POTTS MODELS WITH MAGNETIC FIELD: ARITHMETIC, GEOMETRY, AND COMPUTATION

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ABSTRACT. We give a sheaf theoretic interpretation of Potts models with external magnetic field, in terms of constructible sheaves and their Euler characteristics. We show that the polynomial countability question for the hypersurfaces defined by the vanishing of the partition function is affected by changes in the magnetic field: elementary examples suffice to see non-polynomially countable cases that become polynomially countable after a perturbation of the magnetic field. The same recursive formula for the Grothendieck classes, under edge-doubling operations, holds as in the case without magnetic field, but the closed formulae for specific examples like banana graphs differ in the presence of magnetic field. We give examples of computation of the Euler characteristic with compact support, for the set of real zeros, and find a similar exponential growth with the size of the graph. This can be viewed as a measure of topological and algorithmic complexity. We also consider the computational complexity question for evaluations of the polynomial, and show both tractable and NP-hard examples, using dynamic programming.

CONTENTS

1. Introduction

Several combinatorial graph polynomials have physical significance, either as partition functions of statistical mechanical models on graphs (Ising and Potts models), or as the Kirchhoff and Symanzik polynomials that appear in the parametric form of Feynman integrals in perturbative quantum field theory. In both cases, it is interesting to consider various questions related to the properties of these polynomials and of the hypersurfaces they define. For a survey of the quantum field theory case, we refer the reader to [15], and for the case of Potts models, to [1], [16] and to the general survey [20].

In this paper, we focus on another such polynomial with physical significance: the **V**-polynomial, which gives the partition function of the Potts models with external magnetic field.

After recalling some general facts about these polynomials, we show in §2 that they admit a sheaf theoretic interpretation as the Euler characteristics of a constructible complex $\mathcal{F}_\Gamma^\bullet$ over the graph configuration space $\mathrm{Conf}_\Gamma(X)$ of a smooth projective variety. This addresses a question posed to the second author by Spencer Bloch.

In §3, we consider the hypersurfaces defined by the vanishing of the **V**-polynomial, and the question of whether these varieties are polynomially countable, that is, whether the counting of points over finite fields $\mathbb{F}_p$ is a polynomial in p. In the case of quantum field theory, the analogous polynomial countability question has drawn a lot of attention in recent years, in relation to questions on the occurrence of motives and periods in Feynman integrals. Counterexamples to polynomial countability for the Kirchhoff polynomials of quantum field theory are very elusive, and only occur for combinatorially complicated graphs with a large number of edges and loops (see the recently found examples in [9] and [19]). It is much simpler to find non-polynomially countable examples in the case of the Potts model partition function, see [16]. As expected, even smaller graphs give rise to non-polynomially countable hypersurfaces in the case of the **V**-polynomial. However, a new phenomenon occurs: polynomial countability depends on the magnetic field, and can occasionally be restored by modifying the magnetic field. We illustrate these phenomena in the simplest examples.

In §4 we consider the class in the Grothendieck ring of varieties of the complement of the hypersurface defined by the vanishing of the **V**-polynomial. We show that the same recursive formula for edge-doubling, proved in [1] in the case without magnetic field, continues to hold in this case. However, the presence of magnetic field alters the initial terms of the recursion. We compute the resulting closed form of the class for the case of banana graphs and compare it with the case without magnetic field. As in [1], we then focus on the set of real zeros, and its Euler characteristic with compact support, as a measure of complexity (topological and algorithmic) of the analytic set of real zeros. We provide simple examples where this quantity grows exponentially with the size of the graph.

In §5 we consider a different kind of complexity question regarding the **V**-polynomials, namely the computational complexity of evaluating at a point. Using dynamic programming, we show that line and polygon graphs are tractable, while full binary trees, and trees that limit to the line are NP-hard.

1.1. **The V-polynomial.**
The correspondence between the Tutte polynomial and the partition function assumes a zero-field Hamiltonian [12], which excludes several important cases, including the presence of an external magnetic field. However, there exists a combinatorial

polynomial that is the evaluation of the Potts model with an external field, the **V**-polynomial. In this paper, we will study the algebraic, topological, and computational complexity of the **V**-polynomial.

Let Γ be a finite graph, with edge set $E(\Gamma)$ and vertex set $V(\Gamma)$. A vertex weight on Γ is a function $\omega : V(\Gamma) \to S$, with S a torsion-free abelian semigroup.

We recall from [12] the definition of the **V**-polynomial. It is a polynomial in $\mathbb{Z}[t = (t_e)_{e \in E(\Gamma)}, x = (x_s)_{s \in S}]$, where the t_e are edge variables (edge weights), and the x_s account for the presence of the magnetic field. We view the **V**-polynomial as a map $\mathbf{V}_\Gamma : \mathbb{A}^{\#E(\Gamma)} \times \mathbb{A}^{\#S} \to \mathbb{A}$. For a subset $A \subseteq E(\Gamma)$, we denote by $\Gamma_A \subset \Gamma$ the subgraph of Γ with $V(\Gamma_A) = V(\Gamma)$ and $E(\Gamma_A) = A$. Let $\Gamma_{A,j}$, for $j = 1, \ldots, b_0(\Gamma_A)$ be the connected components of Γ_A. Then the **V**-polynomial is defined as

$$(1.1) \qquad \mathbf{V}_\Gamma(t, x) = \sum_{A \subseteq E(\Gamma)} \prod_{j=1}^{b_0(\Gamma_A)} x_{s_j} \prod_{e \in A} t_e,$$

where $s_j = \sum_{v \in \Gamma_{A,j}} \omega(v)$ is the sum of the weights attached to all the vertices in the j-th component.

The **V**-polynomial is determined recursively by the deletion–contraction relation

- For an edge $e \in E(\Gamma)$ that is not a looping edge,

$$(1.2) \qquad \mathbf{V}_\Gamma(t, x) = \mathbf{V}_{\Gamma \smallsetminus e}(\hat{t}, x) + t_e \, \mathbf{V}_{\Gamma/e}(\hat{t}, x),$$

    with $\hat{t}$ the vector of edge variables with t_e removed.
- For a looping edge e

$$(1.3) \qquad \mathbf{V}_\Gamma(t, x) = (t_e + 1)\mathbf{V}_{\Gamma \smallsetminus e}(\hat{t}, x),$$

- If Γ consists of a set of vertices $V(\Gamma)$ and no edges, $E(\Gamma) = \emptyset$, then

$$(1.4) \qquad \mathbf{V}_\Gamma(t, x) = \prod_{v \in V(\Gamma)} x_{\omega(v)},$$

    where $\omega : V(\Gamma) \to S$ is the vertex weight.

Relations between the **V**-polynomial, the W-polynomial of [17], and the multivariable Tutte polynomial are described in [12].

