Collective Effects

in

Equilibrium and Nonequilibrium Physics

Website: http://cncs.bnu.edu.cn/mccross/Course/

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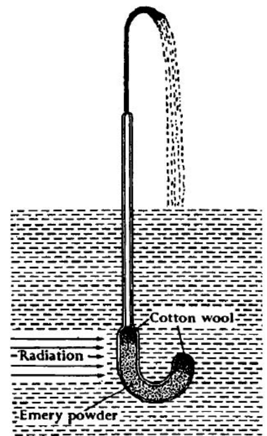
Today's Lecture

Superfluids and superconductors

- What are superfluidity and superconductivity?
- Review of phase dynamics
- Description in terms of a macroscopic phase
- Supercurrents that flow for ever
- Josephson effect
- Four sounds

The Amazing World of Superfluidity and Superconductivity

- Electric currents in loops that flow for ever (measured for $\sim$ decade)
- Beakers of fluid that empty themselves
- Fluids that flow without resistance through tiny holes
- Flow in surface films less than an atomic layer thick
- Flow driven by temperature differences (fountain effect)



History of Superfluidity and Superconductivity

1908 Liquefaction of ^{4}He by Kamerlingh Onnes

1911 Discovery of superconductivity by Onnes (resistance drops to zero)

1933 Meissner effect: superconductors expel magnetic field

1937 Discovery of superfluidity in ^{4}He by Allen and Misener

1938 Connection of superfluidity with Bose-Einstein condensation by London

1955 Feynman's theory of quantized vortices

1956 Onsager and Penrose identify the broken symmetry in superfluidity ODLRO

1957 BCS theory of superconductivity

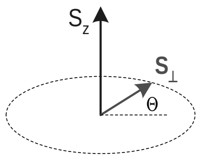
1962 Josephson effect

1973 Discovery of superfluidity in ^{3}He at 2mK by Osheroff, Lee, and Richardson

1986 Discovery of high-T_c superconductors by Bednorz and Müller

1995- Study of superfluidity in ultracold trapped dilute gases

Review of Phase Dynamics with a Conserved Quantity



Rotational symmetry in the XY plane (angle Θ)
The XY and Z components of the spin have different properties:

$$S_z = \Omega^{-1} \sum_{i \text{ in } \Omega} \langle s_{iz} \rangle \qquad \text{is a } \textit{conserved quantity}$$
$$\mathbf{S}_\perp = \Omega^{-1} \sum_{i \text{ in } \Omega} \langle \mathbf{s}_{i\perp} \rangle \qquad \text{is the XY } \textit{order parameter}$$

S_z and Θ are canonically conjugate variables, so that with the free energy

$$F = \int d^d x \left[\frac{1}{2} K (\nabla\Theta)^2 + \frac{S_z^2}{2\chi} - S_z b_z \right]$$

we get

$$\dot{S}_z = -\frac{\delta F}{\delta \Theta} \qquad \text{giving} \qquad \dot{S}_z = -\nabla \cdot \mathbf{j}_{S_z} \quad \text{with} \quad \mathbf{j}_{S_z} = -K \nabla \Theta$$
$$\dot{\Theta} = \frac{\delta F}{\delta S_z} \qquad \text{giving} \qquad \dot{\Theta} = \chi^{-1}(S_z - \chi b_z)$$

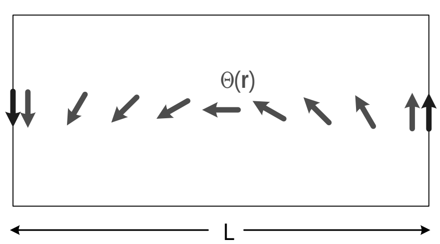
Spin Current

$$\dot{S}_z = -\nabla \cdot \mathbf{j}_{S_z}$$

This is a conservation law with a current $\mathbf{j}_{S_z}$ of the conserved quantity S_z given by a phase gradient

$$\mathbf{j}_{S_z} = -K \nabla \Theta$$

For example



Phase Dynamics

$$\dot{\Theta} = \chi^{-1}(S_z - \chi b_z)$$

- No dynamics in full thermodynamic equilibrium: $S_z = \chi b_{0z}$
- Add an additional external field $b_{1z} = \gamma B_{1z}$

$$\dot{\Theta} = -b_{1z} = -\gamma B_{1z}$$

  the usual precession of a magnetic moment in an applied field (Larmor precession).
- Note that this is an equilibrium state: $S_z \neq \chi(b_{0z} + b_{1z})$ but is a conserved quantity— no approximations
- For formal proof see Halperin and Saslow, Phys. Rev. B 16, 2154 (1977), Appendix: *"the Larmor precession theorem"*

