ROTA–BAXTER ALGEBRAS, SINGULAR HYPERSURFACES, AND RENORMALIZATION ON KAUSZ COMPACTIFICATIONS

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Abstract. We consider Rota-Baxter algebras of meromorphic forms with poles along a (singular) hypersurface in a smooth projective variety and the associated Birkhoff factorization for algebra homomorphisms from a commutative Hopf algebra. In the case of a normal crossings divisor, the Rota-Baxter structure simplifies considerably and the factorization becomes a simple pole subtraction. We apply this formalism to the unrenormalized momentum space Feynman amplitudes, viewed as (divergent) integrals in the complement of the determinant hypersurface. We lift the integral to the Kausz compactification of the general linear group, whose boundary divisor is normal crossings. We show that the Kausz compactification is a Tate motive and the boundary divisor is a mixed Tate configuration. The regularization of the integrals that we obtain differs from the usual renormalization of physical Feynman amplitudes, and in particular it always gives mixed Tate periods.

1. Introduction

In this paper, we consider the problem of extracting periods of algebraic varieties from a class of divergent integrals arising in quantum field theory. The method we present here provides a regularization and extraction of finite values that differs from the usual (renormalized) physical Feynman amplitudes, but whose mathematical interest lies in the fact that it always gives a period of a mixed Tate motive. The motive considered is provided by the Kausz compactification of the general linear group. The regularization procedure we propose is modeled on the algebraic renormalization method, based on Hopf algebras of graphs and Rota–Baxter algebras, as originally developed by Connes and Kreimer [13] and by Ebrahmi-Fard, Guo, and Kreimer [19]. The main difference in our approach is that we apply the formalism to a Rota–Baxter algebra of (even) meromorphic differential forms instead of applying it to a regularization of the integral. The procedure becomes especially simple in cases where the deRham cohomology of the singular hypersurface complement is all realized by forms with logarithmic poles, in which case we replace the divergent integral with a family of convergent integrals obtained by a pole subtraction on the form and by (iterated) Poincaré residues. A similar approach was developed for integrals in configuration spaces by Ceyhan and the first author [12].

In Section 2 we introduce Rota–Baxter algebras of even meromorphic forms, along the lines of [12], and we formulate a general setting for extraction of finite values (regularization and renormalization) of divergent integrals modeled on algebraic renormalization applied to these Rota–Baxter algebras of differential forms.

In Section 3 we discuss the Rota–Baxter algebras of even meromorphic forms in the case of a smooth hypersurface $Y \subset X$. We show that, when restricted to forms with logarithmic poles, the Rota–Baxter operator becomes simply a derivation, and the Birkhoff factorization collapses to a simple pole subtraction, as in the case of log divergent graphs. We show that this simple pole subtraction can lead to too much loss of information about the unrenormalized integrand and we propose considering the additional information of the Poincaré residue and an additional integral associated to the residue.

In Section 4 we consider the case of singular hypersurfaces $Y \subset X$ given by a simple normal crossings divisor. We show that, in this case, the Rota–Baxter operator satisfies a simplified form of the Rota–Baxter identity, which however is not just a derivation. We show that this modified identity still suffices to have a simple pole subtraction $\phi_+(X) = (1 - T)\phi(X)$ in the Birkhoff factorization, even though the negative piece $\phi_-(X)$ becomes more complicated. Again, to avoid too much loss of information in passing from $\phi(X)$ to $\phi_+(X)$, we consider, in addition to the renormalized integral $\int_\sigma \phi_+(X)$, the collection of integrals of the form $\int_{\sigma\cap Y_I} \mathrm{Res}_{Y_I}(\phi(X))$, where Res_{Y_I} is the iterated Poincaré residue ([1], [2]), along the intersection $Y_I = \cap_{j\in I} Y_j$ of components of Y. These integrals are all periods of mixed Tate motives if $\{Y_I\}$ is a mixed Tate configuration, in the sense of [21]. We discuss the question of further generalizations to more general types of singularities, beyond the normal crossings case, via Saito's theory of forms with logarithmic poles [36], by showing that one can also define a Rota–Baxter structure on the Saito complex of forms with logarithmic poles.

In Section 5 we present our main application, which is a regularization (different from the physical one) of the Feynman amplitudes in momentum space, computed on the complement of the determinant hypersurface as in [4]. Since the determinant hypersurface has worse singularities than what we need, we pull back the integral computation to the Kausz compactification [29] of the general linear group, where the boundary divisor that replaces the determinant hypersurface is a simple normal crossings divisor. We show that the (numerical pure) motive of the Kausz compactification is Tate, and that the components of the boundary divisor form a mixed Tate configuration. We discuss extensions of the result to Chow motives.

2. Rota–Baxter algebras of meromorphic forms

We generalize the algebraic renormalization formalism to a setting based on Rota–Baxter algebras of algebraic differential forms on a smooth projective variety with poles along a hypersurface.

2.1. **Rota–Baxter algebras.**

A Rota–Baxter algebra of weight λ is a unital commutative algebra $\mathcal{R}$ together with a linear operator $T : \mathcal{R} \to \mathcal{R}$ satisfying the Rota–Baxter identity

$$T(x)T(y) = T(xT(y)) + T(T(x)y) + \lambda T(xy). \tag{2.1}$$

For example, Laurent polynomials $\mathcal{R} = \mathbb{C}[z, z^{-1}]$ with T the projection onto the polar part are a Rota–Baxter algebra of weight -1.

The Rota–Baxter operator T of a Rota–Baxter algebra of weight -1, satisfying

$$T(x)T(y) + T(xy) = T(xT(y)) + T(T(x)y), \tag{2.2}$$

determines a splitting of $\mathcal{R}$ into $\mathcal{R}_+ = (1 - T)\mathcal{R}$ and $\mathcal{R}_-$, the unitization of $T\mathcal{R}$, where both $\mathcal{R}_\pm$ are algebras. For an introduction to Rota–Baxter algebras we refer the reader to [22].

2.2. **Rota–Baxter algebras of even meromorphic forms.**

Let Y be a hypersurface in a projective variety X, with defining equation $Y = \{f = 0\}$. We denote by $\mathcal{M}^\star_X$ the sheaf of meromorphic differential forms on X, and by $\mathcal{M}^\star_{X,Y}$ the subsheaf of meromorphic forms on with poles (of arbitrary order) along Y. It is a graded-commulative algebra over the field of definition of the varieties X and Y. We can write forms $\omega \in \mathcal{M}^\star_{X,Y}$ as sums $\omega = \sum_{p\geq 0} \alpha_p/f^p$, where the α_p are holomorphic forms.

In particular, we consider forms of even degrees, so that $\mathcal{M}^{\mathrm{even}}_{X,Y}$ is a commutative algebra under the wedge product.

Lemma 2.1. *The commutative algebra $\mathcal{M}^{\mathrm{even}}_{X,Y}$, together with the linear operator $T : \mathcal{M}^{\mathrm{even}}_{X,Y} \to \mathcal{M}^{\mathrm{even}}_{X,Y}$ defined as the polar part*

$$T(\omega) = \sum_{p\geq 1} \alpha_p/f^p, \tag{2.3}$$

is a Rota–Baxter algebra of weight -1.

Proof. For $\omega_1 = \sum_{p\geq 0} \alpha_p/f^p$ and $\omega_2 = \sum_{q\geq 0} \beta_q/f^q$, we have

$$T(\omega_1 \wedge \omega_2) = \sum_{p\geq 0, q\geq 1} \frac{\alpha_p \wedge \beta_q}{f^{p+q}} + \sum_{p\geq 1, q\geq 0} \frac{\alpha_p \wedge \beta_q}{f^{p+q}} - \sum_{p\geq 1, q\geq 1} \frac{\alpha_p \wedge \beta_q}{f^{p+q}},$$

$$T(T(\omega_1) \wedge \omega_2) = \sum_{p\geq 1, q\geq 0} \frac{\alpha_p \wedge \beta_q}{f^{p+q}},$$

$$T(\omega_1 \wedge T(\omega_2)) = \sum_{p\geq 0, q\geq 1} \frac{\alpha_p \wedge \beta_q}{f^{p+q}},$$

$$T(\omega_1) \wedge T(\omega_2) = \sum_{p\geq 1, q\geq 1} \frac{\alpha_p \wedge \beta_q}{f^{p+q}},$$

so that (2.2) is satisfied. □

Equivalently, we have the following description of the Rota–Baxter operator.

Corollary 2.2. *The linear operator*

$$T(\omega) = \alpha \wedge \xi, \quad \text{for } \omega = \alpha \wedge \xi + \eta, \tag{2.4}$$

acting on forms $\omega = \alpha \wedge \xi + \eta$, with α a meromorphic form on X with poles on Y and ξ and η holomorphic forms on X, is a Rota–Baxter operator of weight -1.

Proof. For $\omega_i = \alpha_i \wedge \xi_i + \eta_i$, with $i = 1, 2$, we have

$$T(\omega_1 \wedge \omega_2) = (-1)^{|\alpha_2|\,|\xi_1|} \alpha_1 \wedge \alpha_2 \wedge \xi_1 \wedge \xi_2 + \alpha_1 \wedge \xi_1 \wedge \eta_2 + (-1)^{|\eta_1|\,|\alpha_2|} \alpha_2 \wedge \eta_1 \wedge \xi_2$$

while

$$T(T(\omega_1) \wedge \omega_2) = (-1)^{|\alpha_2|\,|\xi_1|} \alpha_1 \wedge \alpha_2 \wedge \xi_1 \wedge \xi_2 + \alpha_1 \wedge \xi_1 \wedge \eta_2$$

$$T(\omega_1 \wedge T(\omega_2)) = (-1)^{|\alpha_2|\,|\xi_1|} \alpha_1 \wedge \alpha_2 \wedge \xi_1 \wedge \xi_2 + (-1)^{|\eta_1|\,|\alpha_2|} \alpha_2 \wedge \eta_1 \wedge \xi_2$$

and

$$T(\omega_1) \wedge T(\omega_2) = (-1)^{|\alpha_2|\,|\xi_1|} \alpha_1 \wedge \alpha_2 \wedge \xi_1 \wedge \xi_2,$$

where all signs are positive if the forms are of even degree. Thus, the operator T satisfies (2.2). □

The following statement is proved exactly as in Theorem 6.4 of [12] and we omit the proof here.

Lemma 2.3. *Let (X_ℓ, Y_ℓ) for $\ell \geq 1$ be a collection of smooth projective varieties X_ℓ with hypersurfaces Y_ℓ, all defined over the same field of definition. Then the commutative algebra $\bigwedge_\ell \mathcal{M}^{\text{even}}_{X_\ell, Y_\ell}$ is a Rota–Baxter algebra of weight -1 with the polar projection operator T determined by the T_ℓ on each $\mathcal{M}^{\text{even}}_{X_\ell, Y_\ell}$.*

2.3. **Renormalization via Rota–Baxter algebras.** In [13], the BPHZ renormalization procedure of perturbative quantum field theory was reinterpreted as a Birkhoff factorization of loops in the pro-unipotent group of characters of a commutative Hopf algebra of Feynman graphs. This procedure of *algebraic renormalization* was reformulated in more general and abstract terms in [19], using Hopf algebras and Rota–Baxter algebras.

We summarize here quickly the basic setup of algebraic renormalization. We refer the reader to [13], [14], [19], [32] for more details.

The Connes–Kreimer Hopf algebra of Feynman graphs $\mathcal{H}$ is the free commutative algebra with generators 1PI Feynman graphs Γ of the theory, with grading by loop number (or better by number of internal edges)

$$\deg(\Gamma_1 \cdots \Gamma_n) = \sum_i \deg(\Gamma_i), \quad \deg(1) = 0$$

and with coproduct

$$\Delta(\Gamma) = \Gamma \otimes 1 + 1 \otimes \Gamma + \sum_{\gamma \in \mathcal{V}(\Gamma)} \gamma \otimes \Gamma/\gamma, \tag{2.5}$$

where the class $\mathcal{V}(\Gamma)$ consists of all (possibly multiconnected) divergent subgraphs γ such that the quotient graph (identifying each component of γ to a vertex) is still a 1PI Feynman graph of the theory. The antipode is constructed inductively as

$$S(X) = -X - \sum S(X')X''$$

for $\Delta(X) = X \otimes 1 + 1 \otimes X + \sum X' \otimes X''$, with the terms X', X'' of lower degrees.