1.2. The V-polynomial and magnetic field.

The physical interpretation of the **V**-polynomial as partition function of the Potts model with magnetic field comes from rewriting the Fourtuin-Kastelyn representation (1.1) of the polynomial as the partition function

$$Z_\Gamma = \sum_{A \subseteq E(\Gamma)} \prod_{j=1}^{b_0(\Gamma_A)} X_{M_{c_j}} \prod_{a \in A} (e^{-\beta J_e} - 1),$$

where β is a thermodynamic inverse temperature parameter, the J_e are the nearest-neighbor interaction energies along the edges, and M is the magnetic field vector, with

$$X_{M_{c_j}} = \sum_{v \in V(\Gamma_{A,j})} e^{-\beta M_v},$$

with $\Gamma_{A,j}$ the j-th connected component of the graph Γ_A.

2. A SHEAF-THEORETIC INTERPRETATION OF POTTS MODELS WITH MAGNETIC FIELDS

It is known (see [11]) that the values at integers n of the chromatic polynomial of a graph are the Euler characteristics of the graph configuration spaces $\mathrm{Conf}_\Gamma(\mathbb{P}^{n-1})$.

A natural question then arises: whether one can similarly interpret the $\mathbf{V}$-polynomial $\mathbf{V}_\Gamma(t, x)$ of the Potts model with magnetic field as an Euler characteristic of some constructible sheaf (or constructible complex) $\mathcal{F}_\Gamma^\bullet$ on the configuration spaces $\mathrm{Conf}_\Gamma(X)$ of some smooth projective variety X. In this section we give a positive answer to this question. For generalities about constructible sheaves, we refer the reader to the monographs [8] and [13].

The graph configuration spaces are defined as follows. Let X be a smooth projective variety and let Γ be a finite graph with no looping edges and no parallel edges. Then one defines

$$(2.1) \qquad \mathrm{Conf}_\Gamma(X) = X^{\#V(\Gamma)} \smallsetminus \cup_{e \in E(\Gamma)} \Delta_e,$$

where $\Delta_e \subset X^{\#V(\Gamma)}$ is the diagonal $\Delta_e = \{(x_v)_{v \in V(\Gamma)} \,|\, x_{s(e)} = x_{t(e)}\}$, where $\{s(e), t(e)\} = \partial e$. Graph configuration spaces and their compactifications were recently used in Feynman integral computations in relation to periods and motives, [5], [6].

In order to recover the $\mathbf{V}_\Gamma(t, x)$ as Euler characteristic $\chi(\mathrm{Conf}_\Gamma(X), \mathcal{F}_\Gamma^\bullet)$, it suffices to show that the latter satisfies the deletion-contraction relation (1.2) and the normalization (1.4), which completely characterize the $\mathbf{V}$-polynomial.

Theorem 2.1. *Let Γ be a finite graph with no looping edges and no parallel edges. Let S be the set of vertex weights. Let X be a smooth projective variety, and, for each $s \in S$, let $\mathcal{F}_s^\bullet$ be a constructible complex on X with*

$$(2.2) \qquad x_s := \chi(X, \mathcal{F}_s^\bullet).$$

Assume all the $\chi(X, \mathcal{F}_s^\bullet) \neq 0$, for all $s \in S$. For $\Delta_e \subset X_{s(e)} \times X_{t(e)}$ the edge diagonal, and for $\mathcal{F}_{\omega(s(e))}^\bullet$, $\mathcal{F}_{\omega(t(e))}^\bullet$ constructible complexes on X, let

$$(2.3) \qquad t_e := -\frac{\chi(\Delta, \iota_\Delta^!(\mathcal{F}_{\omega(s(e))}^\bullet \overset{L}{\boxtimes} \mathcal{F}_{\omega(t(e))}^\bullet))}{\chi(X, \mathcal{F}_{\omega(s(e))+\omega(t(e))}^\bullet)}.$$

There is a constructible complex $\mathcal{F}_\Gamma^\bullet$ on $\mathrm{Conf}_\Gamma(X)$, determined by the constructible complexes $\mathcal{F}_s^\bullet$ on X, with $s \in S$, such that the $\mathbf{V}$-polynomial satisfies

$$(2.4) \qquad \mathbf{V}_\Gamma(t, x) = \chi(\mathrm{Conf}_\Gamma(X), \mathcal{F}_\Gamma^\bullet).$$

Proof. First note, that if Γ has no parallel edges, then Γ/e has no looping edges. Thus, both deletion and contraction of edges maintain the class of graphs with no looping and no parallel edges. Let e be an edge in Γ. Then

$$\mathrm{Conf}_{\Gamma \smallsetminus e}(X) = \mathrm{Conf}_\Gamma(X) \cup \mathrm{Conf}_{\Gamma/e}(X)$$

is a Whitney regular stratification (see Example 1.8 of [7]). Thus, by theorem 4.1.22 of [8], given a constructible complex $\mathcal{F}^\bullet$ on $\mathrm{Conf}_{\Gamma \smallsetminus e}(X)$, we have

$$\chi(\mathrm{Conf}_\Gamma(X), \mathcal{F}^\bullet) = \chi(\mathrm{Conf}_{\Gamma \smallsetminus e}(X), \mathcal{F}^\bullet) - \chi(\mathrm{Conf}_{\Gamma/e}(X), \iota^! \mathcal{F}^\bullet).$$

Iterating this relation by repeatedly removing and contracting edges, until there are no edges left, we obtain an expression for $\chi(\mathrm{Conf}_\Gamma(X), \mathcal{F}^\bullet)$ as a sum over the set of all subsets $A \subseteq E(\Gamma)$, of terms (with signs) of the form $\chi(\mathrm{Conf}_{\Gamma_A/A}(X), \iota_A^! \mathcal{F}^\bullet)$, where Γ_A is the graph

with vertex set $V(\Gamma_A) = V(\Gamma)$ and edge set $E(\Gamma_A) = A$, while Γ_A/A is the graph obtained from Γ_A by contracting all the edges. This is a graph with $\#V(\Gamma_A/A) = b_0(\Gamma_A)$, the number of connected components, and with $E(\Gamma_A/A) = \emptyset$. We consider on $\mathrm{Conf}_{\Gamma_A/A}(X) = X^{b_0(\Gamma_A)}$ a constructible complex of the form

$$\mathcal{F}^{\bullet}_{\Gamma_A/A} := \overset{L}{\boxtimes}_{j=1,\ldots,b_0(\Gamma_A)} \mathcal{F}^{\bullet}_{s_j},$$

where the $\mathcal{F}^{\bullet}_{s_j}$ are the constructible complexes in the collection $\{\mathcal{F}^{\bullet}_s\}_{s \in S}$, where $s_j = \omega(v_j)$ is the weight of the j-th vertex v_j of Γ_A/A. Then, by the Künneth formula of Corollary 2.3.31 of [8], we have

$$\chi(\mathrm{Conf}_{\Gamma_A/A}(X), \mathcal{F}^{\bullet}_{\Gamma_A/A}) = \prod_{j=1}^{b_0(\Gamma_A)} \chi(X, \mathcal{F}^{\bullet}_{s_j}).$$

Thus, we only need then to compare the Euler characteristic $\chi(\mathrm{Conf}_{\Gamma_A/A}(X), \iota^!_A \mathcal{F}^{\bullet})$ with the Euler characteristic $\chi(\mathrm{Conf}_{\Gamma_A/A}(X), \mathcal{F}^{\bullet}_{\Gamma_A/A})$.