Hydrodynamic Approach

Hydrodynamics: a formal derivation of long wavelength dynamics of conserved quantities and broken symmetry variables in a thermodynamic approach

Starting points

- generalized rigidity: extra contribution to the energy density from gradients of the broken symmetry variable

$$\varepsilon = \frac{1}{2}K(\nabla\Theta)^2$$

- thermodynamic identity

$$d\varepsilon = T\,ds + \mu_z ds_z + \Phi \cdot d(\nabla\Theta) \text{ with } \Phi = K\nabla\Theta$$

- equilibrium phase dynamics (Larmor precession theorem)

$$\dot{\Theta} = \mu_z$$

Derive

- dynamical equations for conserved quantities and broken symmetry variables for slowly varying disturbances

Rigidity and the Thermodynamic Identity

In terms of the energy density

$$d\varepsilon = T ds + \mu_z ds_z + \Phi \cdot d(\nabla\Theta)$$

- *conjugate fields* are

$$\mu_z = \left(\frac{\partial \varepsilon}{\partial s_z}\right)_{s,\nabla\Theta} \qquad \text{and} \qquad \Phi = \left(\frac{\partial \varepsilon}{\partial \nabla\Theta}\right)_{s,\mathbf{s}_z}$$

Or with the free energy density $f = \varepsilon - Ts$

$$df = -s dT + \mu_z ds_z + \Phi \cdot d(\nabla\Theta)$$

- *conjugate fields* are

$$\mu_z = \left(\frac{\partial f}{\partial s_z}\right)_{T,\nabla\Theta} \qquad \text{and} \qquad \Phi = \left(\frac{\partial f}{\partial \nabla\Theta}\right)_{T,\mathbf{s}_z}$$

These give

$$\mu_z = \chi^{-1}(S_z - \chi b_z) \qquad \text{and} \qquad \Phi = K\nabla\Theta$$

Entropy Production

$$T ds = d\varepsilon - \mu_z ds_z - \Phi \cdot d(\nabla\Theta)$$

- Form time derivative of entropy density

$$\frac{ds}{dt} = \frac{1}{T}\frac{d\varepsilon}{dt} - \frac{\mu_z}{T}\frac{ds_z}{dt} - \frac{\Phi}{T} \cdot \frac{d(\nabla\Theta)}{dt}$$

- Conservation laws and dynamics of broken symmetry variable ($\mathbf{j}^\varepsilon$, $\mathbf{j}^{s_z}$ unknown)

$$\frac{ds}{dt} = -\frac{1}{T}\nabla \cdot \mathbf{j}^\varepsilon + \frac{\mu_z}{T}\nabla \cdot \mathbf{j}^{s_z} - \frac{\Phi}{T} \cdot \nabla\mu_z$$

- Entropy production equation

$$\frac{ds}{dt} = -\nabla \cdot \mathbf{j}^s + R \qquad \text{with} \qquad R \geq 0$$

- Identify the entropy current and production

$$\mathbf{j}^s = T^{-1}(\mathbf{j}^\varepsilon - \mu_z \mathbf{j}^{s_z})$$
$$RT = -T^{-1}(\mathbf{j}^\varepsilon - \mu_z \mathbf{j}^{s_z}) \cdot \nabla T - (\mathbf{j}^{s_z} + \Phi) \cdot \nabla\mu_z$$

Equilibrium Dynamics

Entropy Production

$$RT = -T^{-1}(\mathbf{j}^{\varepsilon} - \mu_z \mathbf{j}^{s_z}) \cdot \nabla T - (\mathbf{j}^{s_z} + \Phi) \cdot \nabla \mu_z$$

(strategy: R should be a function of gradients of the conjugate variables)

In the absence of dissipation the rate of entropy production must be zero.