An *algebraic Feynman rule* $\phi : \mathcal{H} \to \mathcal{R}$ is a *homomorphism of commutative algebras* from the Hopf algebra $\mathcal{H}$ of Feynman graphs to a Rota–Baxter algebra $\mathcal{R}$ of weight -1,

$$\phi \in \mathrm{Hom}_{\mathrm{Alg}}(\mathcal{H}, \mathcal{R}).$$

The morphism ϕ by itself does not know about the coalgebra structure of $\mathcal{H}$ and the Rota–Baxter structure of $\mathcal{R}$. These enter in the factorization of ϕ into divergent and finite part.

The Birkhoff factorization of an algebraic Feynman rule consists of a pair of commutative algebra homomorphisms

$$\phi_\pm \in \mathrm{Hom}_{\mathrm{Alg}}(\mathcal{H}, \mathcal{R}_\pm)$$

where $\mathcal{R}_\pm$ is the splitting of $\mathcal{R}$ induced by the Rota–Baxter operator T, with $\mathcal{R}_+ = (1-T)\mathcal{R}$ and $\mathcal{R}_-$ the unitization of $T\mathcal{R}$, satisfying

$$\phi = (\phi_- \circ S) \star \phi_+,$$

where the product $\star$ is dual to the coproduct in the Hopf algebra, $\phi_1 \star \phi_2(X) = \langle \phi_1 \otimes \phi_2, \Delta(X)\rangle$.

As shown in [13], there is an inductive formula for the Birkhoff factorization of an algebraic Feynman rule, of the form

$$\phi_-(X) = -T(\phi(X) + \sum \phi_-(X')\phi(X'')) \quad \text{and} \quad \phi_+(X) = (1-T)(\phi(X) + \sum \phi_-(X')\phi(X'')) \tag{2.6}$$

where $\Delta(X) = 1 \otimes X + X \otimes 1 + \sum X' \otimes X''$.

In the original Connes–Kreimer formulation, this approach is applied to the unrenormalized Feynman amplitudes regularized by dimensional regularization, with the Rota–Baxter algebra consisting of germs of meromorphic functions at the origin, with the operator of projection onto the polar part of the Laurent series.

In the following, we consider the following variant on the Hopf algebra of Feynman graphs.

Definition 2.4. *As an algebra, $\mathcal{H}_{\mathrm{even}}$ is the commutative algebra generated by Feynman graphs of a given scalar quantum field theory that have an even number of internal edges, $\#E(\Gamma) \in 2\mathbb{N}$. The coproduct* (2.5) *on $\mathcal{H}_{\mathrm{even}}$ is similarly defined with the sum over divergent subgraphs γ with even $\#E(\gamma)$, with 1PI quotient.*

Notice that in dimension $D \in 4\mathbb{N}$ all log divergent subgraphs $\gamma \subset \Gamma$ have an even number of edges, since $Db_1(\gamma) = 2\#E(\gamma)$ in this case.

Question 2.5. *Is there a graded-commutative version of Birkhoff factorization involving graded-commutative Rota–Baxter and Hopf algebras?*

Such an extension to the graded-commutative case would be necessary to include the more general case of differential forms of odd degree (associated to Feynman graphs with an odd number of internal edges).

One can approach the question above by using the general setting of [20]:

(1) Let H be any connected filtered cograded Hopf algebra and let A be a (not necessarily commutative) associative algebra equipped with a Rota-Baxter operator of weight $\lambda \neq 0$. The algebraic Birkhoff factorization of any $\phi \in \mathrm{Hom}(H, A)$ was obtained by Ebrahimi-Fard, Guo and Kreimer in [20].

(2) However, if the target algebra A is not commutative, the set of character $\mathrm{Char}(H, A)$ is not a group since it is not closed under convolution product, i.e. if $f, g \in \mathrm{Char}(H, A)$, then $f \star g$ does not necessarily belong to $\mathrm{Char}(H, A)$.

2.4. **Rota–Baxter algebras and Atkinson factorization.** In the following we will discuss some interesting properties of algebraic Birkhoff decomposition when the Rota-Baxter operator satisfies the identity $T(T(x)y) = T(x)y$.

Let $e : H \to A$ be the unit of $\mathrm{Hom}(H, A)$ (under the convolution product) defined by $e(1_H) = 1_A$ and $e(X) = 0$ on $\oplus_{n>0} H_n$.

The main observation can be summaries as follows:

(1) If the Rota-Baxter operator T on A also satisfy the identity $T(T(x)y) = T(x)y$, then on $\ker(e) = \oplus_{n>0} H_n$, the negative part of the Birkhoff factorization φ_- takes the following form:

$$\phi_- = -T(\phi(X)) - \sum T(\phi(X'))\phi(X''), \quad \text{for } \Delta(X) = 1 \otimes X + X \otimes 1 + \sum X' \otimes X''.$$

(2) If T also satisfies $T(xT(y)) = xT(y)$, $\forall x, y \in A$, then the positive part is given by $\phi_+ = (1 - T)(\phi(X))$, $\forall X \in \ker e = \oplus_{n>0} H_n$.

This follows from the properties of the Atkinson Factorization in Rota–Baxter algebras, which we recall below.

Proposition 2.6. *([23]) (Atkinson Factorization) Let (A, T) be a Rota-Baxter algebra of weight $\lambda \neq 0$. Let $\tilde{T} = -\lambda \mathrm{id} - T$ and let $a \in A$. Assume that b_l and b_r are solution of the fixed point equations*

$$b_l = 1 + T(b_l a), \quad b_r = 1 + \tilde{T}(a b_r). \tag{2.7}$$

Then

$$b_l(1 + \lambda a)b_r = 1.$$

Thus

$$1 + \lambda a = b_l^{-1} b_r^{-1} \tag{2.8}$$

if b_l and b_r are invertible.

A Rota-Baxter algebra (A, T) is called complete if there are algebras $A_n \subseteq A, n \geq 0$, such that (A, A_n) is a complete algebra and $T(A_n) \subseteq A_n$.

Proposition 2.7. *([23]) (Existence and uniqueness of the Atkinson Factorization) Let (A, T, A_n) be a complete Rota-Baxter algebra of weight $\lambda \neq 0$. Let $\tilde{T} = -\lambda \mathrm{id} - T$ and let $a \in A_1$.*

(1) Equations (2.7) have unique solutions b_l and b_r. Further b_l and b_r are invertible. Hence Atkinson Factorization (2.8) exists.

(2) If $\lambda \neq 0$ and $T^2 = -\lambda T$ (in particular if $T^2 = -\lambda T$ on A), then there are unique $c_l \in 1 + T(A)$ and $c_r \in 1 + \tilde{T}(A)$ such that

$$1 + \lambda a = c_l c_r.$$

Define

$$(Ta)^{[n+1]} := T((Ta)^{[n]} a) \quad \text{and} \quad (Ta)^{\{n+1\}} = T(a (Ta)^{\{n\}})$$

with the convention that $(Ta)^{[1]} = T(a) = (Ta)^{\{1\}}$ and $(Ta)^{[0]} = 1 = (Ta)^{\{0\}}$.

Proposition 2.8. *Let (A, A_n, T) be a complete filtered Rota-Baxter algebra of weight -1 such that $T^2 = T$. Let $a \in A_1$. If T also satisfies the following identity*

$$T(T(x)y) = T(x)y, \quad \forall x, y \in A, \tag{2.9}$$

then the equation

$$b_l = 1 + T(b_l a). \tag{2.10}$$

has a unique solution

$$1 + T(a)(1-a)^{-1}.$$

Proof. First, we have $(Ta)^{[n+1]} = T(a)a^n$ for $n \geq 0$. In fact, the case when $n = 0$ just follows from the definition. Suppose it is true up to n, then $(Ta)^{[n+2]} = T((Ta)^{[n+1]}a) = T((T(a)a^n)a) = T(T(a)a^{n+1}) = T(a)a^{n+1}$. Arguing as in ([20]), $b_l = \sum_{n=0}^{\infty}(Ta)^{[n]} = 1 + T(a) + T(T(a)a) + \cdots + (Ta)^{[n]} + \cdots$ is the unique solution of (2.10). So

$$\begin{aligned} b_l &= 1 + T(a) + T(a)a + T(a)a^2 + \cdots \\ &= 1 + T(a)(1 + a + a^2 + \cdots) \\ &= 1 + T(a)(1-a)^{-1}. \end{aligned}$$

□

A bialgebra H is called a connected, filtered cograded bialgebra if there are subspaces H_n of H such that (a) $H_pH_q \subseteq \sum_{k \leq p+q} H_k$; (b) $\Delta(H_n) \subseteq \oplus_{p+q=n} H_p \oplus H_q$; (c) $H_0 = \mathrm{im}u (= \mathbb{C})$, where $u : \mathbb{C} \to H$ is the unit of H.

Proposition 2.9. *Let H be a connected filtered cograded bialgebra (hence a Hopf algebra) and let (A, T) be a (not necessarily commutative) Rota-Baxter algebra of weight $\lambda = -1$ with $T^2 = T$. Suppose that T also satisfies (2.9). Let $\phi : H \to A$ be a character, i.e. an algebra homomorphism. Then there are unique maps $\phi_- : H \to T(A)$ and $\phi_+ : H \to \tilde{T}(A)$, where $\tilde{T} = 1 - T$, such that*

$$\phi = \phi_-^{*(-1)} * \phi_+,$$

where $\phi^{*(-1)} = \phi \circ S$, with S the antipode. ϕ_- takes the following form on $\ker e = \oplus_{n>0} H_n$*:*

$$\begin{aligned} \phi_-(X) &= -T(\phi(X)) - \sum_{n=1}^{\infty} (-1)^n \sum T(\phi(X^{(1)}))\phi(X^{(2)})\phi(X^{(3)}) \cdots \phi(X^{(n+1)}) \\ &= -T(\phi(X)) - \sum_{n=1}^{\infty} (-1)^n ((T\phi) \tilde{*} \phi^{\tilde{*}^n})(X). \end{aligned}$$

Here we use the notation $\tilde{\Delta}^{n-1}(X) = \sum X^{(1)} \otimes \cdots \otimes X^{(n)}$, and $\tilde{\Delta}(X) := \Delta(X) - X \otimes 1 - 1 \otimes X$ (which is coassociative), and $\tilde{*}$ is the convolution product defined by $\tilde{\Delta}$. Furthermore, if T satisfies

$$T(xT(y)) = xT(y), \quad \forall x, y \in A, \tag{2.11}$$

then ϕ_+ takes the form on $\ker e = \oplus_{n>0} H_n$*:*

$$\phi_+ = (1 - T)(\phi(X)).$$

Proof. Define $R := \mathrm{Hom}(H, A)$ and

$$P : R \to R, \quad P(f)(X) = T(f(X)), \ f \in \mathrm{Hom}(H, A), X \in H.$$

Then by [23], R is a complete algebra with filtration $R_n = \{f \in \mathrm{Hom}(H, A) | f(H^{(n-1)}) = 0\}, n \geq 0$, and P is a Rota-Baxter operator of weight -1 and $P^2 = P$. Moreover, since T satisfies (2.9), it is easy to check that $P(P(f)g) = P(f)g$ for any $f, g \in \mathrm{Hom}(H, A)$. Let $\phi : H \to A$ be a character. Then $(e - \phi)(1_H) = e(1_H) - \phi(1_H) = 1_A - 1_A = 0$. So $e - \phi \in A_1$. Set $a = e - \phi$, by Proposition