Recall that, if v_1 and v_2 are two vertices in a graph Γ, connected by an edge $e \in E(\Gamma)$ and if v is the corresponding vertex in the contraction graph Γ/e, then the vertex weights satisfy $\omega(v) = \omega(v_1) + \omega(v_2) \in S$, see §2.2. of [12]. Consider the constructible complex $\mathcal{F}^{\bullet} = \overset{L}{\boxtimes}_{v \in V(\Gamma)} \mathcal{F}^{\bullet}_{\omega(v)}$ on $X^{\#V(\Gamma)}$. We have $\chi(X^{\#V(\Gamma)}, \mathcal{F}^{\bullet}) = \prod_v \chi(X, \mathcal{F}^{\bullet}_{\omega(v)})$. When we contract an edge $e \in A$, we obtain, by definition of the variable t_e

$$\chi(X, \iota^!_{\Delta_e}(\mathcal{F}^{\bullet}_{\omega(s(e))} \overset{L}{\boxtimes} \mathcal{F}^{\bullet}_{\omega(t(e))})) = -t_e \, \chi(X, \mathcal{F}^{\bullet}_{\omega(s(e))+\omega(t(e))}).$$

Let $V(\Gamma_{A,j})$ be the set of vertices in the j-th connected component of Γ_A. Inductively, we obtain

$$\chi(\mathrm{Conf}_{\Gamma_A/A}(X), \iota^!_A \mathcal{F}^{\bullet}) = (-1)^{\#A} \prod_{e \in A} t_e \prod_{j=1}^{b_0(\Gamma_A)} \chi(X, \mathcal{F}^{\bullet}_{\sum_{v \in V(\Gamma_{A,j})} \omega(v)}).$$

Thus, we obtain

$$\chi(\mathrm{Conf}_{\Gamma}(X), \mathcal{F}^{\bullet}) = \sum_{A \subseteq E(\Gamma)} \prod_{e \in A} t_e \prod_{j=1}^{b_0(\Gamma_A)} x_j.$$

where $x_j = \chi(X, \mathcal{F}^{\bullet}_{\sum_{v \in V(\Gamma_{A,j})} \omega(v)})$. $\qquad\square$

3. Polynomial countability (arithmetic complexity)

In this section we consider the polynomial countability question for the affine hypersurfaces defined by the vanishing of the **V**-polynomial. We show that one can find non-polynomially countable examples already for the simplest polygonal graphs. However, we also show in a very simple example that one can sometime restore polynomial countability by changes to the magnetic field.

3.1. **The V-hypersurfaces.** Let $Var(\Gamma) \subset E(\Gamma) \times S$ be the set of indices of variables (t, x) the **V**-polynomial of Γ depends on, and let

$$(3.1) \qquad \mathbb{V}_\Gamma = \{(t, x) \in \mathbb{A}^{\#Var(\Gamma)} \,|\, V_\Gamma(t, x) = 0\} \subset \mathbb{A}^{\#Var(\Gamma)}$$

be the affine hypersurface determined by the vanishing of the **V**-polynomial. Since the polynomial has integer coefficients (in fact, all coefficients are equal to 10), we can regard the variety $\mathbb{V}_\Gamma$ as defined over $\mathbb{Q}$, and in fact over $\mathbb{Z}$.

For the reduction of $\mathbb{V}_\Gamma$ modulo a prime p, we let $N_{\mathbb{V}_\Gamma}(q)$ be the counting function that gives the number of $\mathbb{F}_q$-points, for a power $q = p^r$. The hypersurface $\mathbb{V}_\Gamma$ is polynomial countable if, for all p and $q = p^r$, the counting function $N_{\mathbb{V}_\Gamma}(q)$ is a polynomial in q with $\mathbb{Z}$-coefficients.

We can regard the behavior (polynomial or non-polynomial) of the counting function $N_{\mathbb{V}_\Gamma}(q)$ as an indicator of the arithmetic (or motivic) complexity of the hypersurfaces $\mathbb{V}_\Gamma$.

3.2. **The polynomial countability question.** For the Kirchhoff polynomial, all graphs with fewer than 12 edges were verified to give rise to polynomially countable hypersurfaces in [22], using a method of [21]. However, a general result of [3] showed that the graph hypersurfaces additively generate a localization of the Grothendieck ring of varieties, hence most graphs have necessarily non-polynomially countable hypersurfaces. Indeed, more recently, explicit non-polynomially countable examples (starting at 14 edges) were obtained in [9] and [19]. The analogous polynomial countability question for the hypersurfaces defined by the Potts model polynomial is considerably simpler. It was shown in [16] that already the Potts model hypersurface associated to the tetrahedron graph $\Gamma = K_4$ is not polynomially countable. Similarly, hypersurfaces defined by polynomials arising in Ponzano–Regge models of quantum gravity were shown to be non-polynomially countable for the tetrahedron graph, [14]. Clearly, one expects that polynomial countability will fail even more easily in the case of Potts models with external magnetic fields. This is indeed the case, in the sense that it is very easy to find very small graphs for which the **V**-hypersurface is not polynomially countable. However, a new phenomenon occurs: a perturbation of the magnetic field can restore polynomial countability.

3.3. **Constant magnetic field and perturbations.** We discuss here the simplest possible example that illustrates the phenomenon described above. We first consider the triangle graph with constant external magnetic field. We show that the associated **V**-hypersurface is not polynomially countable. We then show that changing the magnetic field can restore polynomial countability.

Lemma 3.1. *The **V**-polynomial for the triangle graph with a constant magnetic field (equal to 1 at each vertex) is given by*

$$(3.2) \qquad x_1^3 + (t_{e_1} + t_{e_2} + t_{e_3})x_2x_1 + (t_{e_1}t_{e_2} + t_{e_2}t_{e_3} + t_{e_1}t_{e_3} + t_{e_1}t_{e_2}t_{e_3})x_3.$$

Proof. This follows directly from 1.1, in the case with a constant magnetic field. $\square$

Theorem 3.2. *The **V**-hypersurface of the triangle graph with a constant magnetic field equal to 1 at each vertex is not polynomially countable.*

Proof. We write (3.2) equivalently as

$$(3.3) \qquad x_1^3 + (t_{e_1} + t_{e_2} + t_{e_3})x_2x_1 + x_3(t_{e_1}(t_{e_2} + t_{e_3} + t_{e_2}t_{e_3}) + t_{e_2}t_{e_3}).$$

We break the counting of zeros of the polynomial over a finite field $\mathbb{F}_p$ into several cases. First, when $x_1 = 0$ then x_2 can be anything. Still assuming $x_1 = 0$, the case when $x_3 = 0$ then contributes p^4 zeros. When $x_3 \neq 0$ (with $x_1 = 0$), then we have

$$(3.4) \qquad t_{e_1}(t_{e_2} + t_{e_3} + t_{e_2}t_{e_3}) + t_{e_2}t_{e_3} = 0.$$

If $t_{e_2} + t_{e_3} + t_{e_2}t_{e_3} = 0$, then $t_{e_2}t_{e_3} = 0$, so both t_{e_2} and t_{e_3} are 0 and t_{e_1} can be anything. Thus, this case contributes $p^2(p-1)$ points. Otherwise, we have

$$t_{e_2} \neq -\frac{t_{e_3}}{t_{e_3} + 1} \quad \text{and} \quad t_{e_1} = -\frac{t_{e_2}t_{e_3}}{t_{e_2} + t_{e_3} + t_{e_2}t_{e_3}},$$

contributing $p^2(p-1)^2$ points.