- Spin current

$$\mathbf{j}^{s_z} = -\Phi = -K \nabla \Theta$$

- Energy current

$$\mathbf{j}^{\varepsilon} = \mu_z \mathbf{j}^{s_z} = -\mu_z K \nabla \Theta$$

- Entropy current

$$\mathbf{j}^{s} = 0$$

We will consider adding dissipation later.

Superfluidity

- superfluidity occurs due to Bose condensation
- the order parameter is "the expectation value of the quantum field operator for destroying a particle" $\Psi = \langle \psi \rangle$
- Ψ is a complex variable: $\Psi = |\Psi| e^{i\Theta}$
  - ◇ $|\Psi|^2$ gives the "condensate density" n_0: the fraction of particles in the zero momentum state is n_0/n
  - ◇ Θ is the phase of the condensate wave function
  - ◇ There are a macroscopic number of particles in a single wave function and so Θ is a macroscopic thermodynamic variable, and is the broken symmetry variable.

Broken Phase (Gauge) Symmetry

- Any phase gives an equivalent state; the ordered state is characterized by a particular phase
- There is an energy cost for gradients of the phase

$$E = \frac{1}{2} n_s \frac{\hbar^2}{m} \int (\nabla\Theta)^2 \, d^d x$$

  - ◇ Stiffness constant K is written as $n_s(\hbar^2/m)$ and n_s is called the *superfluid density*
  - ◇ Stiffness constant *not* the same as the condensate density $n_s \neq n_0$
- Conjugate variable to the phase Θ is the number of particles N
- Currents and dynamics of the phase are coupled to the density, i.e., mass or electric currents
- Currents are present *in equilibrium*, and so are *supercurrents*

Supercurrents by Analogy

- One-to-one correspondence at the quantum operator level

$$\hbar N \equiv S_z \qquad \text{and} \qquad \Theta_{\text{phase}} \equiv -\Theta_{\text{spin}}$$

  (e.g., $\uparrow\equiv$ particle, $\downarrow\equiv$ no particle)
- Gradient of the phase gives a flow of particles

$$\frac{\partial n}{\partial t} = -\nabla \cdot \mathbf{j} \qquad \text{with} \qquad \mathbf{j} = n_s(\hbar/m)\nabla\Theta$$

- Often associate a flow with a velocity: introduce superfluid velocity $\mathbf{v}_s = (\hbar/m)\nabla\Theta$ and then $\mathbf{j} = n_s \mathbf{v}_s$
- Or write in terms of flow of mass

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{g} \qquad \text{with} \qquad \mathbf{g} = \rho_s \mathbf{v}_s, \;\; \rho_s = m n_s$$

Hydrodynamic Derivation

- Free energy expression: generalized rigidity and energy in external potential V

$$f = \frac{\hbar^2 n_s}{2m}(\nabla\Theta)^2 + \frac{1}{2}Kn^2 + Vn$$

(K is bulk modulus)

- Equilibrium phase dynamics (Larmor precession theorem) from dynamics with added constant potential δV:

$$\Psi(V,t) = \Psi(0,t)e^{-i\hbar N\delta V t}$$

gives

$$\hbar\dot{\Theta} = -\delta V \qquad \text{or in general} \qquad \hbar\dot{\Theta} = -\left(\frac{\partial f}{\partial n}\right)_T = -\mu$$

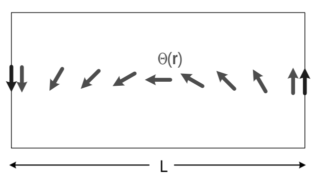
- Entropy production argument from the thermodynamic identity