2.7, we know that there are unique $c_l \in T(A)$ and $c_r \in (1-T)(A)$ such that $\phi = c_l c_r$. Moreover, by Proposition 2.8 we have $\phi_- = b_l = c_l^{-1} = e + T(a)(e-a)^{-1} = e + T(e-\phi)\sum_{n=0}^{\infty}(e-\phi)^n$. We also have $\sum_{n=0}^{\infty}(e-\phi)^n(1_H) = 1_A$ and for any $X \in \ker e = \oplus_{n>0} H_n$, we have $(e-\phi)^0(X) = e(X) = 0$; $(e-\phi)^1(X) = -\phi(X)$; $(e-\phi)^2(X) = \sum (e-\phi)(X')(e-\phi)(X'') = \sum \phi(X')\phi(X'')$. More generally, we have $(e-\phi)^n(X) = (-1)^n \sum \phi(X^{(1)})\phi(X^{(2)})\cdots\phi(X^{(n)}) = (-1)^n \phi^{\tilde{*}^n}(X)$. So for $X \in \ker e = \oplus_{n>0} H_n$,

$$\begin{aligned}
\phi_-(X) &= (T(e-\phi)\sum_{n=0}^{\infty}(e-\phi)^n)(X) \\
&= T(e-\phi)(1_H)\sum_{n=0}^{\infty}(e-\phi)^n(X) + T(e-\phi)(X)\sum_{n=0}^{\infty}(e-\phi)^n(1_H) \\
&\quad + \sum T((e-\phi)(X'))\sum_{n=1}^{\infty}(e-\phi)^n(X'') \\
&= -T(\phi(X)) - \sum T(\phi(X'))\sum_{n=1}^{\infty}(-1)^n \sum \phi((X'')^{(1)})\phi((X'')^{(2)})\cdots\phi((X'')^{(n)}) \\
&= -T(\phi(X)) - \sum_{n=1}^{\infty}(-1)^n \sum T(\phi(X^{(1)}))\phi(X^{(2)})\phi(X^{(3)})\cdots\phi(X^{(n+1)}) \\
&= -T(\phi(X)) - \sum_{n=1}^{\infty}(-1)^n((T\phi)\tilde{*}\phi^{\tilde{*}^n})(X).
\end{aligned}$$

Suppose that T also satisfies Equation (2.11), then for any $a, b \in A$, we have $(1-T)(a)(1-T)(b) = ab - T(a)b - aT(b) + T(a)T(b) = ab - T(T(a)b) - T(aT(b)) + T(a)T(b) = ab - T(ab) = (1-T)(ab)$, as T is a Rota-Baxter operator of weight -1. As shown in ([13]) and ([20]), $\phi_+ = (1-T)(\phi(X) + \sum \phi_-(X')\phi(X''))$. So $\phi_+ = (1-T)(\phi(X)) + \sum(1-T)(\phi_-(X'))(1-T)(\phi(X''))$ by the previous computation. But ϕ_- is in the image of T and $T^2 = T$, so we must have $(1-T)(\phi_-(X')) = 0$, which shows that $\phi_+ = (1-T)(\phi(X))$. ☐

2.5. **A variant of algebraic renormalization.** We consider now a setting inspired by the formalism of the Connes–Kreimer renormalization recalled above. The setting generalizes the one considered in [12] for configuration space integrals and our main application will be to extend the approach of [12] to momentum space integrals.

The main difference with respect to the Connes–Kreimer renormalization is that, instead of renormalizing the Feynman amplitude (regularized so that it gives a meromorphic function), we propose to renormalize the differential form, before integration, and then integrate the renormalized form to obtain a period.

The result obtains by this method differs from the physical renormalization, as we will see in explicit examples in Section 5.9 below. Whenever non-trivial, the convergent integral obtained by the method described here will be a mixed Tate period even in cases where the physical renormalization is not.

The main steps required for our setup are the following.

- For each $\ell \geq 1$, we construct a pair (X_ℓ, Y_ℓ) of a smooth projective variety X_ℓ (defined over $\mathbb{Q}$) whose motive $\mathfrak{m}(X_\ell)$ is mixed Tate (over $\mathbb{Z}$), together with a (singular) hypersurface $Y_\ell \subset X_\ell$.
- We describe the Feynman integrand as a morphism of commutative algebras

$$\phi : \mathcal{H}_{\text{even}} \to \bigwedge_{\ell} \mathcal{M}^{\text{even}}_{X_\ell, Y_\ell}, \quad \phi(\Gamma) = \eta_\Gamma,$$

with η_Γ an algebraic differential form on X_ℓ with polar locus Y_ℓ, for $\ell = b_1(\Gamma)$, and with the Rota–Baxter structure of Lemma 2.3 on the target algebra.

- We express the (unrenormalized) Feynman integrals as a (generally divergent) integral $\int_\sigma \eta_\Gamma$, over a chain σ in X_ℓ.
- We construct a divisor $\Sigma_\ell \subset X_\ell$, that contains the boundary $\partial\sigma$, whose motive $\mathfrak{m}(\Sigma_\ell)$ is mixed Tate (over $\mathbb{Z}$) for all $\ell \geq 1$.
- We perform the Birkhoff decomposition $\phi_\pm$ obtained inductively using the coproduct on $\mathcal{H}$ and the Rota–Baxter operator T (polar part) on $\mathcal{M}^*_{X_\ell, Y_\ell}$.
- This gives a holomorphic form $\phi_+(\Gamma)$ on X_ℓ. The divergent Feynman integral is then replaced by the integral
$$\int_{\Upsilon(\sigma)} \phi_+(\Gamma)$$
which is a period of the mixed Tate motive $\mathfrak{m}(X_\ell, \Sigma_\ell)$.
- In addition to the integral of $\phi_+(\Gamma)$ on X_ℓ we consider integrals on the strata of the complement $X_\ell \smallsetminus Y_\ell$ of the polar part $\phi_-(\Gamma)$, which under suitable conditions will be interpreted as Poincaré residues.

If convergent, the Feynman integral $\int_\sigma \eta_\Gamma$ would be a period of $\mathfrak{m}(X_\ell \smallsetminus Y_\ell, \Sigma_\ell \smallsetminus (\Sigma_\ell \cap Y_\ell))$. The renormalization procedure described above replaces it with a (convergent) integral that is a period of the simpler motive $\mathfrak{m}(X_\ell, \Sigma_\ell)$. By our assumptions on X_ℓ and Σ_ℓ, the motive $\mathfrak{m}(X_\ell, \Sigma_\ell)$ is mixed Tate for all ℓ.

Thus, this strategy eliminates the difficulty of analyzing the motive $\mathfrak{m}(X_\ell \smallsetminus Y_\ell, \Sigma_\ell \smallsetminus (\Sigma_\ell \cap Y_\ell))$ encountered for instance in [4]. The form of renormalization proposed here always produces a mixed Tate period, but at the cost of incurring in a considerable loss of information with respect to the original Feynman integral.

Indeed, a difficulty in the procedure described above is ensuring that the resulting regularized form
$$\phi_+(\Gamma) = (1-T)(\phi(\Gamma) + \sum_{\gamma \subset \Gamma} \phi_-(\gamma) \wedge \phi(\Gamma/\gamma))$$
is nontrivial. This condition may be difficult to control in explicit cases, although we will discuss below an especially simple situation, when one can reduce the problem to forms with logarithmic poles, where using the pole subtraction together with Poicaré residues one can obtain nontrivial periods (although the result one obtains is not equivalent to the physical renormalization of the Feynman amplitude).

An additional difficulty that can cause loss of information with respect to the Feynman integral is coming from the combinatorial conditions on the graph given in [4] that we will use for the embedding into the complement of the determinant hypersurface, see Section 5.9.

3. Rota–Baxter algebras and forms with logarithmic poles

We now focus on the case of meromorphic forms with logarithmic poles, where the Rota–Baxter structure and the renormalization procedure described above drastically simplify.

Lemma 3.1. *Let X be a smooth projective variety and $Y \subset X$ a smooth hypersurface with defining equation $Y = \{f = 0\}$. Let $\Omega^\star_X(\log(Y))$ be the sheaf of algebraic differential forms on X with logarithmic poles along Y. The Rota–Baxter operator T of Lemma 2.1 preserves $\Omega^{\rm even}_X(\log(Y))$ and the pair $(\Omega^{\rm even}_X(\log(Y)), T)$ is a graded Rota–Baxter algebra of degree -1 with the property that, for all $\omega_1, \omega_2 \in \Omega^{\rm even}_X(\log(Y))$, the wedge product $T(\omega_1) \wedge T(\omega_2) = 0$.*

Proof. Forms $\omega \in \Omega^\star_X(\log(Y))$ can be written in canonical form
$$\omega = \frac{df}{f} \wedge \xi + \eta,$$

with ξ and η holomorphic, so that $T(\omega) = \frac{df}{f} \wedge \xi$. We then have (2.2) as in Corollary 2.2 above, with $T(\omega_1) \wedge T(\omega_2) = (-1)^{|\xi_1|+1} \alpha \wedge \alpha \wedge \xi_1 \wedge \xi_2$ where α is the 1-form $\alpha = df/f$ so that $\alpha \wedge \alpha = 0$. □

Lemma 3.1 shows that, when restricted to $\Omega^\star_X(\log(Y))$, the operator T satisfies the simpler identity

$$T(xy) = T(T(x)y) + T(xT(y)). \tag{3.1}$$

This property greatly simplifies the decomposition of the algebra induced by the Rota–Baxter operator. In particular, we get a simplified form of the general result of Proposition 2.9, when taking into account the vanishing $T(x)T(y) = 0$, as shown in Lemma 3.1.

Lemma 3.2. *Let $\mathcal{R}$ be a commutative algebra and $T : \mathcal{R} \to \mathcal{R}$ a linear operator that satisfies the identity* (3.1) *and such that, for all $x, y \in \mathcal{R}$ the product $T(x)T(y) = 0$. Let $\mathcal{R}_+ = \mathrm{Range}(1 - T)$. Then the following properties hold.*

(1) *$\mathcal{R}_+ \subset \mathcal{R}$ is a subalgebra.*
(2) *Both T and $1 - T$ are idempotent, $T^2 = T$ and $(1 - T)^2 = 1 - T$.*

Proof. (1) The product of elements in $\mathcal{R}_+$ can be written as $(1-T)(x) \cdot (1-T)(y) = xy - T(x)y - xT(y) = xy - T(x)y - xT(y) - (T(xy) - T(T(x)y) - T(xT(y))) = (1 - T)(xy - T(x)y - xT(y))$.
(2) The identity (3.1) gives $T(1) = 0$, since taking $x = y = 1$ one obtains $T(1) = 2T^2(1)$ while taking $x = T(1)$ and $y = 1$ gives $T^2(1) = T^3(1)$. Then (3.1) with $y = 1$ gives $T(x) = T(xT(1)) + T(T(x)1) = T^2(x)$ for all $x \in \mathcal{R}$. For $1-T$ we then have $(1-T)^2(x) = x-2T(x)+T^2(x) = (1-T)(x)$, for all $x \in \mathcal{R}$. □

Lemma 3.3. *Let $\mathcal{R}$ be a commutative algebra and $T : \mathcal{R} \to \mathcal{R}$ a linear operator that satisfies the identity* (3.1) *and such that, for all $x, y \in \mathcal{R}$ the product $T(x)T(y) = 0$. If, for all $x, y \in \mathcal{R}$, the identity $T(x)y + xT(y) = T(T(x)y) + T(xT(y))$ holds, then the operator $(1 - T) : \mathcal{R} \to \mathcal{R}_+$ is an algebra homomorphism and the operator T is a derivation on $\mathcal{R}$.*

Proof. We have $(1 - T)(xy) = xy - T(T(x)y) - T(xT(y))$ while $(1 - T)(x) \cdot (1 - T)(y) = xy - T(x)y - xT(y)$. Assuming that, for all $x, y \in \mathcal{R}$, we have $T(T(x)y) + T(xT(y)) = T(x)y + xT(y)$ gives $(1 - T)(xy) = (1 - T)(x) \cdot (1 - T)(y)$. Moreover, the identity (3.1) can be rewritten as $T(xy) = T(x)y + xT(y)$, hence T is just a derivation on $\mathcal{R}$. □

Consider then again the case of a smooth hypersurface Y in $\mathbb{P}^n$. We have the following properties.