Now, suppose that $x_1 \neq 0$. Then, when $t_{e_1} + t_{e_2} + t_{e_3} \neq 0$, we can solve for x_2 with x_3 arbitrary. This case contributes $p^3(p-1)^2$ points. Now suppose that $t_{e_1} = -(t_{e_2} + t_{e_3})$. Then x_2 can be arbitrary. Substituting for t_{e_3}, we have

$$(3.5) \qquad x_1^3 + x_3(t_{e_1}^2 + t_{e_2}^2 + t_{e_1}t_{e_2} + t_{e_1}^2 t_{e_2} + t_{e_1}t_{e_2}^2) = 0.$$

Let us set

$$(3.6) \qquad f(t_{e_1}, t_{e_2}) = t_{e_1}^2 + t_{e_2}^2 + t_{e_1}t_{e_2} + t_{e_1}^2 t_{e_2} + t_{e_1}t_{e_2}^2.$$

Then, whenever $f(t_{e_1}, t_{e_2}) \neq 0$, we have that

$$x_3 = \frac{x_1^3}{f(t_{e_1}, t_{e_2})}.$$

This contributes a number of points equal to $(p-1)p(p^2 - Z[f(t_{e_1}, t_{e_2})])$, where $Z[f(t_{e_1}, t_{e_2})]$ denotes the number of $\mathbb{F}_p$-points of $f(t_{e_1}, t_{e_2}) = 0$. Now, arguing as in [22], if $Z[f(t_{e_1}, t_{e_2})]$ were a polynomial, it would have to be linear, which it is not by inspection. □

We now consider the same graph, but with a variable magnetic field. In particular, we look at the case where the magnetic field is zero at two of the vertices.

Lemma 3.3. *The* **V***-polynomial of the triangle graph with a magnetic field of 1 at one of the vertices and 0 at the others is given by*

$$(3.7) \qquad x_1(x_0^2 + x_0(t_{e_1} + t_{e_2} + t_{e_3}) + t_{e_1}(t_{e_2} + t_{e_3} + t_{e_2}t_{e_3}) + t_{e_2}t_{e_3}).$$

Proof. Again, this follows immediately from 1.1. □

Theorem 3.4. *The* **V***-polynomial of the triangle graph with a magnetic field of 1 at one of the vertices and 0 at the others is polynomially countable.*

Proof. If $x_1 = 0$, then the polynomial (3.7) is zero, with the other variables free, hence contributing p^4 points over $\mathbb{F}_p$. If $x_1 \neq 0$, then suppose $x_0 \neq -t_{e_2} - t_{e_3} - t_{e_3}t_{e_2}$. In this case we have

$$t_{e_1} = \frac{x_0^2 + x_0(t_{e_2} + t_{e_3}) + t_{e_2}t_{e_3}}{x_0 + t_{e_2} + t_{e_3} + t_{e_2}t_{e_3}},$$

with t_{e_2} and with t_{e_3} free. This case thus contributes $(p-1)^2 p^2$ points. If $x_0 = -t_{e_2} - t_{e_3} - t_{e_3}t_{e_2}$, then, after substituting, we need to solve for $t_{e_2}t_{e_3} + t_{e_2}t_{e_3}^2 + t_{e_2}^2 t_{e_3} + t_{e_2}^2 t_{e_3}^2 = 0$. Counting the pairs which satisfy this equation, this contributes a total of $(p-1)p(2(p-2) + p)$ zeros.

Therefore, the total number of zeros of the polynomial is $4p - 7p^2 + 2p^3 + 2p^4$, which is a polynomial in p. □

4. The Euler characteristic (topological complexity)

The polynomial countability question considered in the previous section is closely related to the form of the class $[\mathbb{V}_\Gamma]$ of the hypersurface in the Grothendieck ring of varieties. Recall that, for R a field or ring, the Grothendieck ring $K_0(\mathcal{V}_R)$ of varieties over R is generated by the isomorphism classes $[X]$ of smooth quasi-projective varieties, with the inclusion–exclusion relation $[X] = [Y] + [X \smallsetminus Y]$ for $Y \subset X$ a closed embedding and with the product given by $[X][Y] = [X \times Y]$. The Lefschetz motive $\mathbb{L} = [\mathbb{A}^1]$ is the class of the affine line. If the class of a variety is a polynomial $[X] = \sum_k a_k \mathbb{L}^k$ in the Lefschetz motive, then X is polynomially countable, and the converse is conditional to conjectures on motives.

In the case of the graph polynomials of quantum field theory, [2], and of the Potts model without magnetic field, [1], recursive formulae were obtained for the Grothendieck classes of the hypersurface complements, under simple operations performed on the underlying graph, such as doubling or splitting edges.

In this section we consider the edge-doubling operation and we show that the same recursion continues to hold in the case with magnetic field, but the initial step of the recursion is changed by the presence of the magnetic field. We compute the resulting classes for the banana graphs.

We then compute the Euler characteristic with compact support of the set of real solutions, again in the case of the banana graphs, where we show it grows exponentially with the size of the graph. As shown in [1] (see also [4], [10], [23]), the Euler characteristic with compact support can be used as an estimate of algorithmic complexity of the analytic set of real solutions.

4.1. **Deletion–contraction relation and classes in the Grothendieck ring.** In both the cases of the graph polynomials of quantum field theory and the Potts model partition function, the key ingredient in order to obtain recursive formulae for the Grothendieck classes under iterated edge doubling, is a formula relating the deletion-contraction property of the polynomial to a relation between the classes. One does not have an actual deletion-contraction relation at the level of Grothendieck classes (for example, because if such a relation existed, then the classes would always remain in the Tate part $\mathbb{Z}[\mathbb{L}]$ of the Grothendieck ring, which we know is not the case). However, one obtains from (1.2) a relation

$$(4.1) \qquad [\mathbb{A}^{\#Var(\Gamma)} \smallsetminus \mathbb{V}_\Gamma] = \mathbb{L}[\mathbb{A}^{\#Var(\Gamma)-1} \smallsetminus (\mathbb{V}_{\Gamma \smallsetminus e} \cap \mathbb{V}_{\Gamma/e})] - [\mathbb{A}^{\#Var(\Gamma)-1} \smallsetminus \mathbb{V}_{\Gamma \smallsetminus e}].$$

The remaining cases (bridges and looping edges) are simpler, and give the class $[\mathbb{A}^{\#Var(\Gamma)} \smallsetminus \mathbb{V}_\Gamma]$ as a function of either the class of the deletion or that of the contraction and the Lefschetz motive, see [1].