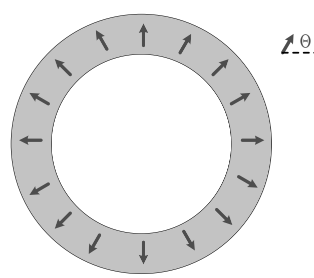
$$d\varepsilon = T ds + \mu dn + \Phi \cdot d(\nabla\Theta) \qquad \text{with} \qquad \Phi = (\hbar^2 n_s/m)\nabla\Theta$$

gives the current of particles

$$\dot{n} = -\nabla \cdot \mathbf{j} \qquad \text{with} \qquad \mathbf{j} = n_s(\hbar/m)\nabla\Theta$$

Currents that Flow Forever





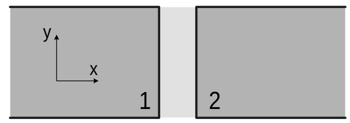
$$g = \rho_s(\hbar/m)(\pi/L)$$

$$g = \rho_s(\hbar/m)(2\pi/L)$$

$$\oint \mathbf{v}_s \cdot \mathbf{dl} = \frac{h}{m}$$

quantum of circulation

Josephson Effect



- Energy depends on phase difference. For weak coupling

$$E = -J_c \cos(\Theta_2 - \Theta_1)$$

- Change in number of particles: current $I = dN_2/dt$

$$I = \frac{dE}{d\Theta_2}$$

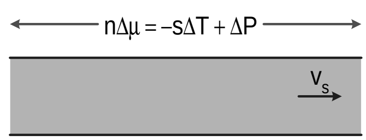
d.c. Josephson effect $\quad I = J_c \sin(\Theta_2 - \Theta_1)$

- Time dependence of phase is given by the potential

$$\hbar \dot{\Theta}_i = -\mu_i$$

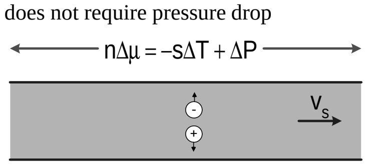
a.c. Josephson effect $\quad \hbar(\dot{\Theta}_2 - \dot{\Theta}_1) = -\Delta\mu$

Breakdown of Superfluidity



$$\hbar \Delta\dot{\Theta} = -\Delta\mu, \qquad v_s = (\hbar/m) n_s \Delta\Theta / L$$

- pressure or temperature difference accelerates superflow
- constant superflow does not require pressure drop



- pressure drop (dissipation) requires passage of vortex topological defects ("quantized vortex lines") across flow channel
- presence of dissipation depends on whether vortices can be produced by thermal activation or other mechanism

Josephson Effect for a Superconductor

- Θ is phase of *pair* wave function
- expressions must be *gauge invariant* in presence of vector potential

For bulk material

$$\text{Supercurrent:} \quad \mathbf{j} = n_s \frac{\hbar}{2m}\left(\nabla\Theta(\mathbf{x}) + \frac{2e}{\hbar c}\mathbf{A}\right)$$

$$\text{Josephson equation:} \quad \hbar\dot{\Theta} = 2eV$$

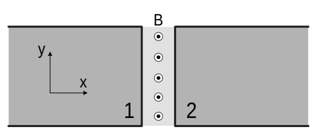
For Josephson junction, current is $I = \int j(y,z)\,dy\,dz$ with

$$j(y,z) = j_c \sin\left(\Theta_2 - \Theta_1 + \frac{2e}{\hbar c}\int_1^2 A_x dx\right)$$

and

$$V = (\hbar/2e)(\dot{\Theta}_2 - \dot{\Theta}_1)$$

Josephson Junction in a Magnetic Field



$$\text{Josephson current density:} \quad j(y,z) = j_c \sin\left(\Theta_2 - \Theta_1 + \frac{2e}{\hbar c}\int_1^2 A_x dx\right)$$

For field $B\hat{\mathbf{z}}$ in junction the vector potential is $\mathbf{A} = -By\hat{\mathbf{x}}$, so that

$$I \propto \int dy\ j_c \sin[\Theta_2 - \Theta_1 - (2e/\hbar c)Byd]$$

giving

$$I = I_c(B)\sin(\Theta_2 - \Theta_1) \qquad \text{with} \qquad I_c(B) = \frac{\sin(\pi\phi/\phi_0)}{\pi\phi/\phi_0}$$

where $\phi = Bld$ is the flux through the junction and $\phi_0 = hc/2e$ is the *flux quantum* $(2.1 \times 10^{-7}\text{gauss cm}^2)$