Proposition 3.4. *Let $Y \subset X$ be a smooth hypersurface in a smooth projective variety. The Rota–Baxter operator $T : \mathcal{M}^{\mathrm{even}}_{\mathbb{P}^n,Y} \to \mathcal{M}^{\mathrm{even}}_{X,Y}$ of weight -1 on meromorphic forms on X with poles along Y restricts to a derivation on the graded algebra $\Omega^{\mathrm{even}}_X(\log(Y))$ of forms with logarithmic poles. Moreover, the operator $1 - T$ is a morphism of commutative algebras from $\Omega^{\mathrm{even}}_X(\log(Y))$ to the algebra of holomorphic forms Ω^{even}_X.*

Proof. It suffices to check that the polar part operator $T : \Omega^{\mathrm{even}}_X(\log(Y)) \to \Omega^{\mathrm{even}}_X(\log(Y))$ satisfies the hypotheses of Lemma 3.3. We have seen that, for all $\omega_1, \omega_2 \in \Omega^{\mathrm{even}}_X(\log(Y))$, the product $T(\omega_1) \wedge T(\omega_2) = 0$. Moreover, for $\omega_i = d\log(f) \wedge \xi_i + \eta_i$, we have $T(\omega_1) \wedge \omega_2 = d\log(f) \wedge \xi_1 \wedge \eta_2$ and $\omega_1 \wedge T(\omega_2) = (-1)^{|\eta_1|} d\log(f) \wedge \eta_1 \wedge \xi_2$, where the ξ_i and η_i are holomorphic, so that we have $T(T(\omega_1) \wedge \omega_2) = T(\omega_1) \wedge \omega_2$ and $T(\omega_1 \wedge T(\omega_2)) = \omega_1 \wedge T(\omega_2)$. Thus, the hypotheses of Lemma 3.3 are satisfied. □

3.1. **Birkhoff factorization and forms with logarithmic poles.** In cases where the pair (X, Y) has the property that the deRham cohomology $H^*_{dR}(X \smallsetminus Y)$ can always be realized by algebraic differential forms with logarithmic poles, the construction above simplifies significantly. Indeed, the Birkhoff factorization becomes essentially trivial, because of Proposition 3.4. In other words, all graphs behave "as if they were log divergent". This can be stated more precisely as follows.

Proposition 3.5. *Let $Y \subset X$ be a smooth hypersurface inside a smooth projective variety and let $\Omega_X^{\rm even}(\log(Y))$ denote the commutative algebra of algebraic differential forms on X of even degree with logarithmic poles on Y. Let $\phi : \mathcal{H} \to \Omega_X^{\rm even}(\log(Y))$ be a morphism of commutative algebras from a commutative Hopf algebra $\mathcal{H}$ to $\Omega_X^{\rm even}(\log(Y))$ with the operator T of pole subtraction. Then for every $X \in \mathcal{H}$ one has*

$$\phi_+(X) = (1-T)\phi(X),$$

while the negative part of the Birkhoff factorization takes the form

$$\phi_-(X) = -T(\phi(X)) - \sum \phi_-(X')\phi(X''),$$

where $\Delta(X) = X \otimes 1 + 1 \otimes X + \sum X' \otimes X''$. Moreover, ϕ_- takes the following nonrecursive form on $\ker e = \oplus_{n>0} H_n$:

$$\begin{aligned} \phi_-(X) &= -T(\phi(X)) - \sum_{n=1}^{\infty} (-1)^n \sum T(\phi(X^{(1)}))\phi(X^{(2)})\phi(X^{(3)}) \cdots \phi(X^{(n+1)}) \\ &= -T(\phi(X)) - \sum_{n=1}^{\infty} (-1)^n ((T\phi)\tilde{*}\phi^{\tilde{*}^n})(X). \end{aligned}$$

Proof. The operator T of pole subtraction is a derivation on $\Omega_X^{\rm even}(\log(Y))$. By (2.6) we have $\phi_+(X) = (1-T)(\phi(X) + \sum \phi_-(X')\phi(X''))$. By Proposition 3.4 we know that, in the case of forms with logarithmic poles along a smooth hypersurface, $1-T$ is an algebra homomorphism, hence $\phi_+(X) = (1-T)(\phi(X)) + \sum (1-T)(\phi_-(X'))(1-T)(\phi(X'')))$, but $\phi_-(X')$ is in the range of T and, again by Proposition 3.4, we have $T^2 = T$, so that the terms in the sum all vanish, since $(1-T)(\phi_-(X')) = 0$. By (2.6) we have $\phi_-(X) = -T(\phi(X) + \sum \phi_-(X')\phi(X'')) = -T\phi(X) - \sum T(\phi_-(X'))\phi(X'') - \sum \phi_-(X')T(\phi(X''))$, because by Proposition 3.4 T is a derivation. The last sum vanishes because $\phi_-(X')$ is in the range of T and we have $T(\eta) \wedge T(\xi) = 0$ for all $\eta, \xi \in \Omega^*_{X_\ell}(\log(Y_\ell))$. Thus, we are left with $\phi_-(X) = -T\phi(X) - \sum T(\phi_-(X'))\phi(X'') = -T\phi(X) - \sum \phi_-(X')\phi(X'')$. The last part follows from Proposition 2.9, since $T(T(\eta) \wedge \xi) = T(\eta) \wedge \xi$. ∎

Notice that this is compatible with the property that $\phi(X) = (\phi_- \circ S \star \phi_+)(X)$ (with the $\star$-product dual to the Hopf algebra coproduct). In fact, this identity is equivalent to $\phi_+ = \phi_- \star \phi$, which means that $\phi_+(X) = \langle \phi_- \otimes \phi, \Delta(X) \rangle = \phi_-(X) + \phi(X) + \sum \phi_-(X')\phi(X'') = (1-T)\tilde{\phi}(X)$ as above. Equivalently, all the nontrivial terms $\phi_-(X')\phi(X'')$ in $\tilde{\phi}(X)$ satisfy $T(\phi_-(X')\phi(X'')) = \phi_-(X')\phi(X'')$, because of the simplified form (4.3) of the Rota–Baxter identity.

Corollary 3.6. *If one has a construction of a character $\phi : \mathcal{H} \to \Omega_X^{\rm even}(\log(Y))$, of the Hopf algebra of Feynman graphs, where $X = X_\ell$ and $Y = Y_\ell$ independently of the number of loops $\ell \geq 1$, then the negative part of the Birkhoff factorization of Proposition 3.5 would take on the simple form*

$$\phi_-(\Gamma) = -\frac{dh}{h} \wedge \left(\xi_\Gamma + \sum_{N \geq 1} (-1)^N \sum_{\gamma_N \subset \cdots \subset \gamma_1 \subset \gamma_0 = \Gamma} \xi_{\gamma_N} \wedge \bigwedge_{j=1}^{N} \eta_{\gamma_{j-1}/\gamma_j} \right), \tag{3.2}$$

where $\phi(\Gamma) = \frac{dh}{h} \wedge \xi_\Gamma + \eta_\Gamma$, and $Y = \{h = 0\}$.

Proof. The result follows from the expression

$$\phi_-(\Gamma) = -T(\phi(\Gamma)) - \sum_{\gamma \subset \Gamma} \phi_-(\gamma)\phi(\Gamma/\gamma),$$

obtained in Proposition 3.5, where $\phi(\Gamma) = \omega_\Gamma = \frac{dh}{h} \wedge \xi_\Gamma + \eta_\Gamma$, so that $T(\phi(\Gamma)) = \frac{dh}{h} \wedge \xi_\Gamma$ and $\phi(\Gamma/\gamma) = \frac{dh}{h} \wedge \xi_{\Gamma/\gamma} + \eta_{\Gamma/\gamma}$. The wedge product of $\phi_-(\gamma) = -T(\phi(\gamma)) - \sum_{\gamma_2 \subset \gamma} \phi_-(\gamma_2)\phi(\gamma/\gamma_2)$ with $\phi(\Gamma/\gamma)$ will give a term $\frac{dh}{h} \wedge \xi_\gamma \wedge \eta_{\Gamma/\gamma}$ and additional terms $\phi_-(\gamma_2)\phi(\gamma/\gamma_2) \wedge \eta_{\Gamma/\gamma}$. Proceeding inductively on these terms, one obtains (3.2). ∎

In the more general case, where X_ℓ and Y_ℓ depend on the loop number $\ell \geq 1$, the form of the negative piece $\phi_-(\Gamma)$ is more complicated, as it will contain forms on the products $X_{\ell(\gamma)} \times X_{\ell(\Gamma/\gamma)}$ with logarithmic poles along $Y_{\ell(\gamma)} \times X_{\ell(\Gamma/\gamma)} \cup X_{\ell(\gamma)} \times Y_{\ell(\Gamma/\gamma)}$.

3.2. **Polar subtraction and the residue.** We have seen that, in the case of a smooth hypersurface $Y \subset X$, the Birkhoff factorization in the algebra of forms with logarithmic poles reduces to a simple pole subtraction, $\phi_+(X) = (1-T)\phi(X)$. If the unrenormalized $\phi(X)$ is a form written as $\alpha + \frac{df}{f} \wedge \beta$, with α and β holomorphic, then $\phi_+(X)$ vanishes identically whenever $\alpha = 0$. In that case, all information about $\phi(X)$ is lost in the process of pole substraction. Suppose that $\int_\sigma \phi(X)$ is the original unrenormalized integral. To maintain some additional information, it is preferable to consider, in addition to the integral $\int_\sigma \phi_+(X)$, also an integral of the form

$$\int_{\sigma \cap Y} \mathrm{Res}_Y(\eta),$$

where $\mathrm{Res}_Y(\eta) = \beta$ is the Poincaré residue of $\eta = \alpha + \frac{df}{f} \wedge \beta$ along Y. It is dual to the Leray coboundary, in the sense that

$$\int_{\sigma \cap Y} \mathrm{Res}_Y(\eta) = \frac{1}{2\pi i} \int_{\mathcal{L}(\sigma \cap Y)} \eta,$$

where the Leray coboundary $\mathcal{L}(\sigma \cap Y)$ is a circle bundle over $\sigma \cap Y$. In this way, even when $\alpha = 0$, one can still retain the nontrivial information coming from the Poincaré residue, which is also expressed as a period.

4. Singular hypersurfaces and meromorphic forms

In our main application, we will need to work with pairs (X, Y) where X is smooth projective, but the hypersurface Y is singular. Thus, we now discuss extensions of the results above to more general situations where $Y \subset X$ is a singular hypersurface in a smooth projective variety X.

Again we denote by $\mathcal{M}^*_{X,Y}$ the sheaf of meromorphic differential forms on X with poles along Y, of arbitrary order, and by $\Omega^*_X(\log(Y))$ the sub-sheaf of forms with logarithmic poles along Y. Let h be a local determination of Y, so that $Y = \{h = 0\}$. We can then locally represent forms $\omega \in \mathcal{M}^*_{X,Y}$ as finite sums $\omega = \sum_{p \geq 0} \omega_p / h^p$, with the ω_p holomorphic. The polar part operator $T : \mathcal{M}^{\mathrm{even}}_{X,Y} \to \mathcal{M}^{\mathrm{even}}_{X,Y}$ can then be defined as in (2.3).

In the case we considered above, with $Y \subset X$ a smooth hypersurface, forms with logarithmic poles can be represented in the form

$$\omega = \frac{dh}{h} \wedge \xi + \eta, \tag{4.1}$$

with ξ and η holomorphic. The Leray residue is given by $\mathrm{Res}(\omega) = \xi$. It is well defined, as the restriction of ξ to Y is independent of the choice of a local equation for Y.