4.2. **Edge doubling and parallel edges recursion.** Let $^e\Gamma$ denote the graph obtained from a given graph Γ by doubling an edge e that is not a looping edge. We also denote by $^m\Gamma$ the graph obtained by adding m edges parallel to e in Γ. Let f denote the new edge resulting from the doubling of an edge e. Using the deletion-contraction relation (1.2) for the **V**-polynomial, we obtain the following expression for the **V**-polynomials of $^e\Gamma$:

$$(4.2) \qquad V_{e\Gamma} = V_{\Gamma \smallsetminus e} + (t_e + t_f + t_e t_f) V_{\Gamma/e}.$$

As in [1], we set $u_e = 1 + t_e$ and $u_f = 1 + t_f$. With this change of variables, we can write (4.2) equivalently as

$$(4.3) \qquad V_{e\Gamma} = V_{\Gamma \smallsetminus e} + (u_e u_f - 1) V_{\Gamma/e}.$$

Let $Var(\Gamma) \subset E(\Gamma) \times S$ be the set of indices of variables (t, x) the **V**-polynomial of Γ depends on. This set depends not only on Γ but also on the assignment of the magnetic field. Let $\mathbb{V}_{e\Gamma} = \{(t, x) \in \mathbb{A}^{\#Var(^e\Gamma)} \,|\, V_{e\Gamma}(t, x) = 0\}$ denote the hypersurface in $\mathbb{A}^{\#Var(^e\Gamma)}$ determined by the vanishing of the **V**-polynomial. Let $\{\mathbb{V}_{e\Gamma}\}$ denote the class in the Grothendieck ring of the hypersurface complement

$$(4.4) \qquad \{\mathbb{V}_{e\Gamma}\} := [\mathbb{A}^{\#Var(^e\Gamma)} \smallsetminus \mathbb{V}_{e\Gamma}] \in K_0(\mathcal{V}_{\mathbb{Z}}).$$

The following result is the analog, in the case with magnetic field, of Theorem 6.1 of [1] for the Potts model without magnetic field. Notice that the presence of the magnetic field does not alter the edge-doubling relation, which remains the same as in the original Potts model case.

Theorem 4.1. *Let $^e\Gamma$ be the graph obtained by doubling an edge e in Γ, that is not a looping edge. Then the class $\{\mathbb{V}_{e\Gamma}\}$ of the hypersurface complement satisfies*

$$(4.5) \qquad \{\mathbb{V}_{e\Gamma}\} = \mathbb{T}\{\mathbb{V}_{\Gamma}\} - (\mathbb{T} + 1)\{Z(V_{\Gamma \smallsetminus e} - V_{\Gamma/e})\},$$

where $\mathbb{T} = [\mathbb{G}_m] = \mathbb{L} - 1$ is the class of the multiplicative group, with $\mathbb{L}$ the Lefschetz motive, and with $Z(V_{\Gamma \smallsetminus e} - V_{\Gamma/e})$ the locus defined by the vanishing of the polynomial $V_{\Gamma \smallsetminus e} - V_{\Gamma/e}$.

Proof. The argument is analogous to Theorem 6.1 of [1]. We break the computation of the Grothendieck class $\{V_{e\Gamma}\}$ into several cases. If $V_{\Gamma/e} = 0$, then $V_{\Gamma \smallsetminus e} \neq 0$, and u_e and u_f are free variables. Thus, this case contributes a term

$$(4.6) \qquad (\mathbb{T} + 1)^2 [\mathbb{V}_{\Gamma/e} \smallsetminus (\mathbb{V}_{\Gamma/e} \cap \mathbb{V}_{\Gamma \smallsetminus e})].$$

If $V_{\Gamma/e} \neq 0$, then we have one of two possibilities. The first is that $V_{\Gamma/e} = V_{\Gamma \smallsetminus e}$, in which case $u_e u_f \neq 0$, contributing a class

$$(4.7) \qquad \mathbb{T}^2 [Z(V_{\Gamma \smallsetminus e} - V_{\Gamma/e}) \cap (\mathbb{A}^{\#Var(\Gamma)-1} \smallsetminus \mathbb{V}_{\Gamma/e})].$$

The other possibility is that $V_{\Gamma/e} \neq V_{\Gamma \smallsetminus e}$. In this case $u_e u_f \neq c$ for some $c \neq 0$. If $u_e u_f = c$, then $u_f \neq 0$ and $u_e = c/u_f$, which gives a $\mathbb{L} - 1$. Therefore, this term contributes a class

$$(\mathbb{T}^2 + \mathbb{T} + 1)[(\mathbb{A}^{\#Var(\Gamma)-1} \smallsetminus \mathbb{V}_{\Gamma/e}) \smallsetminus Z(V_{\Gamma \smallsetminus e} - V_{\Gamma/e})].$$

Thus, the class of the hypersurface complement of $V_{e\Gamma}$ is given by

$$(\mathbb{T} + 1)^2 [\mathbb{V}_{\Gamma/e} \smallsetminus (\mathbb{V}_{\Gamma/e} \cap \mathbb{V}_{\Gamma \smallsetminus e})] + \mathbb{T}^2 [Z(V_{\Gamma \smallsetminus e} - V_{\Gamma/e}) \cap (\mathbb{A}^{\#Var(\Gamma)-1} \smallsetminus \mathbb{V}_{\Gamma/e})]$$

$$+ (\mathbb{T}^2 + \mathbb{T} + 1)[(\mathbb{A}^{\#Var(\Gamma)-1} \smallsetminus \mathbb{V}_{\Gamma/e}) \smallsetminus Z(V_{\Gamma \smallsetminus e} - V_{\Gamma/e})].$$

