 \nabla_\perp \psi]$$

Conclusions

I have introduced the ideas and methods used to understand nonlinear patterns, focussing on the regime near threshold.

Next Lecture: Symmetry Aspects of Nonlinear Patterns

- Analogies with and differences from equilibrium phase transitions
- Phase variable description
- Topological defects