In the next subsection we discuss how this case generalizes to a normal crossings divisor $Y \subset X$ inside a smooth projective variety X. The complex of forms with logarithmic poles extend to the normal crossings divisor case as in [16]. For more general singular hypersurfaces, an appropriate notion of forms with logarithmic poles was introduced by Saito in [36]. The construction of the residue was also generalized to the case where Y is a normal crossings divisor in [16] and for more general singular hypersurfaces in [36].

4.1. **Normal crossings divisors.** The main case of singular hypersurfaces that we focus on for our applications will be simple normal crossings divisors. In fact, while our formulation of the Feynman amplitude in momentum space is based on the formulation of [4], where the unrenormalized Feynman integral lives on the complement of the determinant hypersurface, which has worse singularities, we will reformulate the integral on the Kausz compactification of GL_n where the boundary divisor of the compactification is normal crossings.

If $Y \subset X$ is a simple normal crossings divisor in a smooth projective variety, with Y_j the components of Y, with local equations $Y_j = \{f_j = 0\}$, the complex of forms with logarithmic poles $\Omega^*_X(\log(Y))$ spanned by the forms $\frac{df_j}{f_j}$ and by the holomorphic forms on X.

As in Theorem 6.3 of [12], we obtain that the Rota–Baxter operator of polar projection $T : \mathcal{M}^{\rm even}_{X,Y} \to \mathcal{M}^{\rm even}_{X,Y}$ restricts to a Rota–Baxter operator $T : \Omega^{\rm even}_X(\log(Y)) \to \Omega^{\rm even}_X(\log(Y))$ given by

$$T : \eta \mapsto T(\eta) = \sum_j \frac{df_j}{f_j} \wedge \mathrm{Res}_{Y_j}(\eta), \tag{4.2}$$

where the holomorphic form $\mathrm{Res}_{Y_j}(\eta)$ is the Poincaré residue of η restricted to Y_j.

Unlike the case of a single smooth hypersurface, for a simple normal crossings divisor the Rota–Baxter operator operator T does not satisfy $T(x)T(y) \equiv 0$, since we now have terms like $\frac{df_j}{f_j} \wedge \frac{df_k}{f_k} \neq 0$, for $j \neq k$, so the Rota–Baxter identity for T does not reduce to a derivation, but some of the properties that simplify the Birkhoff factorization in the case of a smooth hypersurface still hold in this case.

Proposition 4.1. *The Rota–Baxter operator T of* (4.2) *satisfies $T^2 = T$ and the Rota–Baxter identity simplifies to the form*

$$T(\eta \wedge \xi) = T(\eta) \wedge \xi + \eta \wedge T(\xi) - T(\eta) \wedge T(\xi). \tag{4.3}$$

The operator $(1 - T) : \mathcal{R} \to \mathcal{R}_+$ is an algebra homomorphism, with $\mathcal{R} = \Omega^{\rm even}_X(\log(Y))$ and $\mathcal{R}_+ = (1 - T)\mathcal{R}$. The Birkhoff factorization of a commutative algebra homomorphism $\phi : \mathcal{H} \to \mathcal{R}$, with $\mathcal{H}$ a commutative Hopf algebra is given by

$$\begin{array}{l} \phi_+(X) = (1 - T)\phi(X) \\ \phi_-(X) = -T(\phi(X) + \sum \phi_-(X')\phi(X'')). \end{array} \tag{4.4}$$

Moreover, ϕ_- takes the following form on $\ker e = \oplus_{n>0} H_n$:

$$\begin{array}{rcl} \phi_-(X) & = & -T(\phi(X)) - \displaystyle\sum_{n=1}^{\infty} (-1)^n \sum T(\phi(X^{(1)}))\phi(X^{(2)})\phi(X^{(3)}) \cdots \phi(X^{(n+1)}) \\ & = & -T(\phi(X)) - \displaystyle\sum_{n=1}^{\infty} (-1)^n ((T\phi) \tilde{*} \phi^{\tilde{*}^n})(X). \end{array}$$

Proof. The argument is the same as in the proof of Theorem 6.3 in [12]. It is clear by construction that T is idempotent and the simplified form (4.3) of the Rota–Baxter identity follows by observing that $T(T(\eta)\wedge\xi) = T(\eta)\wedge\xi$ and $T(\eta\wedge T(\xi)) = \eta\wedge T(\xi)$ as in Theorem 6.3 in [12]. Then one sees that $(1-T)(\eta)\wedge(1-T)(\xi) = \eta\wedge\xi - T(\eta)\wedge\xi - \eta\wedge T(\xi) + T(\eta)\wedge T(\xi) = \eta\wedge\xi - T(\eta\wedge\xi)$ by (4.3). Consider then the Birkhoff factorization. We write $\tilde{\phi}(X) := \phi(X) + \sum \phi_-(X')\phi(X'')$. The fact that $(1-T)$ is an algebra homomorphism then gives $\phi_+(X) = (1-T)(\tilde{\phi}(X)) = (1-T)(\phi(X) + \sum \phi_-(X')\phi(X'')) = (1-T)(\phi(X)) + \sum (1-T)(\phi_-(X'))(1-T)(\phi(X'')))$, with $(1-T)(\phi_-(X')) = -(1-T)T(\tilde{\phi}_-(X')) = 0$, because T is idempotent. The last statement again follows from Proposition 2.9, since we have $T(T(\eta) \wedge \xi) = T(\eta) \wedge \xi$. $\square$

4.2. **Multidimensional residues.** In the case of a simple normal crossings divisor $Y \subset X$, we can proceed as discussed in Section 3.2 for the case of a smooth hypersurface. Indeed, as we have seen in Proposition 4.1, we also have in this case a simple pole subtraction $\phi_+(X) = (1-T)\phi(X)$, even though the negative term $\phi_-(X)$ of the Birkhoff factorization can now be more complicated than in the case of a smooth hypersurface.

The unrenormalized $\phi(X)$ is a form $\eta = \alpha + \sum_j \frac{df_j}{f_j} \wedge \beta_j$, with α and β_j holomorphic and $Y_j = \{f_j = 0\}$ the components of Y. Again, if $\alpha = 0$ we lose all information about $\phi(X)$ in our renormalization of the logarithmic form. To avoid this problem, we can again consider, instead of the single renormalized integral $\int_\sigma \phi_+(X)$, an additional family of integrals

$$\int_{\sigma \cap Y_I} \mathrm{Res}_{Y_I}(\eta),$$

where $Y_I = \cap_{j\in I} Y_j$ is an intersection of components of the divisor Y and $\mathrm{Res}_{Y_I}(\eta)$ is the iterated (or multidimensional, or higher) Poincaré residue of η, in the sense of [1], [2]. These are dual to the iterated Leray coboundaries,

$$\int_{\sigma \cap Y_I} \mathrm{Res}_{Y_I}(\eta) = \frac{1}{(2\pi i)^n} \int_{\mathcal{L}_I(\sigma\cap Y_I)} \eta,$$

where $\mathcal{L}_I = \mathcal{L}_{j_i} \circ \cdots \circ \mathcal{L}_{j_n}$ for $Y_I = Y_{j_1} \cap \cdots \cap Y_{j_n}$.

If arbitrary intersections Y_I of components of Y are all mixed Tate motives, then all these integrals are also periods of mixed Tate motives.

4.3. **Saito's logarithmic forms.** Given a singular reduced hypersurface $Y \subset X$, a differential form ω with logarithmic poles along Y, in the sense of Saito [36], can always be written in the form ([36], (1.1))

$$f\,\omega = \frac{dh}{h} \wedge \xi + \eta, \tag{4.5}$$

where $f \in \mathcal{O}_X$ defines a hypersurface $V = \{f = 0\}$ with $\dim(Y \cap V) \leq \dim(X) - 2$, and with ξ and η holomorphic forms.

In the following, we use the notation ${}^S\Omega^\star_X(\log(Y))$ to denote the forms with logarithmic poles along Y in the sense of Saito, to distinguish it from the more restrictive notion of forms with logarithmic poles $\Omega^\star_X(\log(Y))$ considered above for the normal crossings case.

Following [2], we say that a (reduced) hypersurface $Y \subset X$ has *Saito singularities* if the modules of logarithmic differential forms and vector fields along Y are free. The condition that $Y \subset X$ has Saito singularities is equivalent to the condition that ${}^S\Omega^n_X(\log(Y)) = \bigwedge^n {}^S\Omega^1_X(\log(Y))$, [36].

Let $\mathcal{M}_Y$ denote the sheaf of germs of meromorphic functions on Y. Then setting

$$\mathrm{Res}(\omega) = \frac{1}{f}\,\xi\,|_Y \tag{4.6}$$

defines the residue as a morphism of $\mathcal{O}_X$-modules, for all $q \geq 1$,

$$\mathrm{Res} : {}^S\Omega^q_X(\log(Y)) \to \mathcal{M}_Y \otimes_{\mathcal{O}_Y} \Omega^{q-1}_Y. \tag{4.7}$$

A refinement of (4.7) is given by the following result, [2]. For $Y \subset X$ a reduced hypersurface, and for all $q \geq 1$, there is an exact sequence of $\mathcal{O}_X$-modules

$$0 \to \Omega^{q+1}_X \to {}^S\Omega^{q+1}_X(\log(Y)) \xrightarrow{\mathrm{Res}} \omega^q_Y \to 0. \tag{4.8}$$

Unlike the case of normal crossings divisors, the Saito residue of forms with logarithmic poles is not a holomorphic form, but a *meromorphic* form on Y.

It is natural to ask whether the extraction of polar part from forms with logarithmic poles that we considered here for the case of smooth hypersurfaces and normal crossings divisors extends to more general singular hypersurfaces using Saito's formulation.

Question 4.2. *For more general singular hypersurfaces $Y \subset X$ with Saito singularities, is the Rota–Baxter operator T on even meromorphic forms expressible in terms of Saito residues in the case of forms with logarithmic poles?*

We describe here a possible approach to this question. We introduce an analog of the Rota–Baxter operator considered above, given by the extraction of the polar part. The "polar part" operator, in this more general case, does not maps $\Omega_X^{\rm even}(\log(Y))$ to itself, but we show below that it gives a well defined Rota-Baxter operator of weight -1 on the space of Saito forms ${}^S\Omega_X^{\rm even}(\log(Y))$, and that this operator is a derivation.