This can be simplified to

$$(\mathbb{T} + 1)^2 [\mathbb{V}_{\Gamma/e} \smallsetminus (\mathbb{V}_{\Gamma/e} \cap \mathbb{V}_{\Gamma \smallsetminus e})] - (\mathbb{T} + 1)[Z(V_{\Gamma \smallsetminus e} - V_{\Gamma/e}) \cap (\mathbb{A}^{\#Var(\Gamma)-1} \smallsetminus \mathbb{V}_{\Gamma/e})]$$

$$+(\mathbb{T}^2 + \mathbb{T} + 1)[(\mathbb{A}^{\#Var(\Gamma)-1} \smallsetminus \mathbb{V}_{\Gamma/e})]$$

$$= (\mathbb{T}^2 + 2\mathbb{T} + 1)[\mathbb{V}_{\Gamma/e}] - (\mathbb{T}+1)^2[\mathbb{V}_{\Gamma/e} \cap \mathbb{V}_{\Gamma\smallsetminus e}] - (\mathbb{T}+1)[Z(V_{\Gamma\smallsetminus e} - V_{\Gamma/e}) \cap (\mathbb{A}^{\#Var(\Gamma)-1} \smallsetminus \mathbb{V}_{\Gamma/e})]$$

$$+(\mathbb{T}^2 + \mathbb{T} + 1)[(\mathbb{A}^{\#Var(\Gamma)-1} \smallsetminus \mathbb{V}_{\Gamma/e})]$$

$$= \mathbb{T}[\mathbb{V}_{\Gamma/e}] - (\mathbb{T}+1)^2[\mathbb{V}_{\Gamma/e} \cap \mathbb{V}_{\Gamma\smallsetminus e}] - (\mathbb{T}+1)[Z(V_{\Gamma\smallsetminus e} - V_{\Gamma/e}) \cap (\mathbb{A}^{\#Var(\Gamma)-1} \smallsetminus \mathbb{V}_{\Gamma/e})]$$

$$+(\mathbb{T}^2 + \mathbb{T} + 1)\mathbb{L}^{\#Var(\Gamma)-1}$$

$$= (\mathbb{T}+1)^2(\mathbb{L}^{\#Var(\Gamma)-1} - [\mathbb{V}_{\Gamma/e} \cap \mathbb{V}_{\Gamma\smallsetminus e}]) - \mathbb{T}(\mathbb{L}^{\#Var(\Gamma)-1} - [\mathbb{V}_{G/e}])$$

$$-(\mathbb{T}+1)[Z(V_{\Gamma\smallsetminus e} - V_{\Gamma/e}) \cap (\mathbb{A}^{\#Var(\Gamma)-1} \smallsetminus \mathbb{V}_{\Gamma/e})].$$

We can express $\{\mathbb{V}_{\Gamma/e} \cap \mathbb{V}_{\Gamma\smallsetminus e}\}$ using the deletion-contraction relation (4.1), obtaining

$$(\mathbb{T}+1)(\{\mathbb{V}_\Gamma\} + \{\mathbb{V}_{\Gamma/e}\}) - \mathbb{T}(\{\mathbb{V}_{\Gamma/e}\}) - (\mathbb{T}+1)[Z(V_{\Gamma\smallsetminus e} - V_{\Gamma/e}) \cap (\mathbb{A}^{\#Var(\Gamma)-1} \smallsetminus \mathbb{V}_{\Gamma/e})]$$

$$= (\mathbb{T}+1)(\{\mathbb{V}_\Gamma\}) + \{\mathbb{V}_{\Gamma/e}\} - (\mathbb{T}+1)[Z(V_{\Gamma\smallsetminus e} - V_{\Gamma/e}) \cap (\mathbb{A}^{\#Var(\Gamma)|-1} \smallsetminus \mathbb{V}_{\Gamma/e})].$$

Then, since we have

$$(\mathbb{T}+1)[Z(V_{\Gamma\smallsetminus e} - V_{\Gamma/e}) \cap (\mathbb{A}^{\#Var(\Gamma)-1} \smallsetminus \mathbb{V}_{\Gamma/e})] = Z(V_{\Gamma\smallsetminus e} - V_{\Gamma/e}) \smallsetminus (\mathbb{V}_{\Gamma/e} \cap \mathbb{V}_{\Gamma\smallsetminus e}),$$

applying deletion-contraction once more, we get that the class of the complement is equal to (4.5). $\qquad\square$

As in Theorem 6.4 of [1], we then obtain the recursion relation for multiple parallel edges. Again, this is unaltered with respect to the case without magnetic field.

Corollary 4.2. *Let $^m\Gamma$ be the graph obtained by adding $m \geq 2$ edges parallel to e in Γ. The Grothendieck classes $\{\mathbb{V}_{^m\Gamma}\}$ of the hypersurface complements satisfy the recursion relation*

(4.8) $$\{\mathbb{V}_{^m\Gamma}\} = (2\mathbb{T}+1)\{\mathbb{V}_{^{m-1}\Gamma}\} - \mathbb{T}(\mathbb{T}+1)\{\mathbb{V}_{^{m-2}\Gamma}\}.$$

Proof. As in [1], the relation (4.8) is obtained from the relation for the class $\{\mathbb{V}_{^e\Gamma}\}$ of the graph $^e\Gamma$ obtained by doubling the edge e. $\qquad\square$

Thus, for the case with magnetic field, we obtain the same generating function as in Theorem 6.5 of [1] for the original case without magnetic field.

Corollary 4.3. *The exponential generating function for the hypersurface complements $\{\mathbb{V}_{^m\Gamma}\}$ is*

(4.9) $$\sum_{m\geq 0}\{\mathbb{V}_{^m\Gamma}\}\frac{s^m}{m!} = ((\mathbb{T}+1)\{\mathbb{V}_\Gamma\} - \{\mathbb{V}_{^1\Gamma}\})e^{\mathbb{T}s} + (\{\mathbb{V}_{^1\Gamma}\} - \mathbb{T}\{\mathbb{V}_\Gamma\})e^{(\mathbb{T}+1)s}.$$

Proof. The argument is the same as in Theorem 6.5 of [1]. $\qquad\square$

4.3. **Potts models with magnetic field on banana graphs.** As in [1], we can use the recursive formula for multiple edges described above to compute the Grothendieck classes of the hypersurface complements for the banana graphs. The n-th banana graph is the graph with two vertices of valence n and n parallel edges connecting them.

Lemma 4.4. $^0\Gamma$ *is the graph with two vertices and a single non-looping edge between them. The corresponding Grothendieck class is*

$$(4.10) \qquad \{\mathbb{V}_{0\Gamma}\} = (\mathbb{T}+1)\mathbb{T}^2 + (\mathbb{T}+1)^2\mathbb{T}^2.$$

Proof. The **V**-polynomial of $^0\Gamma$ is given by

$$(4.11) \qquad V_{0\Gamma} = t_e x_{c1+c2} + x_{c1}x_{c2}.$$

If $x_{c1+c2} = 0$, then t_e is free and $x_{c1}x_{c2} \neq 0$, contributing a class

$$(4.12) \qquad \mathbb{L}(\mathbb{L}-1)^2 = (\mathbb{T}+1)\mathbb{T}^2$$

If $x_{c1+c2} \neq 0$, then $t_e \neq -\frac{x_{c1}x_{c2}}{x_{c1+c2}}$ contributing a class

$$(4.13) \qquad \mathbb{L}^2(\mathbb{L}-1)^2 = (\mathbb{T}+1)^2\mathbb{T}^2.$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Lemma 4.5. $^1\Gamma$ *is the graph with two vertices and a pair of parallel edges, e, f, between them. Then the Grothendieck class is*

$$(4.14) \qquad \{\mathbb{V}_{1\Gamma}\} = (\mathbb{T}+1)^2\mathbb{T}^2 + (\mathbb{T}+1)^2\mathbb{T}^3 + \mathbb{T}(\mathbb{T}+1)(\mathbb{T}^2+\mathbb{T}+1).$$