Experimental Discovery of the dc Josephson Effect

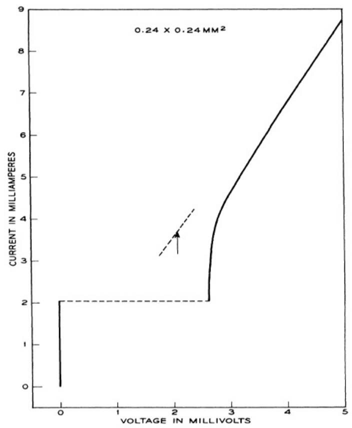


FIG. 1. Current-voltage characteristic for a Pb-I-Pb junction at 1.3°K. The arrow marks the predicted maximum magnitude of the Josephson current.

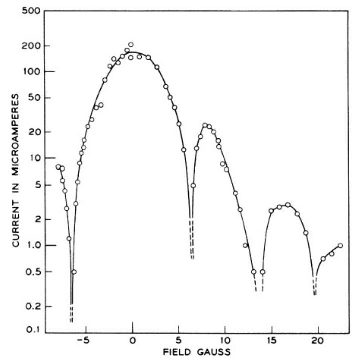
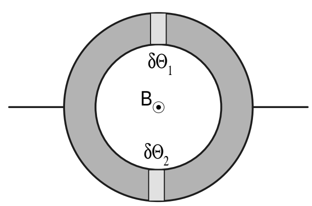


FIG. 3. The field dependence of the Josephson current in a Pb-I-Pb junction at 1.3°K.

Rowell, Phys. Rev. Lett. **11**, 200 (1963)

SQUID

Superconducting Quantum Interference Device



Integrate $\mathbf{j} = n_s \frac{\hbar}{2m}\left(\nabla\Theta(\mathbf{x}) + \frac{2e}{\hbar c}\mathbf{A}\right)$ around whole loop using fact that current $\mathbf{j}$ is small

$$\delta\Theta_1 - \delta\Theta_2 = \frac{2e}{\hbar c}\oint \mathbf{A}\cdot\mathbf{dl} = 2\pi\phi/\phi_0$$

with $\phi = B \times$ area, the flux through the loop.

Total current

$$\begin{aligned} I &= J_c\,[\sin\delta\Theta_1 + \sin\delta\Theta_2] \\ &= 2J_c\sin(\pi\phi/\phi_0)\sin[\tfrac{1}{2}(\delta\Theta_1 + \delta\Theta_2)] \end{aligned}$$

Maximum current varies periodically with applied field — very sensitive magnetometer.

Four Sounds in a Superfluid

Equations of motion for conserved quantities

$$\dot{\rho} = -\nabla \cdot \mathbf{g}$$
$$\dot{\mathbf{g}} = -\nabla P$$
$$\dot{s} = 0$$

and the dynamics of the broken symmetry variable

$$\hbar\dot{\Theta} = -\mu$$

which can be written as

$$\rho\dot{\mathbf{v}}_s = s\nabla T - \nabla P$$

Need to connect the momentum density to the superfluid velocity.

Galilean Invariance

Transform to frame with a velocity $-\mathbf{v}_n$:

- Momentum density

$$\mathbf{g} = \rho_s \mathbf{v}_s{}^{(0)} + \rho \mathbf{v}_n$$

  Define the "normal fluid density" $\rho_n = \rho - \rho_s$ and write the transformed superfluid velocity $\mathbf{v}_s = \mathbf{v}_s^{(0)} + \mathbf{v}_n$