Lemma 4.3. *The set $S_Y := \{f : \dim(\{f = 0\} \cap Y) \leq \dim(X) - 2\}$ is a multiplicative set. Localization of the Saito forms with logarithmic poles gives $S_Y^{-1}\, {}^S\Omega_X(\log(Y)) = {}^S\Omega_X(\log(Y))$.*

Proof. We have $V_{12} = \{f_1 f_2 = 0\} = \{f_1 = 0\} \cup \{f_2 = 0\}$ and $\dim(Y \cap V_{12}) = \dim((Y \cap \{f_1 = 0\}) \cup (Y \cap \{f_2 = 0\})) \leq \dim(X) - 2$, since $\dim(Y \cap \{f_i = 0\}) \leq \dim(X) - 2$ for $i = 1, 2$. Thus, for any $f_1, f_2 \in S_Y$, we have $f_1 f_2 \in S_Y$. Moreover, we have $1 \in S_Y$, hence S_Y is a multiplicative set. The localization of ${}^S\Omega_X^\star(\log(Y))$ at S_Y is just ${}^S\Omega_X^\star(\log(Y))$ itself: in fact, for $\tilde{f}^{-1}\omega \in S_Y^{-1}\, {}^S\Omega_X^\star(\log(Y))$, with $\tilde{f} \in S_Y$ and $\omega \in {}^S\Omega_X^\star(\log(Y))$, expressed as in (4.5), we have

$$f\tilde{f}(\tilde{f}^{-1}\omega) = f\omega = \frac{dh}{h} \wedge \xi + \eta,$$

where $f\tilde{f} \in S_Y$, hence $\tilde{f}^{-1}\omega \in {}^S\Omega_X(\log(Y))$. □

Given a form $\omega \in {}^S\Omega_X^\star(\log(Y))$, which we can write as in (4.5), the residue (4.6) is the image under the restriction map $S_Y^{-1}\Omega_X^\star \to S_Y^{-1}\Omega_Y^\star$ of the form $f^{-1}\xi \in S_Y^{-1}\Omega_X^\star$. Moreover, we have an inclusion $\Omega_X^\star \hookrightarrow {}^S\Omega_X^\star(\log(Y))$, which induces a corresponding map of the localizations $S_Y^{-1}\Omega_X^\star \hookrightarrow S_Y^{-1}\, {}^S\Omega_X^\star(\log(Y)) = {}^S\Omega_X^\star(\log(Y))$. We can then define a linear operator

$$T : {}^S\Omega_X^\star(\log(Y)) \to {}^S\Omega_X^\star(\log(Y)) \wedge S_Y^{-1}\,\Omega_X^\star \hookrightarrow {}^S\Omega_X^\star(\log(Y)) \wedge S_Y^{-1}\, {}^S\Omega_X^\star(\log(Y)) = {}^S\Omega_X^\star(\log(Y))$$

given by

$$T(\omega) = \frac{dh}{h} \wedge \frac{\xi}{f}, \quad \text{for} \quad f\,\omega = \frac{dh}{h} \wedge \xi + \eta. \tag{4.9}$$

Lemma 4.4. *The operator T of (4.9) is a Rota–Baxter operator of weight -1 on ${}^S\Omega_X^{\rm even}(\log(Y))$, which is just given by a derivation, satisfying the Leibnitz rule $T(\omega_1 \wedge \omega_2) = T(\omega_1) \wedge \omega_2 + \omega_1 \wedge T(\omega_2)$.*

Proof. Let

$$f_1\,\omega_1 = \frac{dh}{h} \wedge \xi_1 + \eta_1 \quad f_2\,\omega_2 = \frac{dh}{h} \wedge \xi_2 + \eta_2.$$

Then

$$f_1\, f_2\, \omega_1 \wedge \omega_2 = (\frac{dh}{h} \wedge \xi_1 + \eta_1) \wedge (\frac{dh}{h} \wedge \xi_2 + \eta_2) = \frac{dh}{h} \wedge (\xi_1 \wedge \eta_2 + (-1)^p \eta_1 \wedge \xi_2) + \eta_1 \wedge \eta_2,$$

where $\eta_1 \in \Omega^p(X)$. By Lemma 4.3, we know that $f_1 f_2 \in S_Y$. We have

$$T(\omega_1 \wedge \omega_2) = \frac{dh}{h} \wedge (\frac{\xi_1}{f_1} \wedge \frac{\eta_2}{f_2} + (-1)^p \frac{\eta_1}{f_1} \wedge \frac{\xi_2}{f_2}).$$

Since

$$T(\omega_1) = \frac{dh}{h} \wedge \frac{\xi_1}{f_1}, \quad \text{and} \quad T(\omega_2) = \frac{dh}{h} \wedge \frac{\xi_2}{f_2},$$

we obtain

$$T(\omega_1)\wedge T(\omega_2) = \frac{dh}{h}\wedge\frac{\xi_1}{f_1}\wedge\frac{dh}{h}\wedge\frac{\xi_2}{f_2} = 0$$

Moreover, we have

$$T(\omega_1)\wedge\omega_2 = (\frac{dh}{h}\wedge\frac{\xi_1}{f_1})\wedge\frac{dh}{h}\wedge\frac{\xi_2}{f_2} + \frac{dh}{h}\wedge\frac{\xi_1}{f_1}\wedge\frac{\eta_2}{f_2} = \frac{dh}{h}\wedge\frac{\xi_1}{f_1}\wedge\frac{\eta_2}{f_2},$$

with

$$f_1 f_2(T(\omega_1)\wedge\omega_2) = \frac{dh}{h}\wedge\xi_1\wedge\eta_2,$$

and similarly,

$$\omega_1\wedge T(\omega_2) = (-1)^p\frac{dh}{h}\wedge\frac{\eta_1}{f_1}\wedge\frac{\xi_2}{f_2},$$

hence T satisfies the Leibnitz rule. The operator T also satisfies $T(T(\omega_1)\wedge\omega_2) = T(\omega_1)\wedge\omega_2$, and $T(\omega_1\wedge T(\omega_2)) = \omega_1\wedge T(\omega_2)$, hence the condition that T is a derivation is equivalent to the condition that it is a Rota-Baxter operator of weight -1. ☐

Correspondingly, we have

$$(1-T)\omega = \omega - \frac{dh}{h}\wedge\frac{\xi}{f} = \frac{\eta}{f}\in S_Y^{-1}\,\Omega_X^{\rm even}.$$

Under the restriction map $S_Y^{-1}\,\Omega_X^{\rm even}\to S_Y^{-1}\,\Omega_Y^{\rm even}$ we obtain a form $(1-T)(\omega)|_Y$. It follows that we can define a "subtraction of divergences" operation on $\phi:\mathcal{H}\to {}^S\Omega_X^{\rm even}(\log(Y))$ by taking $\phi_+:\mathcal{H}\to\mathcal{R}_X^{\rm even}(\log(Y))$ given by $\phi_+(a) = (1-T)\phi(a)|_Y$, for $a\in\mathcal{H}$, which maps $\phi(a)=\omega$ to $(1-T)\omega|_Y = f^{-1}\eta|_Y$, where $f\,\omega = \frac{dh}{h}\wedge\xi+\eta$. While this has subtracted the logarithmic pole along Y, it has also created a new pole along $V=\{f=0\}$. Thus, it results again in a meromorphic form. If we consider the restriction to Y of $\phi_+(a) = f^{-1}\,\eta|_Y$, we obtain a meromorphic form with first order poles along a subvariety $V\cap Y$, which is by hypothesis of codimension at least one in Y. Thus, we can conceive of a more complicated renormalization method that progressively subtracts poles on subvarieties of increasing codimension, inside the polar locus of the previous pole subtraction, by iterating this procedure.

5. Compactifications of GL_n and momentum space Feynman integrals

In this section, we restrict our attention to the case of compactifications of PGL_ℓ and of GL_ℓ and we use a formulation of the parametric Feynman integrals of perturbative quantum field theory in terms of (possibly divergent) integrals on a cycle in the complement of the determinant hypersurface [4], to obtain a new method of regularization and renormalization, which always gives rise to a renormalized integral that is a period of a mixed Tate motive, even though a certain loss of information can occur with respect to the physical Feynman integral.

5.1. **The determinant hypersurface.** In the following we use the notation $\hat{\mathcal{D}}_\ell$ and $\mathcal{D}_\ell$, respectively, for the affine and the projective determinant hypersurfaces. Namely, we consider in the affine space $\mathbb{A}^{\ell^2}$, identified with the space of all $\ell\times\ell$-matrices, with coordinates $(x_{ij})_{i,j=1,\dots,\ell}$, the hypersurface

$$\hat{\mathcal{D}}_\ell = \{\det(X)=0\,|\,X=(x_{ij})\}\subset\mathbb{A}^{\ell^2}.$$

Since $\det(X)=0$ is a homogeneous polynomial in the variables (x_{ij}), we can also consider the projective hypersurface $\mathcal{D}_\ell\subset\mathbb{P}^{\ell^2-1}$.

The complement $\mathbb{A}^{\ell^2}\smallsetminus\hat{\mathcal{D}}_\ell$ is identified with the space of invertible $\ell\times\ell$-matrices, namely with GL_ℓ.

5.2. **The Kausz compactification of** GL_n**.** We recall here some basic facts about the Kausz compactification $K\mathrm{GL}_n$ of GL_n, following [29] and the exposition in §11 of [33].

Let $X_0 = \mathbb{P}^{\ell^2-1}$, the projectivization of the space $\mathbb{A}^{\ell^2}$ of square $\ell \times \ell$-matrices. Let Y_i be the locus of matrices of rank i and consider the iterated blowups $X_i = \mathrm{Bl}_{\bar{Y}_i}(X_{i-1})$, with $\bar{Y}_i$ the closure of Y_i in X_{i-1}. It is shown in [38] that the X_i are smooth, and that $X_{\ell-1}$ is a wonderful compactification of PGL_ℓ, in the sense of [15]. Moreover, the Y_i are PGL_i-bundles over a product of Grassmannians. One denotes by $\overline{\mathrm{PGL}_\ell}$ the wonderful compactification of PGL_ℓ obtained in this way.

The Kausz compactification [29] of GL_ℓ is similar to the Vainsencher compactification [38] of PGL_ℓ. One regards $\mathbb{A}^{\ell^2}$ as the big cell in $\mathcal{X}_0 = \mathbb{P}^{\ell^2}$. The iterated sequence of blowups is given in this case by setting $\mathcal{X}_i = \mathrm{Bl}_{\mathcal{Y}_{i-1}\cup\mathcal{H}_i}(\mathcal{X}_{i-1})$, where $\mathcal{Y}_i \subset \mathbb{A}^{\ell^2}$ are the matrices of rank i and $\mathcal{H}_i$ are the matrices at infinity (that is, in $\mathbb{P}^{\ell^2-1} = \mathbb{P}^{\ell^2} \smallsetminus \mathbb{A}^{\ell^2}$) of rank i. It is shown in [29] that the $\mathcal{X}_i$ are smooth and that the blowup loci are disjoint unions of loci that are, respectively, a $\overline{\mathrm{PGL}_i}$-bundle and a $K\mathrm{GL}_i$-bundle over a product of Grassmannians.

As observed in [33], the Kausz compactification is then the closure of GL_ℓ inside the wonderful compactification of $\mathrm{PGL}_{\ell+1}$, see also [27]. The compactification $K\mathrm{GL}_\ell$ is smooth and projective over $\mathrm{Spec}\mathbb{Z}$ (Corollary 4.2 [29]).

The other property of the Kausz compactification that we will be using in the following is the fact that the complement of the dense open set GL_ℓ inside the compactification $K\mathrm{GL}_\ell$ is a normal crossing divisor (Corollary 4.2 [29]).

5.3. **The motive of the Kausz compactification.** We can use the description recalled above of the Kausz compactification, together with the blowup formula, to check that the virtual motive (class in the Grothendieck ring) of the Kausz compactification is Tate.

Proposition 5.1. *Let $K_0(\mathcal{V})$ be the Grothendieck ring of varieties (defined over $\mathbb{Q}$ or over $\mathbb{Z}$) and let $\mathbb{Z}[\mathbb{L}] \subset K_0(\mathcal{V})$ be the Tate subring generated by the Lefschetz motive $\mathbb{L} = [\mathbb{A}^1]$. For all $\ell \geq 1$ the class $[K\mathrm{GL}_\ell]$ is in $\mathbb{Z}[\mathbb{L}]$. Moreover, let $\mathcal{Z}_\ell$ be the normal crossings divisor $\mathcal{Z}_\ell = K\mathrm{GL}_\ell \smallsetminus \mathrm{GL}_\ell$. Then all the unions and intersections of components of $\mathcal{Z}_\ell$ have Grothendieck classes in $\mathbb{Z}[\mathbb{L}]$.*

Proof. We use the blowup formula for classes in the Grothendieck ring: if $\tilde{\mathcal{X}} = \mathrm{Bl}_{\mathcal{Y}}(\mathcal{X})$, where $\mathcal{Y}$ is of codimension $m+1$ in $\mathcal{X}$, then the classes satisfy

$$[\tilde{\mathcal{X}}] = [\mathcal{X}] + \sum_{k=1}^{m} [\mathcal{Y}]\mathbb{L}^k. \tag{5.1}$$