Proof. The **V**-polynomial of this graph is

$$(4.15) \qquad V_{1\Gamma} = x_{c1+c2}(t_e + t_f + t_e t_f) + x_{c1}x_{c2}.$$

If $x_{c1+c2} = 0$, then $x_{c1}x_{c2} \neq 0$ and t_e and t_f are free, contributing a class

$$(4.16) \qquad \mathbb{L}^2(\mathbb{L}-1)^2 = (\mathbb{T}+1)^2\mathbb{T}^2$$

If $x_{c1+c2} \neq 0$, then $x_{c1+x2}(t_f + 1)t_e \neq -x_{c1}x_{c2} - t_f$. So if $t_f \neq -1$, this is equivalent to

$$t_e \neq \frac{-x_{c1}x_{c2} - t_f}{x_{c1+x2}(t_f+1)},$$

contributing a class

$$(4.17) \qquad \mathbb{L}^2(\mathbb{L}-1)^3 = (\mathbb{T}+1)^2\mathbb{T}^3.$$

If $t_f = -1$, then this is equivalent to $x_{c1}x_{c2} \neq -1$, contributing a class

$$(4.18) \qquad \mathbb{L}(\mathbb{L}-1)((\mathbb{L}-1)^2 + \mathbb{L}) = \mathbb{T}(\mathbb{T}+1)(\mathbb{T}^2+\mathbb{T}+1).$$

Therefore, in total, we obtain (4.14). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

We then obtain the following.

Theorem 4.6. *The Grothendieck class $\{\mathbb{V}_{m\Gamma}\}$ of the hypersurface complement for the Potts model with magnetic field on the m-th banana graph is given by*

$$(4.19) \qquad \{\mathbb{V}_{m\Gamma}\} = -(1+\mathbb{T})\mathbb{T}^{m+1} + \mathbb{T}(\mathbb{T}+1)^{m+3}.$$

Proof. Using Lemmata 4.4 and 4.5, together with the generating function (4.9), we obtain

$$\{\mathbb{V}_{^m\Gamma}\} = ((\mathbb{T}+1)((\mathbb{T}+1)\mathbb{T}^2+(\mathbb{T}+1)^2\mathbb{T}^2)-((\mathbb{T}+1)^2\mathbb{T}^2+(\mathbb{T}+1)^2\mathbb{T}^3+\mathbb{T}(\mathbb{T}+1)(\mathbb{T}^2+\mathbb{T}+1)))\mathbb{T}^m$$

$$+(((\mathbb{T}+1)^2\mathbb{T}^2+(\mathbb{T}+1)^2\mathbb{T}^3+\mathbb{T}(\mathbb{T}+1)(\mathbb{T}^2+\mathbb{T}+1))-\mathbb{T}((\mathbb{T}+1)\mathbb{T}^2+(\mathbb{T}+1)^2\mathbb{T}^2))(\mathbb{T}+1)^m$$

Simplifying, we obtain (4.19). $\qquad\square$

Remark 4.7. One should compare (4.19) with the case without magnetic field described in Example 6.6 of [1]. The Grothendieck class of the hypersurface complement of the m-th banana graph, in the case of the Potts model without magnetic field is given by $\mathbb{T}^m + (\mathbb{T}-1)(\mathbb{T}+1)^{m+1}$. While the presence of the magnetic field does not change the form of the recursion and the generating function, it does affect the individual terms, hence the resulting expression as a function of $\mathbb{T}$.

4.4. **The Euler characteristic with compact support.** The Euler characteristic factors through the Grothendieck ring. In the example of the banana graphs, we can then compute the Euler characteristic with compact support $\chi_c(\mathbb{V}_{^m\Gamma}(\mathbb{R}))$ of the set of real zeros $\mathbb{V}_{^m\Gamma}(\mathbb{R})$ of the partition function of the Potts model with magnetic field.

Proposition 4.8. *For the m-th banana graph $^m\Gamma$, the Euler characteristic with compact support $\chi_c(\mathbb{V}_{^m\Gamma}(\mathbb{R}))$ is given by*

(4.20) $$\chi_c(\mathbb{V}_{^m\Gamma}(\mathbb{R})) = (-2)^{m+1} + (-1)^m 2$$

Proof. This follows directly from (4.19) using the fact that $\chi_c(\mathbb{T}) = \chi_c(\mathbb{G}_m(\mathbb{R})) = -2$, and the fact that χ_c defines a ring homomorphism $\chi_c : K_0(\mathcal{V}_{\mathbb{R}}) \to \mathbb{Z}$. $\qquad\square$

The expression for the Euler characteristic with compact support is concise, reflecting the simplicity of the banana graphs, but the rate of growth is exponential, reflecting the complexity of the **V**-polynomial.

5. **V**-POLYNOMIAL EVALUATION (COMPUTATIONAL COMPLEXITY)

The **V**-polynomial grows exponentially with the size of the graph. Thus, rather than considering the complexity of the problem of providing the entire **V**-polynomial of a graph, we focus on other related computational complexity questions. In the literature on the complexity of various combinatorial polynomials of graphs that arise in relation to Potts models, problems of computing a specific coefficient of the polynomial and of evaluating the polynomial at a point have been characterized, see [17]. Since for the **V**-polynomial all the coefficients are equal to one, the relevant complexity problem remains that of evaluating the polynomial at a point. This problem is important because for the cases in which evaluating the polynomial at a point is tractable, determining whether a point is a phase transition is also tractable.

The specific problem is determining, given a graph Γ and the specification of a point where all variables not included are assumed to be zero, whether the value of the **V**-polynomial at that point is greater than a number N.

5.1. **Potts model with magnetic field on the line graph and on polygons.** In [17], computing the partition function for a restricted case of the **W**-polynomial was shown to be NP-hard for the star graph, so we investigate the complexity of more restricted trees. The first tree we consider is the 1-D Potts model or line graph, which is the tree with n consecutive vertices, with every vertex but the first and the last having two neighbors each.

Theorem 5.1. *Evaluating the **V**-polynomial for the line graph is tractable*

Proof. We can see that the Fourtuin-Kastelyn representation (1.1) for the polynomial has exponentially many terms in the size of the line. We label the vertices 1 through n from left to right, and label the edges e_1, e_2, ..., e_{n-1} also from left to right. Then the Fourtuin-Kastelyn representation is

$$(5.1) \qquad \sum_{A \subseteq E} x_{c_1} x_{c_2} ... x_{c_{k(A)}} \prod_{e \in A} t_e$$

where c_i is the sum of the weights of all the vertices in the i-th connected component of A, and $k(A)$ is the number of connected components.