$$\mathbf{g} = \rho_s \mathbf{v}_s + \rho_n \mathbf{v}_n$$

- Entropy current

$$\mathbf{j}^s = s\mathbf{v}_n$$

- Momentum equation can be transformed to

$$\rho_s\dot{\mathbf{v}}_s + \rho_n\dot{\mathbf{v}}_n = -\nabla P$$

  and using the equation for $\dot{\mathbf{v}}_s$ in the form

$$(\rho_s + \rho_n)\dot{\mathbf{v}}_s = s\nabla T - \nabla P$$

  gives

$$\rho_n(\dot{\mathbf{v}}_s - \dot{\mathbf{v}}_n) = s\nabla T$$

First Sound

Usual coupled density and momentum equations

$$\dot{\rho} = -\nabla \cdot \mathbf{g}$$
$$\dot{\mathbf{g}} = -\nabla P$$

and the pressure-density relationship (K is the bulk modulus)

$$\delta P = K\, \delta\rho/\rho$$

These give first sound waves $\propto e^{i(\mathbf{q}\cdot\mathbf{r}-\omega t)}$ propagating with the usual sound speed $\omega = c_1 q$ with

$$c_1 = \sqrt{\frac{K}{\rho}}$$

Second Sound

Coupled counterflow and entropy wave. Use $c_2 \ll c_1 \Rightarrow$ density constant, $\mathbf{g} = 0$

$$\rho_s \mathbf{v}_s + \rho_n \mathbf{v}_n = 0 \qquad \Rightarrow \qquad \mathbf{v}_s - \mathbf{v}_n = -(\rho/\rho_s)\mathbf{v}_n$$

(remember $\rho_s + \rho_n = \rho$).

Entropy equation: $\dot{s} = -s\nabla \cdot \mathbf{v}_n$

Entropy-temperature relationship (C is the specific heat): $\delta s = C\delta T/T$

$$C\dot{T} = sT(\rho_s/\rho)\nabla \cdot (\mathbf{v}_s - \mathbf{v}_n)$$

Counterflow equation

$$\rho_n(\dot{\mathbf{v}}_s - \dot{\mathbf{v}}_n) = s\nabla T$$

These give propagating second sound waves with the speed

$$c_2 = \sqrt{\frac{\rho_s}{\rho_n}\frac{s^2 T}{\rho C}}$$

Fourth Sound

Fluid confined in porous media: no conserved momentum, no Galilean invariance (no $\mathbf{v}_n$), temperature constant

$$\begin{aligned} \dot{\rho} &= -\nabla \cdot \mathbf{g} \\ \mathbf{g} &= \rho_s \mathbf{v}_s \\ \rho \dot{\mathbf{v}}_s &= -\nabla P \\ \delta P &= -K \delta\rho/\rho \end{aligned}$$

These gives a fourth sound wave propagating with the speed

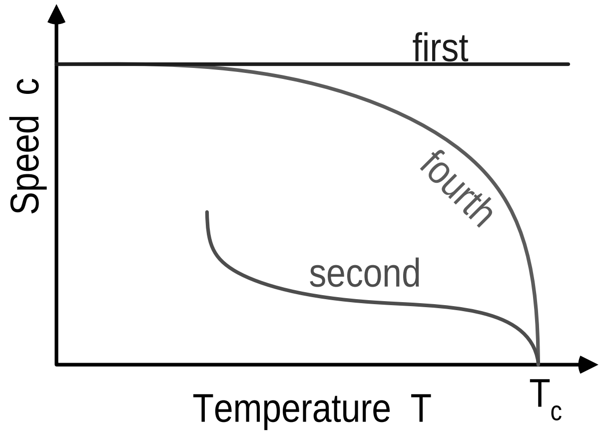
$$c_4 = \sqrt{\frac{\rho_s}{\rho}\frac{K}{\rho}}$$

Third Sound

Wave propagating in thin films down to atomic layer thickness.

Like fourth sound, but involve changes of thickness rather than density, and effective compressibility depends on strength of interaction with surface.

Sounds in Helium-4



$$c_1 = \sqrt{\frac{K}{\rho}}, \qquad c_2 = \sqrt{\frac{\rho_s}{\rho_n}\frac{s^2 T}{\rho C}}, \qquad c_4 = \sqrt{\frac{\rho_s}{\rho}\frac{K}{\rho}}$$

Next Lecture

Onsager theory and the fluctuation-dissipation theorem

- Derivation and discussion
- Application to nanomechanics and biodetectors