The Kausz compactification is obtained as an iterated blowup, starting with a projective space, whose class is in $\mathbb{Z}[\mathbb{L}]$ and blowing up at each step a smooth locus that is a bundle over a product of Grassmannians with fiber either a $K\mathrm{GL}_i$ or a $\overline{\mathrm{PGL}_i}$ for some $i < \ell$. The Grothendieck class of a bundle is the product of the class of the base and the class of the fiber. Classes of Grassmannians (and products of Grassmannians) are in $\mathbb{Z}[\mathbb{L}]$. The classes of the wonderful compactifications $\overline{\mathrm{PGL}_i}$ of PGL_i are also in $\mathbb{Z}[\mathbb{L}]$, since it is known that the motive of these wonderful compactifications are mixed Tate (see for instance [25]). Thus, it suffices to assume, inductively, that the classes $[K\mathrm{GL}_i] \in \mathbb{Z}[\mathbb{L}]$ for all $i < \ell$, and conclude via the blowup formula that $[K\mathrm{GL}_\ell] \in \mathbb{Z}[\mathbb{L}]$.
Consider then the boundary divisor $\mathcal{Z}_\ell = K\mathrm{GL}_\ell \smallsetminus \mathrm{GL}_\ell$. The geometry of the normal crossings divisor $\mathcal{Z}_\ell$ is described explicitly in Theorems 9.1 and 9.3 of [29]. It has components Y_i and Z_i, for $0 \leq i \leq \ell$, that correspond to the blowup loci described above. The multiple intersections $\cap_{i\in I} Y_i \cap \cap_{j\in J} Z_j$ of these components of $\mathcal{Z}_\ell$ are described in turn in terms of bundles over products of flag varieties with fibers that are lower dimensional compactifications $K\mathrm{GL}_i$ and $\overline{\mathrm{PGL}_i}$ and products. Again, flag varieties have cell decompositions, hence their Grothendieck classes are in $\mathbb{Z}[\mathbb{L}]$ and the rest of the argument proceeds as in the previous case. If arbitrary intersections of the

components of $\mathcal{Z}_\ell$ have classes in $\mathbb{Z}[\mathbb{L}]$ then arbitrary unions and unions of intersections also do by inclusion-exclusion in $K_0(\mathcal{V})$. □

Knowing that the Grothendieck class $[K\mathrm{GL}_\ell]$ is in the Tate subring $\mathbb{Z}[\mathbb{L}] \subset K_0(\mathcal{V})$ determines the motive in the category of pure motives with the numerical equivalence. More precisely, we have the following.

Proposition 5.2. *Let $h_{\mathrm{num}}(K\mathrm{GL}_\ell)$ denote the motive of the Kausz compactification $K\mathrm{GL}_\ell$ in the category of pure motives over $\mathbb{Q}$, with the numerical equivalence relation. Then $h_{\mathrm{num}}(K\mathrm{GL}_\ell)$ is in the subcategory generated by the Tate object. The same is true for arbitrary unions and intersections of the components of the boundary divisor $\mathcal{Z}_\ell$ of the compactification.*

Proof. The same argument used in Proposition 5.1 can be upgraded at the level of numerical motives. We replace the blowup formula (5.1) for Grothendieck classes with the corresponding formula for motives, which follows (already at the level of Chow motives) from Manin's identity principle:

$$h(\tilde{X}) = h(X) \oplus \bigoplus_{r=1}^{m} h(Y) \otimes \mathbb{L}^{\otimes r}, \tag{5.2}$$

with $\tilde{X} = \mathrm{Bl}_Y(X)$ the blowup of a smooth subvariety $Y \subset X$ of codimension $m+1$ in a smooth projective variety X, and with $\mathbb{L} = h^2(\mathbb{P}^1)$ is the Lefschetz motive. Moreover, we use the fact that, for numerical motives, the motive of a locally trivial fibration $X \to S$ with fiber Y is given by the product

$$h_{\mathrm{num}}(X) = h_{\mathrm{num}}(Y) \otimes h_{\mathrm{num}}(S), \tag{5.3}$$

see Exercise 13.2.2.2 of [5]. The decomposition (5.3) allows us to describe the numerical motives of the blowup loci of the iterated blowup construction of $K\mathrm{GL}_\ell$ as products of numerical motives of Grassmannians and of lower dimensional compactifications $K\mathrm{GL}_i$ and $\overline{\mathrm{PGL}}_i$. The motive of a Grassmannian can be computed explicitly as in [30], already at the level of Chow motives. If $G(d,n)$ denotes the Grassmannian of d-planes in k^n, the Chow motive $h(G(d,n))$ is given by

$$h(G(d,n)) = \bigoplus_{\lambda \in W^d} \mathbb{L}^{\otimes|\lambda|}, \tag{5.4}$$

where

$$W^d = \{\lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{N}^d \,|\, n-d \geq \lambda_1 \geq \cdots \geq \lambda_d \geq 0\}$$

and $|\lambda| = \sum_i \lambda_i$, see Lemma 3.1 of [30]. The same decomposition into powers of the Lefschetz motive holds at the numerical level. Moreover, we know (also already for Chow motives) that the motives $h(\overline{\mathrm{PGL}}_i)$ of the wonderful compactifications are Tate (see [25]), and we conclude the argument as in Proposition 5.1 by assuming inductively that the motives $h_{\mathrm{num}}(K\mathrm{GL}_i)$ are Tate, for $i < \ell$. The argument for the loci $\cap_{i\in I} Y_i \cap \cap_{j\in J} Z_j$ in $\mathcal{Z}_\ell$ is analogous. □

Remark 5.3. Proposition 5.2 also follows from Proposition 5.1 using the general fact that two numerical motives that have the same class in $K_0(\mathrm{Num}(k)_{\mathbb{Q}})$ are isomorphic as objects in $\mathrm{Num}(k)_{\mathbb{Q}}$, because of the semi-simplicity of the category of numerical motives, together with the existence, for $char(k) = 0$, of a unique ring homorphism (the motivic Euler characteristic) $\chi_{\mathrm{mot}} : K_0(\mathcal{V}_k) \to K_0(\mathrm{Num}(k)_{\mathbb{Q}})$. This is such that, for a smooth projective variety X, $\chi_{\mathrm{mot}}([X]) = [h_{\mathrm{num}}(X)]$, where $h_{\mathrm{num}}(X)$ is the motive of X in $\mathrm{Num}(k)_{\mathbb{Q}}$, see Corollary 13.2.2.1 of [5].

If we want to further upgrade the result of Proposition 5.2 to the level of Chow motives, we run into the difficulty that one no longer necessarily has the decomposition (5.3) for the motive of a locally trivial fibration. However, under some hypotheses on the existence of a cellular structure, one can still obtain a decomposition for motives of bundles, and more generally locally trivial fibrations whose fibers have cell decompositions with suitable properties, see [24], [25], [35].

Question 5.4. *Does the decomposition of the motive $h_{\rm num}(K{\rm GL}_\ell)$ described in the proof of Proposition 5.2 also hold for the Chow motive? In particular, can one obtain an explicit closed formula for the Chow motive $h(K{\rm GL}_\ell)$?*

Remark 5.5. Notice that, arguing as in Remark 5.3, if one assumes the Kimura–O'Sullivan conjecture or Voevodsky's nilpotence conjecture, then the result of Proposition 5.2 would also hold for the Chow motive, by arguing as in Lemma 13.2.1.1 of [5].

5.4. **Feynman integrals in momentum space and non-mixed-Tate examples.** It was shown in [7] that the parametric form of Feynman integrals in perturbative quantum field theory can be formulated as a (possibly divergent) period integral on the complement of a hypersurface defined by the vanishing of a combinatorial polynomial associated to the Feynman graphs. Namely, one writes the (unrenormalized) Feynman amplitudes for a *massless* scalar quantum field theory as integrals

$$U(\Gamma) = \frac{\Gamma(n - D\ell/2)}{(4\pi)^{\ell D/2}} \int_{\sigma_n} \frac{P_\Gamma(t,p)^{-n+D\ell/2}\omega_n}{\Psi_\Gamma(t)^{-n+D(\ell+1)/2}} \tag{5.5}$$

where $n = \#E_\Gamma$ is the number of internal edges, $\ell = b_1(\Gamma)$ is the number of loops, and the domain of integration is a simplex $\sigma_n = \{t \in \mathbb{R}^n_+ | \sum_i t_i = 1\}$. In the integration form, ω_n is the volume form and where P_Γ and Ψ_Γ are polynomials defined as follows. The graph polynomial is defined as

$$\Psi_\Gamma(t) = \sum_T \prod_{e\notin T} t_e$$

while the polynomial P_Γ is given by

$$P_\Gamma(p,t) = \sum_{C\subset\Gamma} s_C \prod_{e\in C} t_e$$

with the sum over cut-sets C (complements of a spanning tree plus one edge) and with variables s_C depending on the external momenta of the graph, $s_C = (\sum_{v\in V(\Gamma_1)} P_v)^2$ with $P_v = \sum_{e\in E_{ext}(\Gamma), t(e)=v} p_e$ for $\sum_{e\in E_{ext}(\Gamma)} p_e = 0$. In the range $-n + D\ell/2 \geq 0$, which includes the log divergent case $n = D\ell/2$, the Feynman amplitude is therefore the integral of an algebraic differential form defined on the complement of the graph hypersurface $\hat{X}_\Gamma = \{t \in \mathbb{A}^n \,|\, \Psi_\Gamma(t) = 0\}$. Divergences occur due to the intersections of the domain of integration σ_n with the hypersurface. Some regularization and renormalization procedure is required to separate the chain of integration from the divergence locus. We refer the reader to [7] (or to [32] for an introductory exposition).

It was originally conjectured by Kontsevich that the graph hypersurfaces $\hat{X}_\Gamma$ would always be mixed Tate motives, which would have explained the pervasive occurrence of multiple zeta values in Feynman integral computations observed in [8]. A general result of [6] disproved the conjecture, while more recent results of [10], [11], [18] showed explicit examples of Feynman graphs that give rise to non-mixed-Tate periods.

5.5. **Determinant hypersurface and parametric Feynman integrals.** In [4] the computation of parametric Feynman integrals was reformulated by replacing the graph hypersurface complement by the complement of the determinant hypersurface.

More precisely, the (affine) graph hypersurface $\hat{X}_\Gamma$ is defined by the vanishing of the graph polynomial Ψ_Γ, which can be written as a determinant

$$\Psi_\Gamma(t) = \det M_\Gamma(t) = \sum_T \prod_{e\notin T} t_e$$

with

$$(M_\Gamma)_{kr}(t) = \sum_{i=0}^n t_i \eta_{ik}\eta_{ir},$$

where the matrix η is given by

$$\eta_{ik} = \begin{cases} \pm 1 & \text{edge } \pm e_i \in \text{ loop } \ell_k \\ 0 & \text{otherwise} \end{cases}$$

One considers then the map

$$\Upsilon : \mathbb{A}^n \to \mathbb{A}^{\ell^2}, \quad \Upsilon(t)_{kr} = \sum_i t_i \eta_{ik} \eta_{ir}$$

that realizes the graph hypersurface as the preimage

$$\hat{X}_\Gamma = \Upsilon^{-1}(\hat{\mathcal{D}}_\ell)$$

of the determinant hypersurface $\hat{\mathcal{D}}_\ell = \{\det(x_{ij}) = 0\}$.
It is shown in [4] that the map

$$\Upsilon : \mathbb{A}^n \smallsetminus \hat{X}_\Gamma \hookrightarrow \mathbb{A}^{\ell^2} \smallsetminus \hat{\mathcal{D}}_\ell \tag{5.6}$$

is an embedding whenever the graph Γ 3-edge-connected with a closed 2-cell embedding of face width ≥ 3.
When the map Υ is an embedding, one can, without loss of information, rewrite the parametric Feynman integral as

$$U(\Gamma) = \int_{\Upsilon(\sigma_n)} \frac{\mathcal{P}_\Gamma(x,p)^{-n+D\ell/2}\omega_\Gamma(x)}{\det(x)^{-n+(\ell+1)D/2}}. \tag{5.7}$$

The question on the nature of periods is then reformulated in [4] by considering a normal crossings divisor $\hat{\Sigma}_\Gamma$ in $\mathbb{A}^{\ell^2}$ with $\Upsilon(\partial\sigma_n) \subset \hat{\Sigma}_\Gamma$ and considering the motive

$$\mathfrak{m}(\mathbb{A}^{\ell^2} \smallsetminus \hat{\mathcal{D}}_\ell, \hat{\Sigma}_\Gamma \smallsetminus (\hat{\Sigma}_\Gamma \cap \hat{\mathcal{D}}_\ell)). \tag{5.8}$$