Thus, to every connected component that appears in a subset of edges A we can associate a product of terms that appears in the sum $\chi_{i,j} = x_{c_{i,j}} \prod_{k=i}^{j} t_{e_k}$, where $c_{i,j} = \sum_{k=i}^{j} w_k$, with w_i the weight of the i-th vertex. We can think of $\chi_{i,j}$ as corresponding to the connected component that starts at vertex i and ends j with the vertices also labeled from left to right. There are only $O(n^2)$ connected components, so the values of these connected components can be computed and stored in a table in polynomial time. Now the sum can be rewritten as

$$(5.2) \qquad \sum_{A \subseteq E} \prod_{k=1}^{k(A)} \chi_{i_k, j_k}.$$

Computing the sum naively still would require a prohibitive amount of time, since there are exponentially many terms. So we use the recursive structure of the line. Let $V_i(x, t)$ be the **V**-polynomial of the sub-line starting at vertex i. Every subset $A \subseteq E$ has a first connected component, namely the connected component containing v_1, which can be represented by $\chi_{1,i}$. Thus, we can write our sum for $V_1(x, t)$ as

$$(5.3) \qquad V_1(x, t) = \sum_{i=1}^{n} \chi_{1,i} V_{i+1}(x, t).$$

Now, we use dynamic programming to compute the sum in $O(n^2)$ time. In round j, assume that we have the values of V_{j+1} through V_n precomputed. Then we can compute

$$(5.4) \qquad V_j(x, t) = \sum_{i=j}^{n} \chi_{j,i} V_{i+1}(x, t)$$

in $n - j$ steps. Therefore, number of steps it takes to compute rounds 1 through n is

$$\sum_{i=1}^{n} i = O(n^2).$$

$\square$

As an easy consequence, we see that polygonal graphs are also tractable.

Corollary 5.2. *Evaluating the **V**-polynomial on the polygonal graphs is tractable.*

Proof. Let $^{(n)}\Gamma$ be the circle graph with n edges. It is tractable since $^{(n)}\Gamma \smallsetminus e$ is the line graph with $n-1$ edges, and $^{(n)}\Gamma/e =^{(n-1)}\Gamma$. So $T(n) \leq T(n-1) + O(n^2)$, where $T(n)$ is the number of steps to evaluate $^{(n)}\Gamma$. Therefore

$$(5.5) \qquad T(n) \leq \sum_{i=1}^{n} O(n^2) \in O(n^3)$$

$\square$

5.2. **Potts model with magnetic field on full binary trees.** The next case we consider is the full binary tree. In contrast to the previous cases, we have the following result.

Theorem 5.3. *Evaluating the **V**-polynomial for the binary tree is NP-hard.*

Proof. We reduce from $\frac{1}{2}$-partition. The problem is, if we are given a finite subset $S \subset \mathbb{N}$, to determine whether we can partition S into two parts with equal sums. Given a subset S of size n, let $M = \sum_{s \in S} s$. Then create a binary tree of height $\lceil \log(n) \rceil + 1$. Now number S and set the vertex weights for the first n leaf nodes to be $s_i \in S$. Set the weights of the rest of the leaf nodes to zero and set the vertex weights of all other nodes to $\lfloor \frac{3}{2}M \rfloor$. We then evaluate the polynomial at a point where we set $x_{s_i} = 1$, $t_e = 1$, $x_{m \lfloor \frac{3}{2}M \rfloor + \lfloor \frac{1}{2}M \rfloor} = 1$, for all m such that $m \leq 2^{\lceil \log(n) \rceil}$, with all the other x variables set to zero.

The **V**-polynomial evaluates to something greater than 0 if and only if there exists a $\frac{1}{2}$-partition for S. To see this, observe the Fourtuin-Kastelyn representation of the polynomial

$$(5.6) \qquad \sum_{A \subseteq E} x_{c_1} x_{c_2} ... x_{c_{k(A)}} \prod_{e \in A} t_e$$

Now, if any term is non-zero, then that means that, since A has edges, for some i we have $x_{c_i} = 1$, with $c_i = m \lfloor \frac{3}{2}M \rfloor + \lfloor \frac{1}{2}M \rfloor$. The only way that this could happen would be as a contribution of $\frac{1}{2}M$ from the leaf nodes.

Conversely, suppose there exists a subset $B \subseteq S$, such that $\sum_{b \in B} b = \frac{1}{2}M$. Then look at the subset A of E in which we take all of the edges above the nodes labelled with elements in B, as well as all of the other edges in the tree. The x_{c_i} variables in that term will correspond either to the unique non-singleton connected component defined by the edge subset B, or to some $s_i \in S$. In the first case, there are an odd number of nodes with vertex weight $\frac{3}{2}M$ and leaf nodes with vertex weights that contribute another $\frac{1}{2}M$. In the second case, all the variables in the term are equal to one. Therefore, we have

$$(5.7) \qquad \sum_{A \subseteq E} x_{c_1} x_{c_2} ... x_{c_{k(A)}} \prod_{e \in A} t_e \geq 1.$$

$\square$

5.3. **Trees limiting to the line graph.** The method used in Theorem 5.3 is sufficiently general that we can extend it to a class of trees that limits to the line graph.

Definition 5.4. *A tree is a $1 + \frac{1}{n}$-ary tree, if $\frac{1}{n}$ of its non-leaf nodes have two children, with the rest having one child.*

For instance, a $1 + 1$-ary tree would be a binary tree. Note that, in general, there are many non-isomorphic $1 + \frac{1}{n}$-ary trees of the same size. However, if two $1 + \frac{1}{n}$-ary trees are the same size, then they have the same number of leaves. We can see this because the number of leaves is equal to the number of nodes with 2 descendents plus 1, by induction. In particular, for all n, the number of leaves increases linearly with the size of the $1 + \frac{1}{n}$-ary tree. The following statement takes into account this ambiguity in the non-isomorphic trees.

Theorem 5.5. *For all $n \in \mathbb{N}$ evaluating the $\boldsymbol{V}$-polynomial on a family of $1 + \frac{1}{n}$-ary trees with elements of every possible size is NP hard.*

Proof. Suppose given an instance of $\frac{1}{2}$-partition of size $|S| = m$. Place the nodes on the leaves of one of the $1 + \frac{1}{n}$-ary trees, T, with a minimal number of nodes, while still having more than m leaves. Set $x_{s_i} = 1$, $t_e = 1$, and $x_{m\lfloor \frac{3}{2}M \rfloor + \lfloor \frac{1}{2}M \rfloor} = 1$, where m is less than the number of nodes of T. Then, by same reasoning as above, the polynomial evaluates to something greater than zero if and only if there exists a $\frac{1}{2}$-partition. $\qquad\square$

This shows that the binary tree is NP-hard not because of its completeness, but because it has a constant proportion of branching.

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