It is well known that the motive of the determinant hypersurface complement $\mathbb{A}^{\ell^2} \smallsetminus \hat{\mathcal{D}}_\ell$ is mixed Tate, with Grothendieck class

$$[\mathbb{A}^{\ell^2} \smallsetminus \hat{\mathcal{D}}_\ell] = \mathbb{L}^{\binom{\ell}{2}} \prod_{i=1}^{\ell} (\mathbb{L}^i - 1).$$

However, as shown in [4], the nature of the motive (5.8) is much more difficult to discern, because of the nature of the intersection between the divisor $\hat{\Sigma}_\Gamma$ and the determinant hypersurface. It is shown in [4] that one can consider a divisor $\hat{\Sigma}_{\ell,g}$ that only depends on $\ell = b_1(\Gamma)$ and on the minimal genus g of the surface S_g realizing the closed 2-cell embedding of Γ,

$$\hat{\Sigma}_{\ell,g} = L_1 \cup \cdots \cup L_{\binom{f}{2}}, \tag{5.9}$$

with $f = \ell - 2g + 1$ and

$$\begin{cases} x_{ij} &= 0 \quad 1 \leq i < j \leq f-1 \\ x_{i1} + \cdots + x_{i,f-1} &= 0 \quad 1 \leq i \leq f-1. \end{cases}$$

It is also shown in [4] that the motives (5.8) are mixed Tate if the *varieties of frames*

$$\mathbb{F}(V_1, \ldots, V_\ell) := \{(v_1, \ldots, v_\ell) \in \mathbb{A}^{\ell^2} \smallsetminus \hat{\mathcal{D}}_\ell \,|\, v_k \in V_k\}$$

are mixed Tate. This question is closely related to the geometry of intersections of unions of Schubert cells in flag varieties and Kazhdan–Lusztig theory.

In this paper we will follow a different approach, which uses the same reformulation of parametric Feynman integrals in momentum space in terms of determinant hypersurfaces, as in [4], but instead of computing the integral in the determinant hypersurface complement, pulls it back to a compactification of GL_ℓ, following the model of computations of Feynman integrals in configuration space described in [12].

5.6. **Pullback to the Kausz compactification and forms with logarithmic poles.** For fixed D, ℓ and for assigned external momenta p, we now consider the algebraic differential form

$$\eta_{\Gamma,D,\ell,p}(x) := \frac{\mathcal{P}_\Gamma(x,p)^{-n+D\ell/2}\omega_\Gamma(x)}{\det(x)^{-n+(\ell+1)D/2}}. \tag{5.10}$$

For simplicity, we write the above as $\eta_\Gamma(x)$. This is defined on the complement of the determinant hypersurface, $\mathbb{A}^{\ell^2} \smallsetminus \hat{\mathcal{D}}_\ell = \mathrm{GL}_\ell$. Thus, by pulling back to the Kausz compactification, we can regard it as an algebraic differential form on

$$K\mathrm{GL}_\ell \smallsetminus \mathcal{Z}_\ell = \mathrm{GL}_\ell,$$

where $\mathcal{Z}_\ell$ is the normal crossings divisor at the boundary of the Kausz compactification.

Let $\mathcal{Z}$ be a simple normal crossings divisor in a smooth projective variety $\mathcal{X}$ such that the complement $\mathcal{U} = \mathcal{X} \smallsetminus \mathcal{Z}$ is affine. Then any deRham cohomology class of $\mathcal{U}$ can be realized using forms with logarithmic poles, [16],

$$H^*(\mathcal{U}) \simeq \mathbb{H}^*(\mathcal{X}, \Omega^*_{\mathcal{X}}(\log(\mathcal{Z}))).$$

This is the case for the Kausz compactifications $K\mathrm{GL}_\ell$.

Thus, there is a form $\beta_\Gamma = \beta_{\Gamma,D,\ell,p}$ on $K\mathrm{GL}_\ell$ with logarithmic poles along the normal crossings divisor $\mathcal{Z}_\ell$, such that

$$[\beta_\Gamma] = [\eta_\Gamma] \in H^*_{dR}(K\mathrm{GL}_\ell \smallsetminus \mathcal{Z}_\ell) = H^*_{dR}(\mathrm{GL}_\ell). \tag{5.11}$$

If we assume that the external momenta p in the polynomial $\mathcal{P}_\Gamma(x,p)$ are rational, then the form $\eta_\Gamma = \eta_{\Gamma,D,\ell,p}(x)$ is an algebraic differential form defined over $\mathbb{Q}$, hence we can also assume that the cohomologous form with logarithmic poles β_Γ is defined over $\mathbb{Q}$.

5.7. **Renormalization in $K\mathrm{GL}_\ell$.** We focus on the case where $X_\ell = K\mathrm{GL}_\ell$, with Y_ℓ the normal crossings divisor of the Kausz compactification and with the proper transform of the divisor $\Sigma_{\ell,g}$ described in (5.9). The morphism $\phi : \mathcal{H} \to \mathcal{M}^*_{X_\ell,Y_\ell}$ assigns to a Feynman graph Γ a meromorphic differential form $\beta_\Gamma = \beta_{\gamma,D,\ell,p}$ with logarithmic poles along Y_ℓ satisfying (5.11).

We then perform the Birkhoff factorization, and we denote by β^+_Γ the regular differential form on $K\mathrm{GL}_\ell$ given by $\phi^+(\Gamma) = \beta^+_\Gamma$. Since we only have logarithmic poles, by Proposition 4.1 the operation becomes a simple pole subtraction and we have $\beta^+_\Gamma = (1-T)\beta_\Gamma$.

We replace the (divergent) integral (5.7) with the renormalized (convergent) integral

$$R(\Gamma) = \int_{\tilde{\Upsilon}(\sigma_n)} \beta^+_{\Gamma,D,\ell,p}, \tag{5.12}$$

where $\tilde{\Upsilon}(\sigma_n)$ is the pullback to $K\mathrm{GL}_\ell$ of the domain of integration $\Upsilon(\sigma_n)$.

5.8. **Nature of the period.** We then discuss the nature of the period obtained by the evaluation of (5.12). We need a preliminary result.

Lemma 5.6. *The divisor $\Sigma_{\ell,g}$ is a mixed Tate configuration.*

Proof. By (5.9), $\Sigma_{\ell,g}$ and any arbitrary union of components are hyperplane arrangements. The Grothendieck class of an arrangement A in $\mathbb{P}^n$ is explicitly given (Theorem 1.1. of [3]) by

$$[A] = [\mathbb{P}^n] - \frac{\chi_{\hat{A}}(\mathbb{L})}{\mathbb{L}-1},$$

where $\chi_{\hat{A}}(t)$ is the characteristic polynomial of the associated central arrangement $\hat{A}$ in $\mathbb{A}^{n+1}$. It then follows by inclusion-exclusion in the Grothendieck ring that all unions and intersections of components of A are mixed Tate. The argument can be lifted from the Grothendieck classes to the motives by arguing as in Remarks 5.3 and 5.5. □

Remark 5.7. The central difficulty in the approach of [4], which is to analyze the nature of the motive of $\Sigma_{\ell,g} \smallsetminus (\Sigma_{\ell,g} \cap \mathcal{D}_\ell)$, is bypassed here by considering only the much simpler motive of $\Sigma_{\ell,g}$.

We then have the following conclusion.

Proposition 5.8. *The integral* (5.12) *is a period of a mixed Tate motive.*

Proof. Since this is an integral of an algebraic differential form defined on the compactification $K\mathrm{GL}_\ell$, the integral (5.12) is a genuine period, in the sense of algebraic geometry, of $K\mathrm{GL}_\ell$. By Proposition 5.1 and Proposition 5.2 we know that the Grothendieck class $[K\mathrm{GL}_\ell]$ and the numerical pure motive $h_{\mathrm{num}}(K\mathrm{GL}_\ell)$ are Tate. Assuming the Kimura-O'Sullivan or the Voevodsky nilpotence conjecture, we can conclude as in Remark 5.5 that the Chow motive $h(K\mathrm{GL}_\ell)$ is also Tate. We also know from Lemma 5.6 that the Chow motive $h(\Sigma_{\ell,g})$ is Tate. There is an embedding of pure motives into mixed motives: see [5] for the details and the subtleties of passing from the cohomological formulation of pure motives to the homological formulation of mixed motives. Under this embedding we obtain objects $\mathfrak{m}(K\mathrm{GL}_\ell)$ and $\mathfrak{m}(\Sigma_{\ell,g})$ ain the subcategory of mixed Tate motives $\mathcal{MTM}(\mathbb{Q})$ inside the Voevodsky triangulated category of mixed motives $\mathcal{DM}(\mathbb{Q})$. Since the divisor $\Sigma_{\ell,g}$ is a mixed Tate configuration, it then follows from [21] that the relative motive $\mathfrak{m}(K\mathrm{GL}_\ell, \Sigma_{\ell,g})$ is also mixed Tate. The statement then follows from [9]. □

One defines the category $\mathcal{MTM}(\mathbb{Z})$ of mixed Tate motives over $\mathbb{Z}$ as mixed Tate motives in $\mathcal{MTM}(\mathbb{Q})$ that are unramified over $\mathbb{Z}$.

Question 5.9. *Are the motives* $\mathfrak{m}(K\mathrm{GL}_\ell)$ *unramified over* $\mathbb{Z}$*?*

This question can be approached in a way analogous to our previous discussion of Question 5.4, namely using the blowup formula and the description of the blowup loci as bundles over products of Grassmannians with fibers that are other $K\mathrm{GL}_i$ and $\overline{PGL_i}$.

Remark 5.10. If the unramified condition holds, then one can conclude from [9] and the previous Proposition 5.8 that the integral (5.12) is a $\mathbb{Q}[2\pi i]$-linear combination of multiple zeta values.

5.9. **Comparison with Feynman integrals.** The result obtained in this way clearly differs from the usual computation of Feynman integrals, where non-mixed-Tate periods are known to occur, [10], [11]. There are several reasons behind this difference, which we now discuss briefly.

There is loss of information in mapping the computation of the Feynman integral from the complement of the graph hypersurface (as in [7], [10], [11]) to the complement of the determinant hypersurface (as in [4]), when the combinatorial conditions on the graph recalled in §5.5 are not satisfied. In such cases the map (5.6) need not be an embedding, hence part of the information contained in the Feynman integral calculation (5.5) will be lost in passing to (5.7).

However, this type of loss of information does not affect some of the cases where non-mixed Tate motives are known to appear in the momentum space Feynman amplitude.

Example 5.11. Let Γ be the graph with 14 edges that gives a counterexample to the Kontsevich polynomial countability conjecture, in Section 1 of [18]. It can be verified by direct inspection that the map $\Upsilon : \mathbb{A}^n \to \mathbb{A}^{\ell^2}$ of (5.6), with $n = \#E(\Gamma)$ and $\ell = b_1(\Gamma)$, is an embedding.

Question 5.12. Is the map $\Upsilon : \mathbb{A}^n \to \mathbb{A}^{\ell^2}$ of (5.6) an embedding for all the currently known explicit counterexamples ([18], [10], [37], [11])?

More seriously, even for integrals where the map (5.6) is an embedding, it is clear that the regularization and renormalization procedure described here, using the Kausz compactification and subtraction of residues for forms with logarithmic poles, is not equivalent to the usual renormalization procedures of the regularized integrals. For instance, our regularized form (hence our regularized integral) can be trivial in cases where the usual regularization and renormalization would give a non-trivial result.

Part of the information loss coming from pole subtraction on the differential form is compensated by keeping track of the residues. However, in our setting these also deliver only mixed Tate periods, so that even when this information is included, one still loses the richer structure of the periods arising from other methods of regularization and renormalization, adopted in the physics literature.

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